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## THEORY OF $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -CONNECTEDNESS

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ABSTRACT. Several specific types of ordinary and generalized connectedness in a generalized topological space have been defined and investigated for various purposes from time to time in the literature of topological spaces. Our recent research in the field of a new type of generalized connectedness in a generalized topological space is reported herein as a starting point for more generalized types.

KEY WORDS AND PHRASES. *Generalized topological space, generalized local connectedness, generalized pathwise connectedness, generalized local pathwise connectedness, generalized simple connectedness, generalized components*

### 1. INTRODUCTION

Among the most important topological properties (briefly,  $\mathcal{T}$ -properties relative to ordinary topology, and  $\mathcal{T}_{\mathfrak{g}}$ -properties relative to generalized topology), the  $\mathcal{T}$ -properties<sup>1</sup> called  $\mathfrak{T}$ -connectedness and  $\mathfrak{g}\text{-}\mathfrak{T}$ -connectedness in  $\mathcal{T}$ -spaces (ordinary and generalized connectedness in ordinary topological spaces) and the  $\mathcal{T}_{\mathfrak{g}}$ -properties called  $\mathfrak{T}_{\mathfrak{g}}$ -connectedness and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness in  $\mathcal{T}_{\mathfrak{g}}$ -spaces (ordinary and generalized connectedness in generalized topological spaces) are no doubt the most important invariant properties [3, 7, 27]. Indeed,  $\mathfrak{T}$ -connectedness is an absolute property of a  $\mathfrak{T}$ -set [3, 13, 31], and  $\mathfrak{g}\text{-}\mathfrak{T}$ -connectedness,  $\mathfrak{T}_{\mathfrak{g}}$ -connectedness and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness, respectively, are absolute properties of a  $\mathfrak{g}\text{-}\mathfrak{T}$ -set, a  $\mathfrak{T}_{\mathfrak{g}}$ -set, and a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -set [15, 27]. Typical examples of  $\mathfrak{g}\text{-}\mathfrak{T}$ -connectedness in  $\mathcal{T}$ -spaces are  $\alpha$ ,  $\beta$ ,  $\gamma$ -connectedness [10, 16, 24]; examples of  $\mathfrak{T}_{\mathfrak{g}}$ -connectedness in  $\mathcal{T}_{\mathfrak{g}}$ -spaces are semi\* $\alpha$ , s, gb-connectedness [7, 14, 26], whereas examples of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness in  $\mathcal{T}_{\mathfrak{g}}$ -spaces are  $\text{bT}^{\mu}$ ,  $\mu\text{-rgb}$ ,  $\pi\text{p}$ -connectedness [4, 21, 30], among others.

In the literature of  $\mathcal{T}_{\mathfrak{g}}$ -spaces, the study of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets by various authors has produced some new classes of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness in  $\mathcal{T}_{\mathfrak{g}}$ -spaces, similar in descriptions to  $\mathfrak{g}\text{-}\mathfrak{T}$ -connectedness in  $\mathcal{T}$ -spaces [5, 4, 17]. By using the  $\theta$ -modification generalized topology and  $\gamma\theta$ -operator introduced by [23], [28] have extended the notion

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<sup>1</sup>Notes to the reader:  $\mathfrak{T} = (\Omega, \mathcal{T})$ ,  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  are topological spaces (briefly,  $\mathcal{T}$ -space and  $\mathcal{T}_{\mathfrak{g}}$ -space) with ordinary and generalized topologies  $\mathcal{T}$  and  $\mathcal{T}_{\mathfrak{g}}$  (briefly, topology and  $\mathfrak{g}$ -topology). Subsets of  $\mathfrak{T}$ ,  $\mathfrak{T}_{\mathfrak{g}}$ , respectively, are called  $\mathfrak{T}$ ,  $\mathfrak{T}_{\mathfrak{g}}$ -sets; subsets of  $\mathcal{T}$ ,  $\mathcal{T}_{\mathfrak{g}}$ , respectively, are called  $\mathcal{T}$ ,  $\mathcal{T}_{\mathfrak{g}}$ -open sets, and their complements are called  $\mathcal{T}$ ,  $\mathcal{T}_{\mathfrak{g}}$ -closed sets. Generalizations of  $\mathfrak{T}$ -sets,  $\mathcal{T}$ -open and  $\mathcal{T}$ -closed sets, respectively, are called  $\mathfrak{g}\text{-}\mathfrak{T}$ -sets,  $\mathfrak{g}\text{-}\mathcal{T}$ -open and  $\mathfrak{g}\text{-}\mathcal{T}$ -closed sets; generalizations of  $\mathfrak{T}_{\mathfrak{g}}$ -sets,  $\mathcal{T}_{\mathfrak{g}}$ -open and  $\mathcal{T}_{\mathfrak{g}}$ -closed sets, respectively, are called  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets,  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}$ -open and  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}$ -closed sets. Connectedness in  $\mathfrak{T}$  with  $\mathfrak{T}$ ,  $\mathfrak{g}\text{-}\mathfrak{T}$ -sets are called  $\mathfrak{T}$ ,  $\mathfrak{g}\text{-}\mathfrak{T}$ -connectedness, respectively; connectedness in  $\mathfrak{T}_{\mathfrak{g}}$  with  $\mathfrak{T}_{\mathfrak{g}}$ ,  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets are called  $\mathfrak{T}_{\mathfrak{g}}$ ,  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness, respectively.

of  $\theta$ -connectedness [1] to the setting of  $\mathcal{T}_{\mathfrak{g}}$ -spaces and studied its  $\mathcal{T}_{\mathfrak{g}}$ -properties accordingly. Based on the work of [10], [5] have introduced a new type of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness in  $\mathcal{T}_{\mathfrak{g}}$ -spaces called hyperconnected and studied the  $\mathcal{T}_{\mathfrak{g}}$ -properties associated with it and its analogue in the generalized sense. In the same year, [29] have introduced, studied and exemplified the notion of extremally  $\mu$ -disconnected  $\mathcal{T}_{\mathfrak{g}}$ -spaces, just to name a few.

In view of the above references, it would appear that, from every new type of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -set introduced in a  $\mathcal{T}_{\mathfrak{g}}$ -space, there can be introduced a new type of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness in the  $\mathcal{T}_{\mathfrak{g}}$ -space. Having introduced a new class of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets and studied from it some  $\mathcal{T}_{\mathfrak{g}}$ -properties in a  $\mathcal{T}_{\mathfrak{g}}$ -space (see *Theories of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -Sets,  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -Maps, and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -Separation Axioms*), it seems, therefore, reasonable to introduce a new type of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness in the  $\mathcal{T}_{\mathfrak{g}}$ -space and discuss its  $\mathcal{T}_{\mathfrak{g}}$ -properties. In this chapter, we attempt to make a contribution to such a development by introducing a new theory, called *Theory of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -Connectedness*, in which it is presented a new generalized version of  $\mathfrak{T}_{\mathfrak{g}}$ -connectedness in terms of the notion of  $\mathfrak{g}\text{-}\mathfrak{T}$ -set, discussing the fundamental properties and giving its characterizations on this ground.

The chapter is organised as follows: In SECT. 2, preliminary notions are described in SECT. 2.1 and the main results of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness in a  $\mathcal{T}_{\mathfrak{g}}$ -space are reported in SECT. 3. In SECT. 4, the establishment of the relationships among various types of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness are discussed in SECT. 4.1. To support the work, a nice application of the concept of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness in a  $\mathcal{T}_{\mathfrak{g}}$ -space is presented in SECT. 4.2. Finally, SECT. 4.3 provides concluding remarks and future directions of the notion of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness in a  $\mathcal{T}_{\mathfrak{g}}$ -space.

## 2. THEORY

2.1. PRELIMINARIES. Though foreign terms presented themselves here are neatly defined in our previous works (see *Theories of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -Sets,  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -Maps, and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -Separation Axioms*), we thought it necessary to recall some of the most essential concepts presented in those works.

Throughout, the universe of discourses is the *master* set  $\mathfrak{U}$ ; it is fixed within the framework of the theory of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness and contains as elements all sets ( $\Lambda$ -sets:  $\Lambda \in \{\Omega, \Sigma\}$  and  $\Gamma$ -subset:  $\Gamma \subseteq \Omega$  or  $\Gamma \subseteq \Sigma$ ;  $\mathcal{T}_{\Lambda}$ ,  $\mathfrak{g}\text{-}\mathcal{T}_{\Lambda}$ ,  $\mathfrak{T}_{\Lambda}$ ,  $\mathfrak{g}\text{-}\mathfrak{T}_{\Lambda}$ -sets;  $\mathcal{T}_{\mathfrak{g},\Lambda}$ ,  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Lambda}$ ,  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ ,  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Lambda}$ -sets) considered in this theory, and  $I_n^0 \stackrel{\text{def}}{=} \{\nu \in \mathbb{N}^0 : \nu \leq n\}$ ; the construction of the index sets  $I_{\infty}^0, I_n^*, I_{\infty}^*$  follows analogously. Let  $\Lambda \in \{\Omega, \Sigma\} \subset \mathfrak{U}$  be a given set and let  $\mathcal{P}(\Lambda) \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g},\nu} \subseteq \Lambda : \nu \in I_{\infty}^*\}$  be the collection of all subsets  $\mathcal{O}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}, \dots$ , of  $\Lambda$ . Then every one-valued map  $\mathcal{T}_{\mathfrak{g},\Lambda} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$  satisfying  $\mathcal{T}_{\mathfrak{g},\Lambda}(\emptyset) = \emptyset$ ,  $\mathcal{T}_{\mathfrak{g},\Lambda}(\mathcal{O}_{\mathfrak{g}}) \subseteq \mathcal{O}_{\mathfrak{g}}$ , and  $\mathcal{T}_{\mathfrak{g},\Lambda}(\bigcup_{\nu \in I_{\infty}^*} \mathcal{O}_{\mathfrak{g},\nu}) = \bigcup_{\nu \in I_{\infty}^*} \mathcal{T}_{\mathfrak{g},\Lambda}(\mathcal{O}_{\mathfrak{g},\nu})$  is called an absolute  $\mathfrak{g}$ -topology on  $\Lambda$ , and  $\mathfrak{T}_{\mathfrak{g},\Lambda} \stackrel{\text{def}}{=} (\Lambda, \mathcal{T}_{\mathfrak{g},\Lambda})$  is called a  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -space, on which no separation axioms are assumed unless otherwise mentioned [12, 11, 25]. For any subset  $\Gamma \subseteq \Lambda$  of  $\Lambda$ , every one-valued restriction map  $\mathcal{T}_{\mathfrak{g},\Gamma} : \mathcal{P}(\Gamma) \mapsto \mathcal{T}_{\mathfrak{g},\Gamma} \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g},\Lambda}\}$ , then, defines a relative  $\mathfrak{g}$ -topology on  $\Gamma$ , and  $\mathfrak{T}_{\mathfrak{g},\Gamma} \stackrel{\text{def}}{=} (\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma})$  is called a  $\mathcal{T}_{\mathfrak{g}}$ -subspace.

The operator  $\text{cl}_{\mathfrak{g},\Lambda} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$  carrying each  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$  into its closure  $\text{cl}_{\mathfrak{g},\Lambda}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g},\Lambda} - \text{int}_{\mathfrak{g},\Lambda}(\mathfrak{T}_{\mathfrak{g},\Lambda} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$  is called a  $\mathfrak{g}$ -closure operator and the operator  $\text{int}_{\mathfrak{g},\Lambda} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$  carrying each  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$  into its interior  $\text{int}_{\mathfrak{g},\Lambda}(\mathcal{S}_{\mathfrak{g}}) = \mathfrak{T}_{\mathfrak{g},\Lambda} - \text{cl}_{\mathfrak{g},\Lambda}(\mathfrak{T}_{\mathfrak{g},\Lambda} \setminus \mathcal{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$  is termed a  $\mathfrak{g}$ -interior operator;

for clarity  $\text{cl}_{\mathfrak{g}}(\cdot)$ ,  $\text{int}_{\mathfrak{g}}(\cdot)$ , respectively, are employed instead of  $\text{cl}_{\mathfrak{g},\Lambda}(\cdot)$ ,  $\text{int}_{\mathfrak{g},\Lambda}(\cdot)$ . Let  $\mathbb{C}_{\Lambda} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$  denotes the absolute complement relative to  $\Lambda \subset \mathfrak{U}$ , and let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$  be any  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -set. Then, the classes

$$(2.1) \quad \begin{aligned} \mathcal{T}_{\mathfrak{g},\Lambda} &\stackrel{\text{def}}{=} \{ \mathcal{O}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g},\Lambda} \}, \\ \neg\mathcal{T}_{\mathfrak{g},\Lambda} &\stackrel{\text{def}}{=} \{ \mathcal{K}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : \mathbb{C}_{\Lambda}(\mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g},\Lambda} \}, \end{aligned}$$

respectively, denote the classes of all  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -open and  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -closed sets relative to  $\mathcal{T}_{\mathfrak{g},\Lambda}$ , and the classes

$$(2.2) \quad \begin{aligned} \mathbb{C}_{\mathcal{T}_{\mathfrak{g},\Lambda}}^{\text{sub}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{ \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g},\Lambda} : \mathcal{O}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}} \}, \\ \mathbb{C}_{\neg\mathcal{T}_{\mathfrak{g},\Lambda}}^{\text{sup}}[\mathcal{S}_{\mathfrak{g}}] &\stackrel{\text{def}}{=} \{ \mathcal{K}_{\mathfrak{g}} \in \neg\mathcal{T}_{\mathfrak{g},\Lambda} : \mathcal{K}_{\mathfrak{g}} \supseteq \mathcal{S}_{\mathfrak{g}} \}, \end{aligned}$$

respectively, denote the classes of  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -open subsets and  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -closed supersets (complements of the  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -open subsets) of the  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$  relative to  $\mathcal{T}_{\mathfrak{g},\Lambda}$ . To this end, the  $\mathfrak{g}$ -closure and the  $\mathfrak{g}$ -interior of a  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  in a  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -space [6] define themselves as

$$(2.3) \quad \text{int}_{\mathfrak{g},\Lambda}(\mathcal{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcup_{\mathcal{O}_{\mathfrak{g}} \in \mathbb{C}_{\mathcal{T}_{\mathfrak{g},\Lambda}}^{\text{sub}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{O}_{\mathfrak{g}}, \quad \text{cl}_{\mathfrak{g},\Lambda}(\mathcal{S}_{\mathfrak{g}}) \stackrel{\text{def}}{=} \bigcap_{\mathcal{K}_{\mathfrak{g}} \in \mathbb{C}_{\neg\mathcal{T}_{\mathfrak{g},\Lambda}}^{\text{sup}}[\mathcal{S}_{\mathfrak{g}}]} \mathcal{K}_{\mathfrak{g}}.$$

The notations  $\text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\cdot)$ ,  $\text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\cdot)$ , and  $\text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\cdot)$ , respectively, mean  $\text{cl}_{\mathfrak{g}}(\text{int}_{\mathfrak{g}}(\cdot))$ ,  $\text{int}_{\mathfrak{g}}(\text{cl}_{\mathfrak{g}}(\cdot))$ , and  $\text{cl}_{\mathfrak{g}}(\text{int}_{\mathfrak{g}}(\text{cl}_{\mathfrak{g}}(\cdot)))$ ; other composition operators are defined in a similar way. Also, the backslash  $\mathfrak{T}_{\mathfrak{g}} \setminus \mathcal{S}_{\mathfrak{g}}$  refers to the set-theoretic difference  $\mathfrak{T}_{\mathfrak{g}} - \mathcal{S}_{\mathfrak{g}}$ . A  $\mathfrak{g}$ -operation on  $\mathcal{P}(\Lambda)$  is a map  $\text{op}_{\mathfrak{g}} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$  satisfying the following statements:

$$(2.4) \quad \begin{aligned} &\forall \mathcal{S}_{\mathfrak{g}} \in \mathcal{P}(\Lambda) \setminus \{\emptyset\}, \exists (\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g},\Lambda} \setminus \{\emptyset\} \times \neg\mathcal{T}_{\mathfrak{g},\Lambda} \setminus \{\emptyset\} : \\ &(\text{op}_{\mathfrak{g}}(\emptyset) = \emptyset) \vee (\neg\text{op}_{\mathfrak{g}}(\emptyset) = \emptyset), \quad (\mathcal{S}_{\mathfrak{g}} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})), \end{aligned}$$

where  $\neg\text{op}_{\mathfrak{g}} : \mathcal{P}(\Lambda) \rightarrow \mathcal{P}(\Lambda)$  is called the "complementary  $\mathfrak{g}$ -operation" on  $\mathcal{P}(\Lambda)$  and, for all  $\mathfrak{T}_{\mathfrak{g}}$ -sets  $\mathcal{S}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g},\nu}, \mathcal{S}_{\mathfrak{g},\mu} \in \mathcal{P}(\Lambda) \setminus \{\emptyset\}$ , the following axioms:

- AX. I.  $(\mathcal{S}_{\mathfrak{g}} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}))$ ,
- AX. II.  $(\text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \text{op}_{\mathfrak{g}} \circ \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})) \vee (\neg\text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \supseteq \neg\text{op}_{\mathfrak{g}} \circ \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}}))$ ,
- AX. III.  $(\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathcal{S}_{\mathfrak{g},\mu} \rightarrow \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\nu}) \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\mu})) \vee (\mathcal{S}_{\mathfrak{g},\mu} \subseteq \mathcal{S}_{\mathfrak{g},\nu} \leftarrow \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\mu}) \supseteq \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\nu}))$ ,
- AX. IV.  $(\text{op}_{\mathfrak{g}}(\bigcup_{\sigma=\nu,\mu} \mathcal{S}_{\mathfrak{g},\sigma}) \subseteq \bigcup_{\sigma=\nu,\mu} \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma})) \vee (\neg\text{op}_{\mathfrak{g}}(\bigcup_{\sigma=\nu,\mu} \mathcal{S}_{\mathfrak{g},\sigma}) \supseteq \bigcup_{\sigma=\nu,\mu} \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\sigma}))$ ,

for some  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -open sets  $\mathcal{O}_{\mathfrak{g}}, \mathcal{O}_{\mathfrak{g},\nu}, \mathcal{O}_{\mathfrak{g},\mu} \in \mathcal{T}_{\mathfrak{g},\Lambda} \setminus \{\emptyset\}$  and  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -closed sets  $\mathcal{K}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g},\nu}, \mathcal{K}_{\mathfrak{g},\mu} \in \neg\mathcal{T}_{\mathfrak{g},\Lambda}$  [8, 22]. The class  $\mathcal{L}_{\mathfrak{g}}[\Omega] = \mathcal{L}_{\mathfrak{g}}^{\omega}[\Lambda] \times \mathcal{L}_{\mathfrak{g}}^{\kappa}[\Omega]$ , where

$$(2.5) \quad \mathcal{L}_{\mathfrak{g}}[\Lambda] \stackrel{\text{def}}{=} \{ \text{op}_{\mathfrak{g},\nu\mu}(\cdot) = (\text{op}_{\mathfrak{g},\nu}(\cdot), \neg\text{op}_{\mathfrak{g},\mu}(\cdot)) : (\nu, \mu) \in I_3^0 \times I_3^0 \}$$

in the  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -space  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ , denotes the class of all possible  $\mathfrak{g}$ -operators and their complementary  $\mathfrak{g}$ -operators in the  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -space  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ . Its elements are defined as:

$$\begin{aligned} \text{op}_{\mathfrak{g}}(\cdot) &\in \mathcal{L}_{\mathfrak{g}}^{\omega}[\Lambda] \stackrel{\text{def}}{=} \{\text{op}_{\mathfrak{g},0}(\cdot), \text{op}_{\mathfrak{g},1}(\cdot), \text{op}_{\mathfrak{g},2}(\cdot), \text{op}_{\mathfrak{g},3}(\cdot)\} \\ &= \{\text{int}_{\mathfrak{g}}(\cdot), \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\cdot), \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\cdot), \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\cdot)\}; \\ \neg \text{op}_{\mathfrak{g}}(\cdot) &\in \mathcal{L}_{\mathfrak{g}}^{\kappa}[\Lambda] \stackrel{\text{def}}{=} \{\neg \text{op}_{\mathfrak{g},0}(\cdot), \neg \text{op}_{\mathfrak{g},1}(\cdot), \neg \text{op}_{\mathfrak{g},2}(\cdot), \neg \text{op}_{\mathfrak{g},3}(\cdot)\} \\ (2.6) \quad &= \{\text{cl}_{\mathfrak{g}}(\cdot), \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}}(\cdot), \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\cdot), \text{int}_{\mathfrak{g}} \circ \text{cl}_{\mathfrak{g}} \circ \text{int}_{\mathfrak{g}}(\cdot)\}. \end{aligned}$$

A  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -set  $\mathcal{S}_{\mathfrak{g},\Lambda} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$  in a  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -space is called a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -set if and only if there exist a pair  $(\mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}) \in \mathcal{T}_{\mathfrak{g},\Lambda} \times \neg \mathcal{T}_{\mathfrak{g},\Lambda}$  of  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -open and  $\mathcal{T}_{\mathfrak{g},\Lambda}$ -closed sets, and a  $\mathfrak{g}$ -operator  $\text{op}_{\mathfrak{g}}(\cdot) \in \mathcal{L}_{\mathfrak{g}}[\Lambda]$  such that the following statement holds:

$$(2.7) \quad (\exists \xi) [(\xi \in \mathcal{S}_{\mathfrak{g}}) \wedge ((\mathcal{S}_{\mathfrak{g}} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g}})))] .$$

The  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda}$  is said to be of category  $\nu$  if and only if it belongs to the following class of  $\mathfrak{g}$ - $\nu$ - $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -sets:

$$(2.8) \quad \begin{aligned} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g},\Lambda}] &\stackrel{\text{def}}{=} \{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : (\exists \mathcal{O}_{\mathfrak{g}}, \mathcal{K}_{\mathfrak{g}}, \text{op}_{\mathfrak{g},\nu}(\cdot)) \\ &[(\mathcal{S}_{\mathfrak{g}} \subseteq \text{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})) \vee (\mathcal{S}_{\mathfrak{g}} \supseteq \neg \text{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}}))]\}. \end{aligned}$$

Accordingly, the classes of  $\mathfrak{g}$ - $\nu$ - $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -open and  $\mathfrak{g}$ - $\nu$ - $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -closed sets, respectively, are defined by

$$(2.9) \quad \begin{aligned} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g},\Lambda}] &\stackrel{\text{def}}{=} \{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : (\exists \mathcal{O}_{\mathfrak{g}}, \text{op}_{\mathfrak{g},\nu}(\cdot)) [\mathcal{S}_{\mathfrak{g}} \subseteq \text{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g}})]\}, \\ \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g},\Lambda}] &\stackrel{\text{def}}{=} \{\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g},\Lambda} : (\exists \mathcal{K}_{\mathfrak{g}}, \text{op}_{\mathfrak{g},\nu}(\cdot)) [\mathcal{S}_{\mathfrak{g}} \supseteq \neg \text{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g}})]\}. \end{aligned}$$

From these classes, the following relation holds:

$$(2.10) \quad \begin{aligned} \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Lambda}] &= \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g},\Lambda}] \\ &= \bigcup_{\nu \in I_3^0} (\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g},\Lambda}] \cup \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g},\Lambda}]) \\ &= (\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g},\Lambda}]) \cup (\bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g},\Lambda}]) \\ &= \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Lambda}] \cup \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Lambda}]. \end{aligned}$$

By omitting the subscript  $\mathfrak{g}$  in almost all symbols of the above definitions, very similar definitions are obtained but in a  $\mathcal{T}_{\Lambda}$ -space. Thus,

$$(2.11) \quad (\exists \xi) [(\xi \in \mathcal{S}) \wedge ((\mathcal{S} \subseteq \text{op}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg \text{op}(\mathcal{K})))] .$$

is the condition which characterises a  $\mathfrak{T}_{\Lambda}$ -set  $\mathcal{S} \subset \mathfrak{T}_{\Lambda}$  in a  $\mathcal{T}_{\Lambda}$ -space as a  $\mathfrak{g}$ - $\mathfrak{T}_{\Lambda}$ -set; it is called a  $\mathfrak{g}$ - $\mathfrak{T}_{\Lambda}$ -set  $\mathcal{S} \subset \mathfrak{T}_{\Lambda}$  of category  $\nu$  if and only if it belongs to the following class of  $\mathfrak{g}$ - $\nu$ - $\mathfrak{T}_{\Lambda}$ -sets:

$$(2.12) \quad \begin{aligned} \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\Lambda}] &\stackrel{\text{def}}{=} \{\mathcal{S} \subset \mathfrak{T}_{\Lambda} : (\exists \mathcal{O}, \mathcal{K}, \text{op}_{\nu}(\cdot)) \\ &[(\mathcal{S} \subseteq \text{op}_{\nu}(\mathcal{O})) \vee (\mathcal{S} \supseteq \neg \text{op}_{\nu}(\mathcal{K}))]\}. \end{aligned}$$

Evidently, the classes of  $\mathfrak{g}$ - $\nu$ - $\mathfrak{T}_{\Lambda}$ -open and  $\mathfrak{g}$ - $\nu$ - $\mathfrak{T}_{\Lambda}$ -closed sets, respectively, are defined by

$$(2.13) \quad \begin{aligned} \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\Lambda}] &\stackrel{\text{def}}{=} \{\mathcal{S} \subset \mathfrak{T}_{\Lambda} : (\exists \mathcal{O}, \text{op}_{\nu}(\cdot)) [\mathcal{S} \subseteq \text{op}_{\nu}(\mathcal{O})]\}, \\ \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\Lambda}] &\stackrel{\text{def}}{=} \{\mathcal{S} \subset \mathfrak{T}_{\Lambda} : (\exists \mathcal{K}, \text{op}_{\nu}(\cdot)) [\mathcal{S} \supseteq \neg \text{op}_{\nu}(\mathcal{K})]\}. \end{aligned}$$

The classes  $O[\mathfrak{T}_{\mathfrak{g},\Lambda}]$  and  $K[\mathfrak{T}_{\mathfrak{g},\Lambda}]$  denote the families of  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -open and  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ -closed sets, respectively, in  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ , with  $S[\mathfrak{T}_{\mathfrak{g},\Lambda}] = O[\mathfrak{T}_{\mathfrak{g},\Lambda}] \cup K[\mathfrak{T}_{\mathfrak{g},\Lambda}]$ ; the classes  $O[\mathfrak{T}_{\Lambda}]$  and  $K[\mathfrak{T}_{\Lambda}]$  denote the families of  $\mathfrak{T}$ -open and  $\mathfrak{T}_{\Lambda}$ -closed sets, respectively, in  $\mathfrak{T}_{\Lambda}$ , with  $S[\mathfrak{T}_{\Lambda}] = O[\mathfrak{T}_{\Lambda}] \cup K[\mathfrak{T}_{\Lambda}]$ . (Whenever we feel that the subscript  $\Lambda \in \{\Omega, \Sigma\}$  is understood from the context, it will be omitted for clarity.)

A  $(\mathfrak{T}_{\Omega}, \mathfrak{T}_{\Sigma})$ -map and a  $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -map, respectively, are mappings in the usual sense between  $\mathcal{T}$ -spaces and  $\mathcal{T}_{\mathfrak{g}}$ -spaces. Thus,  $\pi : \mathfrak{T}_{\Omega} \rightarrow \mathfrak{T}_{\Sigma}$  is a  $(\mathfrak{T}_{\Omega}, \mathfrak{T}_{\Sigma})$ -map between  $\mathcal{T}$ -spaces  $\mathfrak{T}_{\Omega} = (\Omega, \mathcal{T}_{\Omega})$  and  $\mathfrak{T}_{\Sigma} = (\Sigma, \mathcal{T}_{\Sigma})$ , and  $\pi : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  is a  $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -map between  $\mathcal{T}_{\mathfrak{g}}$ -spaces  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ .

A map  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\Omega} \rightarrow \mathfrak{T}_{\Sigma}$  is called a  $\mathfrak{g}$ - $(\mathfrak{T}_{\Omega}, \mathfrak{T}_{\Sigma})$ -map if and only if, for every  $(\mathcal{O}_{\omega}, \mathcal{K}_{\omega}) \in \mathcal{T}_{\Omega} \times \neg\mathcal{T}_{\Omega}$ , there exists  $(\mathcal{O}_{\sigma}, \mathcal{K}_{\sigma}) \in \mathcal{T}_{\Sigma} \times \neg\mathcal{T}_{\Sigma}$  such that the following statement holds:

$$(2.14) \quad [\pi_{\mathfrak{g}}(\mathcal{O}_{\omega}) \subseteq \text{op}(\mathcal{O}_{\sigma})] \vee [\pi_{\mathfrak{g}}(\mathcal{K}_{\omega}) \supseteq \neg \text{op}(\mathcal{K}_{\sigma})].$$

It is said to be of category  $\nu$  if and only if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-M}[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}]$  where,

$$(2.15) \quad \mathfrak{g}\text{-}\nu\text{-M}[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}] \stackrel{\text{def}}{=} \{ \pi_{\mathfrak{g}} : (\forall \mathcal{O}_{\omega}, \mathcal{K}_{\omega})(\exists \mathcal{O}_{\sigma}, \mathcal{K}_{\sigma}, \mathbf{op}_{\nu}(\cdot)) \\ [(\pi_{\mathfrak{g}}(\mathcal{O}_{\omega}) \subseteq \mathbf{op}_{\nu}(\mathcal{O}_{\sigma})) \vee (\pi_{\mathfrak{g}}(\mathcal{K}_{\omega}) \supseteq \neg \mathbf{op}_{\nu}(\mathcal{K}_{\sigma}))] \}.$$

Write  $\mathfrak{g}\text{-}\nu\text{-M}[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}] = \mathfrak{g}\text{-}\nu\text{-M}_O[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}] \cup \mathfrak{g}\text{-}\nu\text{-M}_K[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}]$  where,

$$(2.16) \quad \mathfrak{g}\text{-}\nu\text{-M}_O[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}] \stackrel{\text{def}}{=} \{ \pi_{\mathfrak{g}} : (\forall \mathcal{O}_{\omega})(\exists \mathcal{O}_{\sigma}, \mathbf{op}_{\nu}(\cdot)) [\pi_{\mathfrak{g}}(\mathcal{O}_{\omega}) \subseteq \mathbf{op}_{\nu}(\mathcal{O}_{\sigma})] \}, \\ \mathfrak{g}\text{-}\nu\text{-M}_K[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}] \stackrel{\text{def}}{=} \{ \pi_{\mathfrak{g}} : (\forall \mathcal{K}_{\omega})(\exists \mathcal{K}_{\sigma}, \mathbf{op}_{\nu}(\cdot)) [\pi_{\mathfrak{g}}(\mathcal{K}_{\omega}) \supseteq \mathbf{op}_{\nu}(\mathcal{K}_{\sigma})] \}.$$

Then, if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-M}_O[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}]$ , it is said to be a  $\mathfrak{g}\text{-}\nu$ - $(\mathfrak{T}_{\Omega}, \mathfrak{T}_{\Sigma})$ -open map; if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-M}_K[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}]$ , it is said to be a  $\mathfrak{g}\text{-}\nu$ - $(\mathfrak{T}_{\Omega}, \mathfrak{T}_{\Sigma})$ -closed map. As in the foregoing discussion, the establishment of  $\mathfrak{g}\text{-M}[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}] = \mathfrak{g}\text{-M}_O[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}] \cup \mathfrak{g}\text{-M}_K[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}]$  is immediate.

A map  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  is called a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -map if and only if, for every  $(\mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega}) \in \mathcal{T}_{\mathfrak{g},\Omega} \times \neg\mathcal{T}_{\mathfrak{g},\Omega}$ , there exists  $(\mathcal{O}_{\mathfrak{g},\sigma}, \mathcal{K}_{\mathfrak{g},\sigma}) \in \mathcal{T}_{\mathfrak{g},\Sigma} \times \neg\mathcal{T}_{\mathfrak{g},\Sigma}$  such that the following statement holds:

$$(2.17) \quad [\pi_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\omega}) \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma})] \vee [\pi_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\omega}) \supseteq \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\sigma})].$$

It is said to be of category  $\nu$  if and only if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  where,

$$(2.18) \quad \mathfrak{g}\text{-}\nu\text{-M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \stackrel{\text{def}}{=} \{ \pi_{\mathfrak{g}} : (\forall \mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega})(\exists \mathcal{O}_{\mathfrak{g},\sigma}, \mathcal{K}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)) \\ [(\pi_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\omega}) \subseteq \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\sigma})) \vee (\pi_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\omega}) \supseteq \neg \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\sigma}))] \}.$$

Let  $\mathfrak{g}\text{-}\nu\text{-M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] = \mathfrak{g}\text{-}\nu\text{-M}_O[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \cup \mathfrak{g}\text{-}\nu\text{-M}_K[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  where,

$$(2.19) \quad \mathfrak{g}\text{-}\nu\text{-M}_O[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \stackrel{\text{def}}{=} \{ \pi_{\mathfrak{g}} : (\forall \mathcal{O}_{\mathfrak{g},\omega})(\exists \mathcal{O}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)) \\ [\pi_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\omega}) \subseteq \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\sigma})] \}, \\ \mathfrak{g}\text{-}\nu\text{-M}_K[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \stackrel{\text{def}}{=} \{ \pi_{\mathfrak{g}} : (\forall \mathcal{K}_{\mathfrak{g},\omega})(\exists \mathcal{K}_{\mathfrak{g},\sigma}, \mathbf{op}_{\mathfrak{g},\nu}(\cdot)) \\ [\pi_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\omega}) \supseteq \mathbf{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\sigma})] \}.$$

Then, if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-M}_O[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ , it is called a  $\mathfrak{g}\text{-}\nu$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -open map; if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-M}_K[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ , it is called a  $\mathfrak{g}\text{-}\nu$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -closed map. The establishment of  $\mathfrak{g}\text{-M}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] = \mathfrak{g}\text{-M}_O[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \cup \mathfrak{g}\text{-M}_K[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  is immediate.

A map  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  is said to be  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -continuous if and only if, for every  $(\mathcal{O}_{\mathfrak{g},\sigma}, \mathcal{K}_{\mathfrak{g},\sigma}) \in \mathcal{T}_{\mathfrak{g},\Sigma} \times \neg\mathcal{T}_{\mathfrak{g},\Sigma}$ , there exists  $(\mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega}) \in \mathcal{T}_{\mathfrak{g},\Omega} \times \neg\mathcal{T}_{\mathfrak{g},\Omega}$  such that the following statement holds:

$$(2.20) \quad [\pi_{\mathfrak{g}}^{-1}(\mathcal{O}_{\mathfrak{g},\sigma}) \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\omega})] \vee [\pi_{\mathfrak{g}}^{-1}(\mathcal{K}_{\mathfrak{g},\sigma}) \supseteq \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\omega})].$$

It is said to be of category  $\nu$  if and only if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  where,

$$(2.21) \quad \mathfrak{g}\text{-}\nu\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \stackrel{\text{def}}{=} \left\{ \pi_{\mathfrak{g}} : (\forall \mathcal{O}_{\mathfrak{g},\sigma}, \mathcal{K}_{\mathfrak{g},\sigma}) (\exists \mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega}, \text{op}_{\mathfrak{g},\nu}(\cdot)) \right. \\ \left. [(\pi_{\mathfrak{g}}^{-1}(\mathcal{O}_{\mathfrak{g},\sigma}) \subseteq \text{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\omega})) \vee (\pi_{\mathfrak{g}}^{-1}(\mathcal{K}_{\mathfrak{g},\sigma}) \supseteq \neg \text{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\omega}))] \right\}.$$

Obviously,  $\mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] = \bigcup_{\nu \in I_{\mathfrak{g}}^0} \mathfrak{g}\text{-}\nu\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ .

A map  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  is said to be  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -irresolute if and only if, for every  $(\mathcal{O}_{\mathfrak{g},\sigma}, \mathcal{K}_{\mathfrak{g},\sigma}) \in \mathcal{T}_{\mathfrak{g},\Sigma} \times \neg\mathcal{T}_{\mathfrak{g},\Sigma}$ , there exists  $(\mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega}) \in \mathcal{T}_{\mathfrak{g},\Omega} \times \neg\mathcal{T}_{\mathfrak{g},\Omega}$  such that the following statement holds:

$$(2.22) \quad [\pi_{\mathfrak{g}}^{-1}(\text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\sigma})) \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\omega})] \vee [\pi_{\mathfrak{g}}^{-1}(\neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\sigma})) \supseteq \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\omega})].$$

It is said to be a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -irresolute map of category  $\nu$  if and only if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  where,

$$(2.23) \quad \mathfrak{g}\text{-}\nu\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \stackrel{\text{def}}{=} \left\{ \pi_{\mathfrak{g}} : (\forall \mathcal{O}_{\mathfrak{g},\sigma}, \mathcal{K}_{\mathfrak{g},\sigma}) (\exists \mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega}, \text{op}_{\mathfrak{g},\nu}(\cdot)) \right. \\ \left. [(\pi_{\mathfrak{g}}^{-1}(\text{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\sigma})) \subseteq \text{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\omega})) \vee (\pi_{\mathfrak{g}}^{-1}(\neg \text{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\sigma})) \supseteq \neg \text{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\omega}))] \right\}.$$

Evidently,  $\mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] = \bigcup_{\nu \in I_{\mathfrak{g}}^0} \mathfrak{g}\text{-}\nu\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ . The classes  $M_O[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}]$  and  $M_K[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}]$  denote the families of  $\mathfrak{T}$ -open and  $\mathfrak{T}$ -closed maps, respectively, from  $\mathfrak{T}_{\Omega}$  into  $\mathfrak{T}_{\Sigma}$ , with  $M[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}] = M_O[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}] \cup M_K[\mathfrak{T}_{\Omega}; \mathfrak{T}_{\Sigma}]$ . The classes  $M_O[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  and  $M_K[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  denote the families of  $\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{T}_{\mathfrak{g}}$ -closed maps, respectively, from  $\mathfrak{T}_{\mathfrak{g},\Omega}$  into  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ , with  $M[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] = M_O[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \cup M_K[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ .

**DEFINITION 2.1** ( $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -Separation,  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -Connected). A  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separation of category  $\nu$  of two nonempty  $\mathfrak{T}_{\mathfrak{g}}$ -sets  $\mathcal{R}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$  of a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is realised if and only if there exists either a pair  $(\mathcal{U}_{\mathfrak{g},\xi}, \mathcal{U}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$  of nonempty  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -open sets or a pair  $(\mathcal{V}_{\mathfrak{g},\xi}, \mathcal{V}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  of nonempty  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -closed sets such that:

$$(2.24) \quad \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \mathcal{R}_{\mathfrak{g}} \sqcup \mathcal{S}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \mathcal{R}_{\mathfrak{g}} \sqcup \mathcal{S}_{\mathfrak{g}} \right).$$

Two nonempty  $\mathfrak{T}_{\mathfrak{g}}$ -sets  $\mathcal{R}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}} \subseteq \mathfrak{T}_{\mathfrak{g}}$  of a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  which are not  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated of category  $\nu$  are said to be  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected of category  $\nu$ .

The definitions of classes of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated sets of category  $\nu$  follow.

**DEFINITION 2.2.** Let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . Then:

- I. The  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  is said to be  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected of category  $\nu$  if and only if it belongs to the following class of  $\mathfrak{g}$ - $\nu$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected sets:

$$(2.25) \quad \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \left\{ \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : (\forall (\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]) \right. \\ \left. \left[ \neg \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right) \wedge \neg \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right) \right] \right\}.$$

- II. The  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  is said to be  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated of category  $\nu$  if and only if it belongs to the following class of  $\mathfrak{g}$ - $\nu$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated sets:

$$(2.26) \quad \mathfrak{g}\text{-}\nu\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \left\{ \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}} : (\exists (\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]) \right. \\ \left. \left[ \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right) \right] \right\}.$$

The dependence of  $\mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-}\nu\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$  on both  $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  is immediate. Thus, to define the pairs  $(\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g}}], \nu\text{-D}[\mathfrak{T}_{\mathfrak{g}}])$ ,  $(\mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g}}], \mathfrak{g}\text{-}\nu\text{-D}[\mathfrak{T}_{\mathfrak{g}}])$ , and  $(\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g}}], \nu\text{-D}[\mathfrak{T}_{\mathfrak{g}}])$ , respectively, it suffices to let them be dependent on the pairs  $(\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}], \nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}])$ ,  $(\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}], \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}])$ , and  $(\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}], \nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}])$ ; the characters of these classes are found in earlier chapters (see *Theories of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -Sets,  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -Maps, and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -Separation Axioms*). The notations  $\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$  stand for  $\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_{\mathfrak{g}}^0} \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] = \bigcup_{\nu \in I_{\mathfrak{g}}^0} \mathfrak{g}\text{-}\nu\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ , respectively.

REMARK 2.3. In defining the classes  $\mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathfrak{g}\text{-}\nu\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ , it is clear that by the statement  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  is meant a pair of nonempty  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -open and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -closed sets. Furthermore, by  $\Omega \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  or  $\Omega \in \mathfrak{g}\text{-}\nu\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$  is meant a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connection of category  $\nu$  or a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separation of category  $\nu$  of the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is realised.

DEFINITION 2.4. Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space. Then:

- I.  $\mathfrak{T}_{\mathfrak{g}}$  is called a  $\mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)})$  if and only if it is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected of category  $\nu$ .
- II.  $\mathfrak{T}_{\mathfrak{g}}$  is called a  $\mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$  if and only if it is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated of category  $\nu$ .

In the sequel, by  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(LC)} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(LC)})$ ,  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(PC)} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(PC)})$ ,  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(LPC)} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(LPC)})$ , and  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(SC)} \stackrel{\text{def}}{=} (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(SC)})$ , respectively, are meant *locally*, *pathwise*, *locally pathwise*, and *simply*  $\mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces. Finally, by a  $\mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(A)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(A)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(A)})$  is meant  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(A)} = \bigvee_{\nu \in I_{\mathfrak{g}}^0} \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(A)} = (\Omega, \bigvee_{\nu \in I_{\mathfrak{g}}^0} \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(A)}) = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(A)})$ , where  $A \in \{C, LC, PC, LPC, SC, D\}$ . In the following sections, the main results of the theory of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness are presented.

### 3. MAIN RESULTS

If for all  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  neither  $\mathcal{U}_{\mathfrak{g},\xi} \sqcup \mathcal{V}_{\mathfrak{g},\zeta} = \Omega$  nor  $\mathcal{U}_{\mathfrak{g},\zeta} \sqcup \mathcal{V}_{\mathfrak{g},\xi} = \Omega$  is satisfied, then a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is said to be  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated. Hence, the following theorem:



**THEOREM 3.1.** *If  $\mathfrak{g}\text{-}\mathfrak{T}_g^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_g^{(D)})$  be a  $\mathfrak{g}\text{-}\mathcal{T}_g^{(D)}$ -space, then there exists a pair  $(\mathcal{U}_{g,\xi}, \mathcal{V}_{g,\zeta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \times \mathfrak{g}\text{-K}[\mathfrak{T}_g]$  or a pair  $(\mathcal{U}_{g,\zeta}, \mathcal{V}_{g,\xi}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \times \mathfrak{g}\text{-K}[\mathfrak{T}_g]$  such that*

$$(3.1) \quad [\mathcal{U}_{g,\xi} \sqcup \mathcal{V}_{g,\zeta} = \Omega] \vee [\mathcal{U}_{g,\zeta} \sqcup \mathcal{V}_{g,\xi} = \Omega].$$

**PROOF.** Let  $\mathfrak{g}\text{-}\mathfrak{T}_g^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_g^{(D)})$  be a  $\mathfrak{g}\text{-}\mathcal{T}_g^{(D)}$ -space. Then, there exists a pair  $(\mathcal{U}_{g,\lambda}, \mathcal{V}_{g,\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \times \mathfrak{g}\text{-K}[\mathfrak{T}_g]$  such that

$$\left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{g,\lambda} = \Omega \right) \vee \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{g,\lambda} = \Omega \right).$$

If  $\mathcal{U}_{g,\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_g]$  then  $\mathcal{U}_{g,\zeta} = \mathbb{C}(\mathcal{U}_{g,\xi}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_g]$ , and if  $\mathcal{V}_{g,\xi} \in \mathfrak{g}\text{-K}[\mathfrak{T}_g]$  then  $\mathcal{V}_{g,\zeta} = \mathbb{C}(\mathcal{V}_{g,\xi}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_g]$ . Thus, if  $\mathcal{U}_{g,\xi} \in \mathfrak{g}\text{-O}[\mathfrak{T}_g]$ , it suffices to set  $\mathcal{V}_{g,\zeta} = \mathbb{C}(\mathcal{U}_{g,\xi})$ , and if  $\mathcal{V}_{g,\xi} \in \mathfrak{g}\text{-K}[\mathfrak{T}_g]$ , it suffices to set  $\mathcal{U}_{g,\zeta} = \mathbb{C}(\mathcal{V}_{g,\xi})$ . By substitutions, it follows, then, that

$$[\mathcal{U}_{g,\xi} \sqcup \mathcal{V}_{g,\zeta} = \Omega] \vee [\mathcal{U}_{g,\zeta} \sqcup \mathcal{V}_{g,\xi} = \Omega],$$

which was to be proved. Q.E.D.

**REMARK 3.2.** Given  $(\mathcal{R}_g, \mathcal{S}_g) \subseteq \mathfrak{T}_g$  and  $\neg\text{op}_{g,\nu} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ , the statement  $(\mathcal{R}_g \cap \neg\text{op}_{g,\nu}(\mathcal{S}_g)) \cup (\neg\text{op}_{g,\nu}(\mathcal{R}_g) \cap \mathcal{S}_g) = \emptyset$ , when  $\nu = 0$ , may be called the *Hausdorff-Lennes Separation Condition* in the  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$ .

If a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  is  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected, then either  $\mathbb{C}(\mathcal{U}_{g,\lambda}) = \mathcal{U}_{g,\eta}$ , so that  $\mathcal{U}_{g,\lambda} \in \mathfrak{g}\text{-K}[\mathfrak{T}_g]$  or,  $\mathbb{C}(\mathcal{V}_{g,\lambda}) = \mathcal{V}_{g,\eta}$ , so that  $\mathcal{V}_{g,\lambda} \in \mathfrak{g}\text{-O}[\mathfrak{T}_g]$ , where  $(\lambda, \eta) \in \{(\xi, \zeta), (\zeta, \xi)\}$ . Therefore,  $\mathfrak{T}_g$  is  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected if it has no nonempty proper  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -set  $\mathcal{S}_g \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_g]$ . Hence, these theorems follow:

**THEOREM 3.3.** *If a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  has a nonempty proper  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -open-closed set  $\mathcal{S}_g \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_g]$ , then it is a  $\mathfrak{g}\text{-}\mathcal{T}_g^{(D)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_g^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_g^{(D)})$ :*

$$(3.2) \quad \exists \mathcal{S}_g \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_g] \Rightarrow \mathfrak{g}\text{-}\mathfrak{T}_g^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_g^{(D)}).$$

**PROOF.** Let  $\mathcal{S}_g \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_g]$  be a nonempty proper  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -open-closed set in  $\mathfrak{T}_g$ . Then, there exists  $(\mathcal{U}_{g,\lambda}, \mathcal{V}_{g,\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \times \mathfrak{g}\text{-K}[\mathfrak{T}_g]$  such that

$$[\mathcal{U}_{g,\xi} \supseteq \mathcal{S}_g \supseteq \mathbb{C}(\mathcal{U}_{g,\zeta})] \vee [\mathbb{C}(\mathcal{V}_{g,\xi}) \supseteq \mathcal{S}_g \supseteq \mathcal{V}_{g,\zeta}].$$

Consequently, the following relation holds:

$$\left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{g,\lambda} \supseteq \mathcal{S}_g \cup \mathcal{U}_{g,\zeta} \supseteq \mathbb{C}(\mathcal{U}_{g,\zeta}) \cup \mathcal{U}_{g,\zeta} \right) \vee \left( \mathbb{C}(\mathcal{V}_{g,\xi}) \cup \mathcal{V}_{g,\xi} \supseteq \mathcal{S}_g \cup \mathcal{V}_{g,\xi} \supseteq \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{g,\lambda} \right).$$

Since  $\mathbb{C}(\mathcal{U}_{g,\zeta}) \cup \mathcal{U}_{g,\zeta} = \Omega$ ,  $\mathcal{S}_g \cup \mathcal{U}_{g,\zeta} = \Omega$  and, consequently,  $\bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{g,\lambda} = \Omega$ ; observe that,  $\mathcal{U}_{g,\xi} = \mathcal{S}_g = \mathbb{C}(\mathcal{U}_{g,\zeta})$  because  $\mathcal{S}_g \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_g]$  is a nonempty proper  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -open-closed set in  $\mathfrak{T}_g$ . Since  $\mathbb{C}(\mathcal{V}_{g,\xi}) \cup \mathcal{V}_{g,\xi} = \Omega$  and  $\mathcal{S}_g \in \mathfrak{g}\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_g]$ ,



$\mathbb{C}(\mathcal{V}_{\mathfrak{g},\xi}) = \mathcal{S}_{\mathfrak{g}} = \mathcal{V}_{\mathfrak{g},\zeta}$ . Therefore,  $\mathbb{C}(\mathcal{V}_{\mathfrak{g},\xi}) \cup \mathcal{V}_{\mathfrak{g},\xi} = \mathcal{S}_{\mathfrak{g}} \cup \mathcal{V}_{\mathfrak{g},\xi} = \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda}$ . By substitutions, it consequently follows that

$$\left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \Omega \right) \vee \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \Omega \right),$$

which was to be proved. Q.E.D.

The converse of the above theorem also holds as demonstrated below.

**THEOREM 3.4.** *If  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$  be a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space, then it has a nonempty proper  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open-closed set  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ :*

$$(3.3) \quad \exists \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \Leftrightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}).$$

**PROOF.** Let  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$  be a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space. Then, there exists a pair  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that

$$\left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \Omega \right) \vee \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \Omega \right).$$

But  $\bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \Omega$  implies either  $\mathcal{U}_{\mathfrak{g},\xi} = \mathbb{C}(\mathcal{U}_{\mathfrak{g},\zeta})$  or  $\mathcal{U}_{\mathfrak{g},\zeta} = \mathbb{C}(\mathcal{U}_{\mathfrak{g},\xi})$ , and on the other hand,  $\bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \Omega$  implies either  $\mathcal{V}_{\mathfrak{g},\xi} = \mathbb{C}(\mathcal{V}_{\mathfrak{g},\zeta})$  or  $\mathcal{V}_{\mathfrak{g},\zeta} = \mathbb{C}(\mathcal{V}_{\mathfrak{g},\xi})$ . Consequently, there exists a  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  such that

$$[\mathcal{U}_{\mathfrak{g},\xi} \supseteq \mathcal{S}_{\mathfrak{g}} \supseteq \mathbb{C}(\mathcal{U}_{\mathfrak{g},\zeta})] \vee [\mathbb{C}(\mathcal{V}_{\mathfrak{g},\xi}) \supseteq \mathcal{S}_{\mathfrak{g}} \supseteq \mathcal{V}_{\mathfrak{g},\zeta}].$$

Hence,  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)}$  has a nonempty proper  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open-closed set  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ ; this completes the proof of the theorem. Q.E.D.

Combined together, the above theorems establish the necessary and sufficient conditions for a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  to be  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected, and hence the following corollary.

**COROLLARY 3.5.** *Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space. Then it is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$  if and only if  $\mathfrak{T}_{\mathfrak{g}}$  has a nonempty proper  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open-closed set  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ :*

$$(3.4) \quad \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}) \Leftrightarrow \exists \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}].$$

A  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separation is realised if the only  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets in  $\mathfrak{T}_{\mathfrak{g}}$  which are both  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open-closed sets are the improper  $\mathfrak{T}_{\mathfrak{g}}$ -sets  $\emptyset, \Omega \subseteq \mathfrak{T}_{\mathfrak{g}}$ . The theorem follows.

**THEOREM 3.6.** *A  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is said to be a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$  if the only  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets in  $\mathfrak{T}_{\mathfrak{g}}$  which are both  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open-closed sets are the improper  $\mathfrak{T}_{\mathfrak{g}}$ -sets  $\emptyset, \Omega \subseteq \mathfrak{T}_{\mathfrak{g}}$ .*

**PROOF.** Let  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  be a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -set in  $\mathfrak{T}_{\mathfrak{g}}$ . Then, there exists  $(\mathcal{U}_{\mathfrak{g}}, \mathcal{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that  $\mathcal{U}_{\mathfrak{g}} \supseteq \mathcal{S}_{\mathfrak{g}} \supseteq \mathcal{V}_{\mathfrak{g}}$ . Consequently,  $\mathbb{C}(\mathcal{U}_{\mathfrak{g}}) \subseteq \mathbb{C}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathbb{C}(\mathcal{V}_{\mathfrak{g}})$ . Since  $(\mathbb{C}(\mathcal{V}_{\mathfrak{g}}), \mathbb{C}(\mathcal{U}_{\mathfrak{g}})) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ , it follows that,  $\mathbb{C}(\mathcal{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ . Since  $\mathcal{S}_{\mathfrak{g}} \cap \mathbb{C}(\mathcal{S}_{\mathfrak{g}}) = \emptyset$ , implying  $\mathcal{S}_{\mathfrak{g}} \cup \mathbb{C}(\mathcal{S}_{\mathfrak{g}}) = \Omega$ , it results, obviously, that,

$$[(\mathcal{S}_{\mathfrak{g}}, \mathbb{C}(\mathcal{S}_{\mathfrak{g}})) \in (\emptyset, \Omega)] \vee [(\mathcal{S}_{\mathfrak{g}}, \mathbb{C}(\mathcal{S}_{\mathfrak{g}})) \in (\Omega, \emptyset)].$$

This completes the proof of the theorem. Q.E.D.

The logical relationship between  $\mathfrak{T}_g$ -connectedness and  $g\text{-}\mathfrak{T}_g$ -connectedness is contained in the following theorem.

**THEOREM 3.7.** *If  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  be a  $g\text{-}\mathcal{T}_g^{(C)}$ -space  $g\text{-}\mathfrak{T}_g^{(C)} = (\Omega, g\text{-}\mathcal{T}_g^{(C)})$ , then it is also a  $\mathcal{T}_g^{(C)}$ -space  $\mathfrak{T}_g^{(C)} = (\Omega, \mathcal{T}_g^{(C)})$ :*

$$(3.5) \quad g\text{-}\mathfrak{T}_g^{(C)} = (\Omega, g\text{-}\mathcal{T}_g^{(C)}) \Rightarrow \mathfrak{T}_g^{(C)} = (\Omega, \mathcal{T}_g^{(C)}).$$

**PROOF.** Let  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  be a  $\mathcal{T}_g^{(D)}$ -space  $\mathfrak{T}_g^{(D)} = (\Omega, \mathcal{T}_g^{(D)})$ . Then it has a nonempty proper  $\mathfrak{T}_g$ -open-closed set  $\mathcal{S}_g \in \mathcal{O}[\mathfrak{T}_g] \cap \mathcal{K}[\mathfrak{T}_g]$  in  $\mathfrak{T}_g^{(D)}$ . Since  $\mathcal{O}[\mathfrak{T}_g] \cap \mathcal{K}[\mathfrak{T}_g] \subseteq g\text{-}\mathcal{O}[\mathfrak{T}_g] \cap g\text{-}\mathcal{K}[\mathfrak{T}_g]$ , it follows that,  $\mathcal{S}_g \in g\text{-}\mathcal{O}[\mathfrak{T}_g] \cap g\text{-}\mathcal{K}[\mathfrak{T}_g]$  in  $\mathfrak{T}_g^{(D)}$ . This proves that  $\mathfrak{T}_g^{(D)}$  is also a  $g\text{-}\mathcal{T}_g^{(D)}$ -space  $g\text{-}\mathfrak{T}_g^{(D)} = (\Omega, g\text{-}\mathcal{T}_g^{(D)})$ . In other words, if  $\mathfrak{T}_g$  is a  $g\text{-}\mathcal{T}_g^{(C)}$ -space  $g\text{-}\mathfrak{T}_g^{(C)} = (\Omega, g\text{-}\mathcal{T}_g^{(C)})$ , then it is also a  $\mathcal{T}_g^{(C)}$ -space  $\mathfrak{T}_g^{(C)} = (\Omega, \mathcal{T}_g^{(C)})$ , and the proof is complete. Q.E.D.

By virtue of the above theorem, the following corollary follows.

**COROLLARY 3.8.** *If  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  be a  $\mathcal{T}_g^{(D)}$ -space  $\mathfrak{T}_g^{(D)} = (\Omega, \mathcal{T}_g^{(D)})$ , then it is also a  $g\text{-}\mathcal{T}_g^{(D)}$ -space  $g\text{-}\mathfrak{T}_g^{(D)} = (\Omega, g\text{-}\mathcal{T}_g^{(D)})$ :*

$$(3.6) \quad g\text{-}\mathfrak{T}_g^{(D)} = (\Omega, g\text{-}\mathcal{T}_g^{(D)}) \Leftarrow \mathfrak{T}_g^{(D)} = (\Omega, \mathcal{T}_g^{(D)}).$$

A  $\mathfrak{T}_g$ -set  $\mathcal{S}_g \subset \mathfrak{T}_g$  of a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  is  $g\text{-}\mathfrak{T}_g$ -connected if and only if it is  $g\text{-}\mathfrak{T}_g$ -connected as a  $\mathcal{T}_g$ -subspace. The theorem follows.

**THEOREM 3.9.** *If  $\mathcal{S}_g \subset \mathfrak{T}_{g,\Gamma}$  is a  $\mathfrak{T}_g$ -set of a  $\mathcal{T}_g$ -subspace  $\mathfrak{T}_{g,\Gamma} = (\Gamma, \mathcal{T}_{g,\Gamma})$  of a  $\mathcal{T}_g$ -space  $\mathfrak{T}_{g,\Omega} = (\Omega, \mathcal{T}_{g,\Omega})$ , then  $\mathcal{S}_g$  is  $g\text{-}\mathfrak{T}_{g,\Gamma}$ -connected if and only if it is  $g\text{-}\mathfrak{T}_{g,\Omega}$ -connected:*

$$(3.7) \quad \mathcal{S}_g \in g\text{-}\mathcal{Q}[\mathfrak{T}_{g,\Gamma}] \Leftrightarrow \mathcal{S}_g \in g\text{-}\mathcal{Q}[\mathfrak{T}_{g,\Omega}].$$

**PROOF.** *Necessity.* Let  $\mathcal{S}_g \subset \mathfrak{T}_{g,\Gamma}$  be a  $\mathfrak{T}_g$ -set of a  $\mathcal{T}_g$ -subspace  $\mathfrak{T}_{g,\Gamma} = (\Gamma, \mathcal{T}_{g,\Gamma})$  of a  $\mathcal{T}_g$ -space  $\mathfrak{T}_{g,\Omega} = (\Omega, \mathcal{T}_{g,\Omega})$  and suppose that  $\mathcal{S}_g \in g\text{-}\mathcal{Q}[\mathfrak{T}_{g,\Gamma}]$ . Then,  $\mathcal{S}_g \notin g\text{-}\mathcal{D}[\mathfrak{T}_{g,\Gamma}]$  and, hence, for all  $(\mathcal{U}_{g,\lambda}, \mathcal{V}_{g,\lambda})_{\lambda=\xi,\zeta} \in g\text{-}\mathcal{O}[\mathfrak{T}_{g,\Gamma}] \times g\text{-}\mathcal{K}[\mathfrak{T}_{g,\Gamma}]$ ,

$$\neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{g,\lambda} = \mathcal{S}_g\right) \wedge \neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{g,\lambda} = \mathcal{S}_g\right),$$

But,  $\mathcal{T}_{g,\Gamma} \times \neg\mathcal{T}_{g,\Gamma} \subseteq \mathcal{T}_{g,\Omega} \times \neg\mathcal{T}_{g,\Omega}$ , and on the other hand,  $\mathbf{op}_{g,\Gamma}(\cdot) \in \mathcal{L}_g[\Gamma]$  implies  $\mathbf{op}_{g,\Gamma}(\mathcal{O}_{g,\lambda}) = \Gamma \cap \mathbf{op}_{g,\Omega}(\mathcal{O}_{g,\lambda}) = \mathbf{op}_{g,\Omega}(\mathcal{O}_{g,\lambda})$  and  $\neg\mathbf{op}_{g,\Gamma}(\mathcal{K}_{g,\lambda}) = \Gamma \cap \neg\mathbf{op}_{g,\Omega}(\mathcal{K}_{g,\lambda}) = \neg\mathbf{op}_{g,\Omega}(\mathcal{K}_{g,\lambda})$  for any  $(\mathcal{O}_g, \mathcal{K}_g) \in \mathcal{T}_{g,\Gamma} \times \neg\mathcal{T}_{g,\Gamma}$ , where  $\mathbf{op}_{g,\Omega}(\cdot) \in \mathcal{L}_g[\Omega]$ . Thus, for all  $(\mathcal{U}_{g,\lambda}, \mathcal{V}_{g,\lambda})_{\lambda=\xi,\zeta} \in g\text{-}\mathcal{O}[\mathfrak{T}_{g,\Omega}] \times g\text{-}\mathcal{K}[\mathfrak{T}_{g,\Omega}]$ ,

$$\neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{g,\lambda} = \mathcal{S}_g\right) \wedge \neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{g,\lambda} = \mathcal{S}_g\right).$$

Consequently,  $\mathcal{S}_g \notin g\text{-}\mathcal{D}[\mathfrak{T}_{g,\Omega}]$  and, hence,  $\mathcal{S}_g \in g\text{-}\mathcal{Q}[\mathfrak{T}_{g,\Omega}]$ .

*Sufficiency.* Conversely, suppose that  $\mathcal{S}_g \in g\text{-}\mathcal{Q}[\mathfrak{T}_{g,\Omega}]$ . This implies that  $\mathcal{S}_g \notin g\text{-}\mathcal{D}[\mathfrak{T}_{g,\Omega}]$ . Therefore, for all  $(\mathcal{U}_{g,\lambda}, \mathcal{V}_{g,\lambda})_{\lambda=\xi,\zeta} \in g\text{-}\mathcal{O}[\mathfrak{T}_{g,\Omega}] \times g\text{-}\mathcal{K}[\mathfrak{T}_{g,\Omega}]$ ,

$$\neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{g,\lambda} = \mathcal{S}_g\right) \wedge \neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{g,\lambda} = \mathcal{S}_g\right).$$

But,  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \subseteq (\mathcal{S}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}}) \subseteq \mathfrak{T}_{\mathfrak{g},\Gamma} \times \mathfrak{T}_{\mathfrak{g},\Gamma}$  implies, evidently,  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Gamma}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Gamma}]$ , since  $\mathfrak{op}_{\mathfrak{g},\Gamma}(\cdot) \in \mathcal{L}_{\mathfrak{g}}[\Gamma]$  is equivalent to  $\mathfrak{op}_{\mathfrak{g},\Gamma}(\cdot) = \Gamma \cap \mathfrak{op}_{\mathfrak{g},\Omega}(\cdot) \in \mathcal{L}_{\mathfrak{g}}[\Omega]$ . Consequently, for all  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Gamma}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Gamma}]$ ,

$$\neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}}\right) \wedge \neg\left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}}\right),$$

Therefore,  $\mathcal{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Gamma}]$  and, thus,  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Gamma}]$ . Q.E.D.

There are some very fundamental  $\mathcal{T}_{\mathfrak{g}}$ -properties of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected sets which follow from the next theorem.

**THEOREM 3.10.** *If  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  be a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set of a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$ , then there exists  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that*

$$(3.8) \quad \left(\bigvee_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g}} \subseteq \mathcal{U}_{\mathfrak{g},\lambda})\right) \vee \left(\bigvee_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g}} \subseteq \mathcal{V}_{\mathfrak{g},\lambda})\right).$$

**PROOF.** Let  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  be a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set in a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$ . Then, for all  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ ,

$$\begin{aligned} & \neg\left(\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}}\right) \wedge \neg\left(\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}}\right) \\ \Rightarrow & \neg\left(\bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda} = \emptyset\right) \wedge \neg\left(\bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} = \emptyset\right) \\ \Rightarrow & \left(\bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda} \neq \emptyset\right) \wedge \left(\bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} \neq \emptyset\right). \end{aligned}$$

Since  $\mathfrak{T}_{\mathfrak{g}}$  is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$ , there exists, therefore, pairs  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that

$$\begin{aligned} & \left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \Omega\right) \vee \left(\bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \Omega\right) \\ \Rightarrow & \left(\bigsqcup_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\lambda}) = \mathcal{S}_{\mathfrak{g}}\right) \vee \left(\bigsqcup_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g}} \cap \mathcal{V}_{\mathfrak{g},\lambda}) = \mathcal{S}_{\mathfrak{g}}\right) \\ \Rightarrow & \left(\bigvee_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g}} \subseteq \mathcal{U}_{\mathfrak{g},\lambda})\right) \vee \left(\bigvee_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g}} \subseteq \mathcal{V}_{\mathfrak{g},\lambda})\right). \end{aligned}$$

Since  $\bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda}, \bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} \neq \emptyset$  hold, and, moreover,  $\mathfrak{T}_{\mathfrak{g}}$  is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$ , the proof at once follows. Q.E.D.

Equivalently stated, the following proposition states that, any  $\mathfrak{T}_{\mathfrak{g}}$ -set which is contained in a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set is also a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected.

**PROPOSITION 3.11.** *Let  $\mathcal{R}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be  $\mathfrak{T}_{\mathfrak{g}}$ -sets in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . If  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  and  $\mathcal{R}_{\mathfrak{g}}$  satisfies*

$$(3.9) \quad [\mathcal{R}_{\mathfrak{g}} \subseteq \mathfrak{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})] \vee [\mathcal{R}_{\mathfrak{g}} \subseteq \neg \mathfrak{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})],$$

then  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ .

PROOF. Let  $\mathcal{R}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be  $\mathfrak{T}_{\mathfrak{g}}$ -sets in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ , where  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , and, by hypothesis,  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ . Since  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ , there exists, then,  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\rho},\zeta_{\rho}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that,

$$\left( \bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}} \mathcal{U}_{\mathfrak{g},\lambda} = \mathcal{R}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}} \mathcal{V}_{\mathfrak{g},\lambda} = \mathcal{R}_{\mathfrak{g}} \right).$$

Since  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , it must be contained in either of  $\mathcal{U}_{\mathfrak{g},\xi_{\rho}}, \mathcal{U}_{\mathfrak{g},\zeta_{\rho}}$ , or in either of  $\mathcal{V}_{\mathfrak{g},\xi_{\rho}}, \mathcal{V}_{\mathfrak{g},\zeta_{\rho}}$ . Consequently,

$$\begin{aligned} & \left( \bigvee_{\lambda=\xi_{\rho},\zeta_{\rho}} (\mathcal{S}_{\mathfrak{g}} \subseteq \mathcal{U}_{\mathfrak{g},\lambda}) \right) \vee \left( \bigvee_{\eta=\xi_{\rho},\zeta_{\rho}} (\mathcal{S}_{\mathfrak{g}} \subseteq \mathcal{V}_{\mathfrak{g},\eta}) \right) \\ \Rightarrow & \left( \bigvee_{\lambda=\xi_{\rho},\zeta_{\rho}} (\text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathcal{U}_{\mathfrak{g},(\rho,\lambda)}) \right) \vee \left( \bigvee_{\eta=\xi_{\rho},\zeta_{\rho}} (\neg \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathcal{V}_{\mathfrak{g},(\rho,\eta)}) \right), \end{aligned}$$

where  $\mathcal{U}_{\mathfrak{g},(\rho,\lambda)} = \text{op}_{\mathfrak{g}}(\mathcal{U}_{\mathfrak{g},\lambda})$  and  $\mathcal{V}_{\mathfrak{g},(\rho,\eta)} = \neg \text{op}_{\mathfrak{g}}(\mathcal{V}_{\mathfrak{g},\eta})$  for every pair  $(\lambda, \eta) \in \{(\xi_{\rho}, \zeta_{\rho}), (\zeta_{\rho}, \xi_{\rho})\}$ . With no loss of generality, let it be supposed that

$$[\text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathcal{U}_{\mathfrak{g},(\rho,\lambda)}] \vee [\neg \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathcal{V}_{\mathfrak{g},(\rho,\eta)}]$$

holds for a  $(\lambda, \eta) \in \{(\xi_{\rho}, \zeta_{\rho}), (\zeta_{\rho}, \xi_{\rho})\}$ . Then, since the relations  $\mathcal{R}_{\mathfrak{g}} = \bigsqcup_{\sigma=\lambda,\eta} \mathcal{U}_{\mathfrak{g},\sigma} \subseteq \bigsqcup_{\sigma=\lambda,\eta} \mathcal{U}_{\mathfrak{g},(\rho,\sigma)}$  and  $\emptyset = \bigcap_{\sigma=\lambda,\eta} \mathcal{V}_{\mathfrak{g},\sigma} \supseteq \bigcap_{\sigma=\lambda,\eta} \mathcal{V}_{\mathfrak{g},(\rho,\sigma)}$  hold, it follows that,

$$\begin{aligned} \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \cap \mathcal{U}_{\mathfrak{g},\eta} & \subseteq \mathcal{U}_{\mathfrak{g},(\rho,\lambda)} \cap \mathcal{U}_{\mathfrak{g},\eta} \subseteq \bigcap_{\sigma=\lambda,\eta} \mathcal{U}_{\mathfrak{g},(\rho,\sigma)} = \emptyset; \\ \neg \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \cap \mathcal{V}_{\mathfrak{g},\lambda} & \subseteq \mathcal{V}_{\mathfrak{g},(\rho,\eta)} \cap \mathcal{V}_{\mathfrak{g},\lambda} \subseteq \bigcap_{\sigma=\lambda,\eta} \mathcal{V}_{\mathfrak{g},\sigma} = \emptyset. \end{aligned}$$

Therefore,  $\text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \cap \mathcal{U}_{\mathfrak{g},\eta}, \neg \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \cap \mathcal{V}_{\mathfrak{g},\lambda} = \emptyset$ . On the other hand, since  $\mathcal{R}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  satisfies  $[\mathcal{R}_{\mathfrak{g}} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})] \vee [\mathcal{R}_{\mathfrak{g}} \subseteq \neg \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}})]$ , it results that,

$$\begin{aligned} \mathcal{U}_{\mathfrak{g},\eta} & = \mathcal{R}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\eta} = \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \cap \mathcal{U}_{\mathfrak{g},\eta}, \\ \mathcal{V}_{\mathfrak{g},\lambda} & = \mathcal{R}_{\mathfrak{g}} \cap \mathcal{V}_{\mathfrak{g},\lambda} = \neg \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \cap \mathcal{U}_{\mathfrak{g},\lambda}. \end{aligned}$$

From these and  $\text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \cap \mathcal{U}_{\mathfrak{g},\eta}, \neg \text{op}_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g}}) \cap \mathcal{V}_{\mathfrak{g},\lambda} = \emptyset$ , it follows that,  $\mathcal{U}_{\mathfrak{g},\eta}, \mathcal{V}_{\mathfrak{g},\lambda} = \emptyset$ , which contradict the hypothesis that  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ . Q.E.D.

The following proposition states that, if it be given a collection of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected sets with non-void intersection, then  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness is preserved under the operation of union.

PROPOSITION 3.12. *Let  $\{\mathcal{S}_{\mathfrak{g},\nu} : \nu \in I_n^*\} \subseteq \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  be a collection of  $n \geq 1$   $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected sets in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . If  $\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \neq \emptyset$ , then  $\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  in  $\mathfrak{T}_{\mathfrak{g}}$ :*

$$(3.10) \quad \bigcap_{\nu \in I_n^*} (\mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]) \neq \emptyset \Rightarrow \bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}].$$

PROOF. Let  $\{\mathcal{S}_{\mathfrak{g},\nu} : \nu \in I_n^*\} \subseteq \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  be a collection of  $n \geq 1$   $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected sets in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ , and suppose, by hypothesis, that  $\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ , where  $\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \neq \emptyset$ . Since  $\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ , there exists, then,  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that

$$\left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \right) \vee \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \right).$$

Since  $\bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \neq \emptyset$ , there exists a unit  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\{\eta\} \subset \mathfrak{T}_{\mathfrak{g}}$  satisfying  $\{\eta\} \subseteq \bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \neq \emptyset$ . But, by hypothesis,  $\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ . Consequently,

$$\left( \bigvee_{\lambda=\xi,\zeta} \left( \{\eta\} \subseteq \mathcal{U}_{\mathfrak{g},\lambda} \cap \left( \bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \right) \right) \right) \vee \left( \bigvee_{\lambda=\xi,\zeta} \left( \{\eta\} \subseteq \mathcal{V}_{\mathfrak{g},\lambda} \cap \left( \bigcap_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \right) \right) \right).$$

Clearly, for every  $\nu \in I_n^*$ ,

$$\begin{aligned} & \left( \bigvee_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g},\nu} \cap \mathcal{U}_{\mathfrak{g},\lambda} \neq \emptyset) \right) \vee \left( \bigvee_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g},\nu} \cap \mathcal{V}_{\mathfrak{g},\lambda} \neq \emptyset) \right) \\ & \Rightarrow \left( \bigvee_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathcal{U}_{\mathfrak{g},\lambda}) \right) \vee \left( \bigvee_{\lambda=\xi,\zeta} (\mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathcal{V}_{\mathfrak{g},\lambda}) \right). \end{aligned}$$

Therefore,

$$\left( \bigvee_{\lambda=\xi,\zeta} \left( \bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathcal{U}_{\mathfrak{g},\lambda} \right) \right) \vee \left( \bigvee_{\lambda=\xi,\zeta} \left( \bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \subseteq \mathcal{V}_{\mathfrak{g},\lambda} \right) \right),$$

which contradicts the hypothesis that  $\bigcup_{\nu \in I_n^*} \mathcal{S}_{\mathfrak{g},\nu} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ . Q.E.D.

Stated differently, the following proposition states that, if every two-point  $\mathfrak{T}_{\mathfrak{g}}$ -set of a  $\mathfrak{T}_{\mathfrak{g}}$ -set is a  $\mathfrak{T}_{\mathfrak{g}}$ -subset of some  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected subset of the  $\mathfrak{T}_{\mathfrak{g}}$ -set, then the  $\mathfrak{T}_{\mathfrak{g}}$ -set is also a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set.

**PROPOSITION 3.13.** *Let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . If every two-point  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{Q}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}$  satisfies the relation  $\mathcal{Q}_{\mathfrak{g}} \subseteq \mathcal{R}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}$ , where  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , then  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ :*

$$(3.11) \quad \mathcal{Q}_{\mathfrak{g}} \cap \mathcal{S}_{\mathfrak{g}} \supseteq \mathcal{Q}_{\mathfrak{g}} \cap \mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}] \Rightarrow \mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}].$$

PROOF. Let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  and suppose that every two-point  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{Q}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}$  satisfies the relation  $\mathcal{Q}_{\mathfrak{g}} \subseteq \mathcal{R}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}$ , where  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , and by hypothesis,  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ . Then, there exists  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that

$$\left( \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right).$$

Since  $\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda} \neq \emptyset$  for every  $\lambda \in \{\xi_{\sigma}, \zeta_{\sigma}\}$ , assume that

$$\begin{aligned} \{\xi\} &= \mathcal{Q}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\xi_{\sigma}} = \mathcal{Q}_{\mathfrak{g}} \cap \mathcal{V}_{\mathfrak{g},\xi_{\sigma}}, \\ \{\zeta\} &= \mathcal{Q}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g},\zeta_{\sigma}} = \mathcal{Q}_{\mathfrak{g}} \cap \mathcal{V}_{\mathfrak{g},\zeta_{\sigma}}, \quad \mathcal{Q}_{\mathfrak{g}} = \{\xi\} \cup \{\zeta\}. \end{aligned}$$

In other words,  $\mathcal{Q}_{\mathfrak{g}} \subset \times_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{U}_{\mathfrak{g}, \lambda}$  or  $\mathcal{Q}_{\mathfrak{g}} \subset \times_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{V}_{\mathfrak{g}, \lambda}$ . Since  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , for all  $(\mathcal{U}_{\mathfrak{g}, \lambda}, \mathcal{V}_{\mathfrak{g}, \lambda})_{\lambda=\xi_{\rho}, \zeta_{\rho}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ ,

$$\begin{aligned} & \neg \left( \bigsqcup_{\lambda=\xi_{\rho}, \zeta_{\rho}} \mathcal{U}_{\mathfrak{g}, \lambda} = \mathcal{R}_{\mathfrak{g}} \right) \wedge \neg \left( \bigsqcup_{\lambda=\xi_{\rho}, \zeta_{\rho}} \mathcal{V}_{\mathfrak{g}, \lambda} = \mathcal{R}_{\mathfrak{g}} \right) \\ \Rightarrow & \neg \left( \bigsqcup_{\lambda=\xi_{\rho}, \zeta_{\rho}} (\mathcal{Q}_{\mathfrak{g}} \cap \mathcal{U}_{\mathfrak{g}, \lambda}) = \mathcal{Q}_{\mathfrak{g}} \right) \wedge \neg \left( \bigsqcup_{\lambda=\xi_{\rho}, \zeta_{\rho}} (\mathcal{Q}_{\mathfrak{g}} \cap \mathcal{V}_{\mathfrak{g}, \lambda}) = \mathcal{Q}_{\mathfrak{g}} \right) \\ \Rightarrow & \neg (\mathcal{Q}_{\mathfrak{g}} = \mathcal{Q}_{\mathfrak{g}}) \wedge \neg (\mathcal{Q}_{\mathfrak{g}} = \mathcal{Q}_{\mathfrak{g}}), \end{aligned}$$

which contradicts the hypothesis that  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ .

Q.E.D.

A  $\mathcal{T}_{\mathfrak{g}}$ -space is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected if any two-point  $\mathfrak{T}_{\mathfrak{g}}$ -set can be enclosed in some  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected set, and hence the following proposition.

**PROPOSITION 3.14.** *If any two-point  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{Q}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  can be enclosed in some  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected set  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , then the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)})$ :*

$$(3.12) (\forall \mathcal{Q}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}) (\exists \mathcal{S}_{\mathfrak{g}} = \mathcal{S}_{\mathfrak{g}} \cup \mathcal{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]) \Rightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}).$$

**PROOF.** Let  $\xi \in \mathfrak{T}_{\mathfrak{g}}$  be fixed and, for every  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ , let  $\mathcal{Q}_{\mathfrak{g}(\xi, \zeta)} \subset \mathfrak{T}_{\mathfrak{g}}$  be a two-point  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected set in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  containing  $\xi, \zeta \in \mathfrak{T}_{\mathfrak{g}}$ . Then,  $\bigcup_{\zeta \in \mathfrak{T}_{\mathfrak{g}}} \mathcal{Q}_{\mathfrak{g}(\xi, \zeta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  and, by hypothesis, it is the entire  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ . Hence  $\mathfrak{T}_{\mathfrak{g}}$  is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)})$ . Q.E.D.

The theorem given below states that, any pair of nonempty  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets which is contained in some pair of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated sets is also  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected.

**THEOREM 3.15.** *Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space. If  $(\mathcal{S}_{\mathfrak{g}, \alpha}, \mathcal{S}_{\mathfrak{g}, \beta}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$  be a pair of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated sets and  $(\mathcal{R}_{\mathfrak{g}, \alpha}, \mathcal{R}_{\mathfrak{g}, \beta}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$  be a pair of nonempty  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets satisfying  $(\mathcal{R}_{\mathfrak{g}, \alpha}, \mathcal{R}_{\mathfrak{g}, \beta}) \subseteq (\mathcal{S}_{\mathfrak{g}, \alpha}, \mathcal{S}_{\mathfrak{g}, \beta})$ , then  $(\mathcal{R}_{\mathfrak{g}, \alpha}, \mathcal{R}_{\mathfrak{g}, \beta}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ :*

$$(3.13) (\mathcal{S}_{\mathfrak{g}, \lambda} = \mathcal{R}_{\mathfrak{g}, \lambda} \cup \mathcal{S}_{\mathfrak{g}, \lambda})_{\lambda=\alpha, \beta} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \Rightarrow (\mathcal{R}_{\mathfrak{g}, \lambda})_{\lambda=\alpha, \beta} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}].$$

**PROOF.** Let  $(\mathcal{S}_{\mathfrak{g}, \alpha}, \mathcal{S}_{\mathfrak{g}, \beta}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$  be a pair of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated sets and let  $(\mathcal{R}_{\mathfrak{g}, \alpha}, \mathcal{R}_{\mathfrak{g}, \beta}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$  be a pair of nonempty  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets satisfying  $(\mathcal{R}_{\mathfrak{g}, \alpha}, \mathcal{R}_{\mathfrak{g}, \beta}) \subseteq (\mathcal{S}_{\mathfrak{g}, \alpha}, \mathcal{S}_{\mathfrak{g}, \beta})$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . Then, there exists  $(\mathcal{U}_{\mathfrak{g}, \lambda}, \mathcal{V}_{\mathfrak{g}, \lambda})_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that

$$\begin{aligned} & \left( \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{U}_{\mathfrak{g}, \lambda} = \bigsqcup_{\eta=\alpha, \beta} \mathcal{S}_{\mathfrak{g}, \eta} \right) \vee \left( \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{V}_{\mathfrak{g}, \lambda} = \bigsqcup_{\eta=\alpha, \beta} \mathcal{S}_{\mathfrak{g}, \eta} \right) \\ \Leftrightarrow & \left( \bigcap_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{U}_{\mathfrak{g}, \lambda} = \emptyset \right) \vee \left( \bigcap_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{V}_{\mathfrak{g}, \lambda} = \emptyset \right). \end{aligned}$$

Since  $(\mathcal{R}_{\mathfrak{g}, \alpha}, \mathcal{R}_{\mathfrak{g}, \beta}) \subseteq (\mathcal{S}_{\mathfrak{g}, \alpha}, \mathcal{S}_{\mathfrak{g}, \beta})$ ,  $\bigcap_{\eta=\alpha, \beta} \mathcal{R}_{\mathfrak{g}, \eta} \subseteq \bigcap_{\eta=\alpha, \beta} \mathcal{S}_{\mathfrak{g}, \eta}$ . If the relation  $\bigcup_{\eta=\alpha, \beta} \mathcal{S}_{\mathfrak{g}, \eta} = \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{U}_{\mathfrak{g}, \lambda}$  is satisfied, then  $\bigcap_{\eta=\alpha, \beta} \mathcal{R}_{\mathfrak{g}, \eta} \subseteq \bigcap_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{U}_{\mathfrak{g}, \lambda} = \emptyset$ ; if

$\bigsqcup_{\eta=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\eta} = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda}$ , then  $\bigcap_{\eta=\alpha,\beta} \mathcal{R}_{\mathfrak{g},\eta} \subseteq \bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} = \emptyset$  holds. Hence, there exists  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\rho},\zeta_{\rho}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that

$$\left( \bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}} \mathcal{U}_{\mathfrak{g},\lambda} = \bigsqcup_{\eta=\alpha,\beta} \mathcal{R}_{\mathfrak{g},\eta} \right) \vee \left( \bigsqcup_{\lambda=\xi_{\rho},\zeta_{\rho}} \mathcal{V}_{\mathfrak{g},\lambda} = \bigsqcup_{\eta=\alpha,\beta} \mathcal{R}_{\mathfrak{g},\eta} \right).$$

This proves the theorem. Q.E.D.

The basic relation between  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separateness follows:

**THEOREM 3.16.** *In order that a  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  of a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected it is necessary and sufficient that there exists no  $(\mathcal{R}_{\mathfrak{g},\alpha}, \mathcal{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$  or  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that it be expressible as*

$$(3.14) \quad \left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{R}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}} \right).$$

**PROOF.** *Necessity.* Let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set in the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  and let there exists  $(\mathcal{R}_{\mathfrak{g},\lambda}, \mathcal{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that

$$\left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{R}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}} \right).$$

Since  $(\mathcal{R}_{\mathfrak{g},\lambda}, \mathcal{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ , there exists  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that  $(\mathcal{R}_{\mathfrak{g},\lambda}, \mathcal{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} = (\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta}$ . Consequently,

$$\left( \bigsqcup_{\lambda=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\lambda=\alpha,\beta} \mathcal{V}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right).$$

This shows that  $(\mathcal{R}_{\mathfrak{g},\lambda}, \mathcal{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ . Hence,  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ . The condition of the theorem is, therefore, necessary.

*Sufficiency.* Conversely, suppose that  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$ , there exists, then, a pair  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that

$$\left( \bigsqcup_{\lambda=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\lambda=\alpha,\beta} \mathcal{V}_{\mathfrak{g},\lambda} = \mathcal{S}_{\mathfrak{g}} \right).$$

But,  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ . Therefore, it follows that there exists  $(\mathcal{R}_{\mathfrak{g},\lambda}, \mathcal{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that the relations expressible by  $(\mathcal{R}_{\mathfrak{g},\lambda}, \mathcal{S}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} = (\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta}$  hold. Hence, the  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  is expressible as

$$\left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{R}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}} \right).$$

The condition of the theorem is, therefore, sufficient. Q.E.D.

If  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ , then  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -open sets in  $\mathcal{S}_{\mathfrak{g}}$  are clearly also in  $\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ , and conversely. Likewise, if  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ , then  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -closed sets in  $\mathcal{S}_{\mathfrak{g}}$  are clearly also in  $\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ , and conversely. Hence, an immediate consequence of the above theorem is the following corollary:



COROLLARY 3.17. Let  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g}}]$  be a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -set in a  $\mathfrak{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . Then:

- I. If  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ , then in order that  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , it is necessary and sufficient that there exists no  $(\mathcal{R}_{\mathfrak{g},\alpha}, \mathcal{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$  such that it be expressible as  $\mathcal{S}_{\mathfrak{g}} = \bigsqcup_{\lambda=\alpha,\beta} \mathcal{R}_{\mathfrak{g},\lambda}$ .
- II. If  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ , then in order that  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , it is necessary and sufficient that there exists no  $(\mathcal{R}_{\mathfrak{g},\alpha}, \mathcal{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that it be expressible as  $\mathcal{S}_{\mathfrak{g}} = \bigsqcup_{\lambda=\alpha,\beta} \mathcal{R}_{\mathfrak{g},\lambda}$ .

The following remark contains classifications of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness with respect to openness and closedness.

REMARK 3.18. Suppose  $\bigsqcup_{\sigma=\alpha,\beta} \mathcal{R}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}}$  hold, then it is no error to call  $\mathcal{S}_{\mathfrak{g}}$  a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected open set if  $(\mathcal{R}_{\mathfrak{g},\alpha}, \mathcal{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ , and a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected closed set if  $(\mathcal{R}_{\mathfrak{g},\alpha}, \mathcal{R}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ .

From the above corollary, it would appear that  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness depends on the existence of certain  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated sets or, equivalently, on the existence of certain disjoint  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open, closed sets. As another simple ways of characterizing  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness, the proposition follows.

PROPOSITION 3.19. A  $\mathfrak{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$  if and only if any one of the following statements holds:

- I.  $\exists (\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in (\mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2 : \bigsqcup_{\lambda=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\lambda} = \Omega;$
- II.  $\exists (\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in (\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2 : \bigsqcup_{\lambda=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\lambda} = \Omega;$
- III.  $\exists (\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in (\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2 : \bigsqcup_{\lambda=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\lambda} = \Omega.$

PROOF. *Necessity.* Let  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)})$  be a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space. Then, there exists  $\mathcal{S}_{\mathfrak{g}} \in (\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]) \setminus \{\emptyset, \Omega\}$ . Consequently, there exists  $(\mathcal{U}_{\mathfrak{g}}, \mathcal{V}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that

$$\mathcal{U}_{\mathfrak{g}} \supseteq \mathcal{S}_{\mathfrak{g}} \supseteq \mathcal{V}_{\mathfrak{g}} \Rightarrow \mathfrak{C}(\mathcal{U}_{\mathfrak{g}}) \subseteq \mathfrak{C}(\mathcal{S}_{\mathfrak{g}}) \subseteq \mathfrak{C}(\mathcal{V}_{\mathfrak{g}}).$$

Therefore,  $\mathfrak{C}(\mathcal{S}_{\mathfrak{g}}) \in (\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]) \setminus \{\emptyset, \Omega\}$ . Hence,

$$(\mathcal{S}_{\mathfrak{g}}, \mathfrak{C}(\mathcal{S}_{\mathfrak{g}})) \in (\mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2 \cup (\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2 \cup (\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2.$$

*Sufficiency.* Conversely, suppose that

$$\exists (\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in (\mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2 \cup (\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2 \cup (\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2,$$

such that  $\Omega = \bigsqcup_{\lambda=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\lambda}$ . Then if  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in (\mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2$ , there exists  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that,

$$\begin{aligned} & \left( \bigsqcup_{\lambda=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\lambda} \right) \vee \left( \bigsqcup_{\lambda=\alpha,\beta} \mathcal{V}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\lambda} \right) \\ \Rightarrow & \left( \bigsqcup_{\lambda=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\lambda} = \Omega \right) \vee \left( \bigsqcup_{\lambda=\alpha,\beta} \mathcal{V}_{\mathfrak{g},\lambda} = \Omega \right). \end{aligned}$$

If  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in (\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2$ , there exists  $(\mathcal{U}_{\mathfrak{g},\alpha}, \mathcal{U}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$  such that  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \subseteq (\mathcal{U}_{\mathfrak{g},\alpha}, \mathcal{U}_{\mathfrak{g},\beta})$ . Consequently, it follows that  $\Omega = \bigsqcup_{\lambda=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\lambda} \subseteq \bigsqcup_{\lambda=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\lambda}$ . Hence,  $\bigsqcup_{\lambda=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\lambda} = \Omega$ .

If  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in (\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \setminus \{\emptyset\})^2$ , there exists  $(\mathcal{V}_{\mathfrak{g},\alpha}, \mathcal{V}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$  such that  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \supseteq (\mathcal{V}_{\mathfrak{g},\alpha}, \mathcal{V}_{\mathfrak{g},\beta})$ . Thus, it results that  $\emptyset = \bigcap_{\lambda=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\lambda} \supseteq \bigcap_{\lambda=\alpha,\beta} \mathcal{V}_{\mathfrak{g},\lambda}$ . Hence,  $\bigsqcup_{\lambda=\alpha,\beta} \mathcal{V}_{\mathfrak{g},\lambda} = \Omega$ . These complete the proof of the proposition. Q.E.D.

The following lemma is a useful tool for the proof of the theorem following it.

LEMMA 3.20. *Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space, and let  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  be a pair of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected sets in  $\mathfrak{T}_{\mathfrak{g}}$ . If there exists a unit  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\{\xi\} \subset \mathfrak{T}_{\mathfrak{g}}$  such that  $\bigcap_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma} = \{\xi\}$ , then  $\bigcup_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  in  $\mathfrak{T}_{\mathfrak{g}}$ :*

$$(3.15) \quad \exists \{\xi\} = \bigcap_{\sigma=\alpha,\beta} (\mathcal{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]) \Rightarrow \bigcup_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}].$$

PROOF. Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space, let  $\mathcal{S}_{\mathfrak{g}} = \bigcup_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma}$ , and suppose that there exists a unit  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\{\xi\} \subset \mathfrak{T}_{\mathfrak{g}}$  such that  $\{\xi\} = \bigcap_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma}$ , where  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , and assume that

$$\left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}} \right) \vee \left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{V}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}} \right),$$

for some  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\alpha,\beta} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ . Since  $\{\xi\} \subseteq \mathcal{S}_{\mathfrak{g}}$ ,

$$\left( \bigvee_{\sigma=\alpha,\beta} (\{\xi\} \subseteq \mathcal{U}_{\mathfrak{g},\sigma}) \right) \vee \left( \bigvee_{\sigma=\alpha,\beta} (\{\xi\} \subseteq \mathcal{V}_{\mathfrak{g},\sigma}) \right),$$

meaning that, with respect to  $(\mathcal{U}_{\mathfrak{g},\alpha}, \mathcal{U}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ , either  $\xi \in \mathcal{U}_{\mathfrak{g},\alpha}$  or  $\xi \in \mathcal{U}_{\mathfrak{g},\beta}$ ; with respect to  $(\mathcal{V}_{\mathfrak{g},\alpha}, \mathcal{V}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ , either  $\xi \in \mathcal{V}_{\mathfrak{g},\alpha}$  or  $\xi \in \mathcal{V}_{\mathfrak{g},\beta}$ . Therefore, set

$$(\{\xi\} \subseteq \mathcal{U}_{\mathfrak{g},\alpha}) \vee (\{\xi\} \subseteq \mathcal{V}_{\mathfrak{g},\alpha}).$$

Clearly,  $\mathcal{U}_{\mathfrak{g},\beta}, \mathcal{V}_{\mathfrak{g},\beta} \neq \emptyset$ ;  $\mathcal{U}_{\mathfrak{g},\beta} \subseteq \bigsqcup_{\sigma=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}}$  and,  $\mathcal{V}_{\mathfrak{g},\beta} \subseteq \bigsqcup_{\sigma=\alpha,\beta} \mathcal{V}_{\mathfrak{g},\sigma} = \mathcal{S}_{\mathfrak{g}}$ . Therefore, for at least a  $\sigma \in \{\alpha, \beta\}$ ,

$$(\mathcal{U}_{\mathfrak{g},\beta} \cap \mathcal{S}_{\mathfrak{g},\sigma} \neq \emptyset) \vee (\mathcal{V}_{\mathfrak{g},\beta} \cap \mathcal{S}_{\mathfrak{g},\sigma} \neq \emptyset).$$

Choose a  $\eta \in \{\alpha, \beta\}$ . Then, for every  $\sigma \in \{\alpha, \beta\}$ ,

$$(\mathcal{U}_{\mathfrak{g},\sigma} \cap \mathcal{S}_{\mathfrak{g},\eta} \subseteq \mathcal{U}_{\mathfrak{g},\sigma}) \vee (\mathcal{V}_{\mathfrak{g},\sigma} \cap \mathcal{S}_{\mathfrak{g},\eta} \subseteq \mathcal{V}_{\mathfrak{g},\sigma}).$$

Therefore, with respect to  $(\mathcal{U}_{\mathfrak{g},\alpha}, \mathcal{U}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ ,  $\mathcal{U}_{\mathfrak{g},\alpha} \cap \mathcal{S}_{\mathfrak{g},\eta}$  and  $\mathcal{U}_{\mathfrak{g},\beta} \cap \mathcal{S}_{\mathfrak{g},\eta}$  are  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated sets; with respect to  $(\mathcal{V}_{\mathfrak{g},\alpha}, \mathcal{V}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ ,  $\mathcal{V}_{\mathfrak{g},\alpha} \cap \mathcal{S}_{\mathfrak{g},\eta}$  and  $\mathcal{V}_{\mathfrak{g},\beta} \cap \mathcal{S}_{\mathfrak{g},\eta}$  are also  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated sets. Consequently,

$$\begin{aligned} & \left( \mathcal{S}_{\mathfrak{g},\eta} \cap \left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{U}_{\mathfrak{g},\sigma} \right) = \mathcal{S}_{\mathfrak{g},\eta} \right) \vee \left( \mathcal{S}_{\mathfrak{g},\eta} \cap \left( \bigsqcup_{\sigma=\alpha,\beta} \mathcal{V}_{\mathfrak{g},\sigma} \right) = \mathcal{S}_{\mathfrak{g},\eta} \right) \\ & \Rightarrow \left( \bigsqcup_{\sigma=\alpha,\beta} (\mathcal{U}_{\mathfrak{g},\sigma} \cap \mathcal{S}_{\mathfrak{g},\eta}) = \mathcal{S}_{\mathfrak{g},\eta} \right) \vee \left( \bigsqcup_{\sigma=\alpha,\beta} (\mathcal{V}_{\mathfrak{g},\sigma} \cap \mathcal{S}_{\mathfrak{g},\eta}) = \mathcal{S}_{\mathfrak{g},\eta} \right). \end{aligned}$$

Therefore,  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \notin \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , contrary to hypothesis. Hence, it follows that  $\bigcup_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma}$  must be  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected in  $\mathfrak{T}_{\mathfrak{g}}$ , that is  $\bigcup_{\sigma=\alpha,\beta} \mathcal{S}_{\mathfrak{g},\sigma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ . Q.E.D.

For the case of  $n \geq 1$   $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected sets, the theorem follows.

**THEOREM 3.21.** *Let  $\mathcal{S}_{g,1}, \mathcal{S}_{g,2}, \dots, \mathcal{S}_{g,n} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  be  $n \geq 1$   $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected sets in  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$ . If, for every  $(\alpha, \beta) \in I_n^* \times I_n^*$ , there exists a unit  $\mathfrak{T}_g$ -set  $\{\xi\} \subset \mathfrak{T}_g$  such that  $\bigcap_{\sigma=\alpha,\beta} \mathcal{S}_{g,\sigma} = \{\xi\}$ , then  $\bigcup_{\sigma \in I_n^*} \mathcal{S}_{g,\sigma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  in  $\mathfrak{T}_g$ :*

$$(3.16) \quad \exists \{\xi\} = \bigcap_{\sigma \in \{\alpha,\beta\} \subseteq I_n^* \times I_n^*} (\mathcal{S}_{g,\sigma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]) \Rightarrow \bigcup_{\sigma \in I_n^*} \mathcal{S}_{g,\sigma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g].$$

**PROOF.** Let  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  be a  $\mathcal{T}_g$ -space, let  $\mathcal{S}_g = \bigcup_{\sigma \in I_n^*} \mathcal{S}_{g,\sigma}$ , and suppose that, for every  $(\alpha, \beta) \in I_n^* \times I_n^*$ , there exists a unit  $\mathfrak{T}_g$ -set  $\{\xi\} \subset \mathfrak{T}_g$  such that  $\bigcap_{\sigma=\alpha,\beta} \mathcal{S}_{g,\sigma} = \{\xi\}$ , where  $\mathcal{S}_{g,1}, \mathcal{S}_{g,2}, \dots, \mathcal{S}_{g,n} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  are  $n \geq 1$   $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected sets in  $\mathfrak{T}_g$ . If  $(\xi_\alpha, \xi_\beta) \in \mathcal{S}_g \times \mathcal{S}_g$  be any pair of elements of  $\mathcal{S}_g$ , then there is a pair  $(\mathcal{S}_{g,\alpha}, \mathcal{S}_{g,\beta}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g] \times \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  of  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected sets such that  $(\xi_\alpha, \xi_\beta) \in (\mathcal{S}_{g,\alpha}, \mathcal{S}_{g,\beta})$ . Set  $\mathcal{Q}_{g,(\alpha,\beta)} = \{\xi_\alpha, \xi_\beta\}$  and  $\mathcal{R}_{g,(\alpha,\beta)} = \bigcup_{\sigma=\alpha,\beta} \mathcal{S}_{g,\sigma}$ ; clearly,  $\bigcup_{\sigma=\alpha,\beta} \mathcal{S}_{g,\sigma} \neq \emptyset$  by hypothesis. Then, for every  $(\alpha, \beta) \in I_n^* \times I_n^*$ , the relation  $\mathcal{Q}_{g,(\alpha,\beta)} \subseteq \mathcal{R}_{g,(\alpha,\beta)} \subseteq \mathcal{S}_g$  holds. Since, for every  $(\alpha, \beta) \in I_n^* \times I_n^*$ , there exists a unit  $\mathfrak{T}_g$ -set  $\{\xi\} \subset \mathfrak{T}_g$  such that  $\bigcap_{\sigma=\alpha,\beta} \mathcal{S}_{g,\sigma} = \{\xi\}$ , it follows that  $\mathcal{R}_{g,(\alpha,\beta)} = \bigcup_{\sigma=\alpha,\beta} \mathcal{S}_{g,\sigma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  in  $\mathfrak{T}_g$ . Since, for every  $(\alpha, \beta) \in I_n^* \times I_n^*$ ,  $\mathcal{Q}_{g,(\alpha,\beta)} \subseteq \mathcal{S}_g$  is a two-point  $\mathfrak{T}_g$ -set satisfying the relation  $\mathcal{Q}_{g,(\alpha,\beta)} \subseteq \mathcal{R}_{g,(\alpha,\beta)} \subseteq \mathcal{S}_g$ , where  $\mathcal{R}_{g,(\alpha,\beta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  in  $\mathfrak{T}_g$ , it follows that  $\mathcal{S}_g = \bigcup_{\sigma \in I_n^*} \mathcal{S}_{g,\sigma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  in  $\mathfrak{T}_g$ . This proves the theorem. Q.E.D.

When a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  is  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -separated, it is natural that we should attempt to obtain some information about the various  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected sets into which it can be  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -separated. The maximal  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected sets of the  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  are particularly interesting.

**DEFINITION 3.22.** If  $\zeta \in \mathcal{S}_g \subset \mathfrak{T}_g$  is a point of a  $\mathfrak{T}_g$ -set in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$ , then

$$(3.17) \quad \mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta] \stackrel{\text{def}}{=} \{\xi \in \mathcal{S}_g : (\exists \mathcal{R}_g \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]) [(\xi, \zeta) \in \mathcal{R}_g^2 \subseteq \mathcal{S}_g^2]\}$$

is called the " $\mathfrak{g}\text{-}\mathfrak{T}_g$ -component of  $\mathcal{S}_g$  corresponding to  $\zeta$ ."

According to this definition, a  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -component is nonempty,  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected, and is not a proper  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -set of any  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected set of a  $\mathcal{T}_g$ -space. The theorem follows.

**THEOREM 3.23.** *For each point  $\zeta \in \mathfrak{T}_g$  in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$ , the  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -component  $\mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]$  of  $\mathcal{S}_g$  corresponding to  $\zeta$  is the largest  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected set in  $\mathfrak{T}_g$  which contains the point  $\zeta$ :*

$$(3.18) \quad (\forall \zeta \in \mathfrak{T}_g) (\nexists \mathcal{R}_g \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]) [\mathcal{R}_g \supset \mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]].$$

**PROOF.** Let  $(\xi, \zeta) \in (\mathcal{R}_{g,\zeta} \setminus \{\zeta\}) \times \mathcal{R}_{g,\zeta}$  in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$ , where  $\mathcal{R}_{g,\zeta} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  is any  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected set which contains  $\zeta \in \mathfrak{T}_g$ , and  $\xi \in \mathcal{R}_{g,\zeta}$ . Clearly,  $\mathcal{Q}_{g,(\xi,\zeta)} = \{\xi, \zeta\} \subseteq \mathcal{R}_{g,\zeta}$  and, therefore,  $\xi \in \mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]$ , implying  $\mathcal{R}_{g,\zeta} \subseteq \mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]$ . To prove the  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connectedness of  $\mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]$ , consider an arbitrary point  $\eta \in \mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]$ . Since  $\mathcal{Q}_{g,(\eta,\zeta)} = \{\eta, \zeta\} \subseteq \mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]$ , there exists a  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected  $\mathcal{R}_{g,\eta} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  such that  $\mathcal{Q}_{g,(\eta,\zeta)} = \{\eta, \zeta\} \subseteq \mathcal{R}_{g,\eta}$ . Therefore,  $\mathcal{R}_{g,\eta} \subseteq \mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]$  and, consequently,  $\mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta] = \bigcup_{\eta \in \mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]} \mathcal{R}_{g,\eta}$ . But, this is the union of the collection  $\{\mathcal{R}_{g,\eta} : \eta \in \mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta]\} \subseteq \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  of  $\mathfrak{g}\text{-}\mathfrak{T}_g$ -connected sets with a common point  $\zeta \in \mathfrak{T}_g$ . Hence,  $\mathfrak{g}\text{-C}_{\mathcal{S}_g}[\zeta] \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$ . Q.E.D.

In a  $\mathcal{T}_{\mathfrak{g}}$ -space,  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -components are  $\mathfrak{g}$ - $\mathfrak{T}$ -closed sets, as demonstrated in the following theorem.

**THEOREM 3.24.** *For each point  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ , the  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -component  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \subset \mathfrak{T}_{\mathfrak{g}}$  of  $\mathcal{S}_{\mathfrak{g}}$  corresponding to  $\zeta$  is a  $\mathfrak{g}$ - $\mathfrak{T}$ -closed set of  $\mathfrak{T}_{\mathfrak{g}}$ :*

$$(3.19) \quad (\forall \zeta \in \mathfrak{T}_{\mathfrak{g}}) [\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]].$$

**PROOF.** Let  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \subset \mathfrak{T}_{\mathfrak{g}}$  be the  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -component of  $\mathcal{S}_{\mathfrak{g}}$  corresponding to  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ . Then,  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  is the largest  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set in  $\mathfrak{T}_{\mathfrak{g}}$  containing the point  $\zeta$ . Suppose that  $\xi \in \neg \text{op}_{\mathfrak{g}}(\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta])$ . Since  $\neg \text{op}_{\mathfrak{g}}(\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  is a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set, and  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  is the largest  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set in  $\mathfrak{T}_{\mathfrak{g}}$  which contains the point  $\zeta$ ,  $\mathcal{Q}_{\mathfrak{g},(\xi,\zeta)} = \{\xi, \zeta\} \subseteq \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$ . Hence,  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \supseteq \neg \text{op}_{\mathfrak{g}}(\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta])$ , meaning that  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  must be a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -closed set in  $\mathfrak{T}_{\mathfrak{g}}$ . This proves the theorem. Q.E.D.

A central fact about the  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -components of a  $\mathcal{T}_{\mathfrak{g}}$ -space is that, to each point  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  there corresponds a unique  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -component  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  of  $\mathcal{S}_{\mathfrak{g}}$ . This fact is contained in the following theorem.

**THEOREM 3.25.** *The class  $\{\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] : \zeta \in \mathfrak{T}_{\mathfrak{g}}\}$  of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -components of a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  forms a partition of  $\mathfrak{T}_{\mathfrak{g}}$ :*

$$(3.20) \quad \{\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] : \zeta \in \mathfrak{T}_{\mathfrak{g}}\} \Rightarrow \bigsqcup_{\zeta \in \mathfrak{T}_{\mathfrak{g}}} \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] = \Omega.$$

**PROOF.** Let  $\{\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] : \zeta \in \mathfrak{T}_{\mathfrak{g}}\}$  be the class of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -components of a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . Clearly,  $\Omega = \bigcup_{\zeta \in \mathfrak{T}_{\mathfrak{g}}} \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$ . Let  $\eta \in \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \cap \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\xi]$ . Then, since  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta], \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\xi] \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  and contain the point  $\eta \in \mathfrak{T}_{\mathfrak{g}}$ , it follows that,  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\eta] \supseteq \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  and  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\eta] \supseteq \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\xi]$ . But  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta], \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\xi]$  are  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -components and, hence,  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] = \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\eta] = \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\xi]$ . This shows that distinct  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -components are disjoint or, equivalently,  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \cap \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\xi] \neq \emptyset$  implies  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] = \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\xi]$ . Q.E.D.

By virtue of this theorem, it thus follows that, each  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$  belongs to a unique  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -component  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  of  $\mathcal{S}_{\mathfrak{g}}$ . The corollary follows.

**COROLLARY 3.26.** *For each point  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ , there corresponds a unique  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -component  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  of  $\mathcal{S}_{\mathfrak{g}}$  containing it:*

$$(3.21) \quad (\forall \zeta \in \mathfrak{T}_{\mathfrak{g}}) (\exists! \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]) [\zeta \in \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]].$$

A  $\mathcal{T}_{\mathfrak{g}}$ -space that is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected has at most one  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -component, as demonstrated in the following proposition.

**PROPOSITION 3.27.** *If  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)})$ , then it has at most one  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -component  $\mathfrak{g}\text{-C}_{\Omega}[\zeta] = \Omega$ :*

$$(3.22) \quad \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}) \Leftrightarrow \exists! \mathfrak{g}\text{-C}_{\Omega}[\zeta] = \Omega.$$

**PROOF.** Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)})$ , and let it be supposed that it has  $\alpha \in I_{\infty}^*$   $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -components  $\mathfrak{g}\text{-C}_{\Omega}[\zeta_1], \mathfrak{g}\text{-C}_{\Omega}[\zeta_2], \dots, \mathfrak{g}\text{-C}_{\Omega}[\zeta_{\alpha}]$ . Then,  $\bigcup_{\mu \in I_{\alpha}^*} \mathfrak{g}\text{-C}_{\Omega}[\zeta_{\mu}] = \Omega$  because  $\bigcap_{\mu \in I_{\alpha}^*} \mathfrak{g}\text{-C}_{\Omega}[\zeta_{\mu}] = \emptyset$ . Hence,  $\mathfrak{T}_{\mathfrak{g}}$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated, which contradicts the fact that  $\mathfrak{T}_{\mathfrak{g}}$  is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)})$ . Q.E.D.

The combination of an additional concept called *path* with the notion of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness will bring forth a further refinement of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness called *pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness*.

DEFINITION 3.28. A path from an initial point  $\xi \in \mathfrak{T}_{\mathfrak{g}}$  to a terminal point  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is a  $([0, 1], \mathfrak{T}_{\mathfrak{g}})$ -continuous map  $\varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$  with  $(\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)) = (\xi, \zeta)$ . A  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  of a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is said to be pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected if and only if, for every  $(\xi, \zeta) \in \mathcal{S}_{\mathfrak{g}} \times \mathcal{S}_{\mathfrak{g}}$ ,

$$(3.23) \quad (\exists \mathcal{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]) (\exists \varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}) [\mathcal{S}_{\mathfrak{g}} \supseteq \mathcal{Q}_{\mathfrak{g}} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta}|_{[0,1]})].$$

Evidently,  $\text{im}(\varphi_{\mathfrak{g},\zeta}|_{[0,1]})$  signifies the *image* of the  $([0, 1], \mathfrak{T}_{\mathfrak{g}})$ -continuous map  $\varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$  from the initial point  $\xi = \varphi_{\mathfrak{g},\zeta}(0)$  to the terminal point  $\zeta = \varphi_{\mathfrak{g},\zeta}(1)$ . The following theorem is an immediate consequence of the above definition.

THEOREM 3.29. A subset  $\Gamma \subseteq \Omega$  of  $\Omega$  of a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$ , with the absolute  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g},\Omega} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ , is said to be pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected if and only if, with the relative  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g},\Gamma} : \mathcal{P}(\Gamma) \mapsto \mathcal{T}_{\mathfrak{g},\Gamma} = \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g},\Omega}\}$ , the  $\mathcal{T}_{\mathfrak{g}}$ -subspace  $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma})$  is pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected.

PROOF. *Necessity.* Let  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space, and suppose that a subset  $\Gamma \subseteq \Omega$  of  $\Omega$ , with the absolute  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g},\Omega} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ , is pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected in  $\mathfrak{T}_{\mathfrak{g},\Omega}$ . Then, for every  $(\xi, \zeta) \in \Gamma \times \Gamma \subseteq \Omega \times \Omega$ ,

$$(\exists \mathcal{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]) (\exists \varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g},\Omega}) [\Gamma \supseteq \mathcal{Q}_{\mathfrak{g},\omega} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta}|_{[0,1]})].$$

Since  $\mathcal{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  and  $\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}] \supseteq \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Gamma}]$ , it follows that  $\mathcal{Q}_{\mathfrak{g},\omega} \cap \Gamma = \mathcal{Q}_{\mathfrak{g},\gamma}$  for every  $\mathcal{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Gamma}]$ . Since  $\{\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)\} \subset \Gamma \times \Gamma$ , it also follows that  $\Gamma \supseteq \mathcal{Q}_{\mathfrak{g},\gamma} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta}|_{[0,1]})$ . Therefore, for every  $(\xi, \zeta) \in \Gamma \times \Gamma$ ,

$$(\exists \mathcal{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Gamma}]) (\exists \varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g},\Gamma}) [\Gamma \supseteq \mathcal{Q}_{\mathfrak{g},\gamma} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta}|_{[0,1]})].$$

Hence, with the relative  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g},\Gamma} : \mathcal{P}(\Gamma) \mapsto \mathcal{T}_{\mathfrak{g},\Gamma} = \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g},\Omega}\}$ , the  $\mathcal{T}_{\mathfrak{g}}$ -subspace  $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma})$  is pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected.

*Sufficiency.* Conversely, suppose that, with the relative  $\mathfrak{g}$ -topology given by  $\mathcal{T}_{\mathfrak{g},\Gamma} : \mathcal{P}(\Gamma) \mapsto \mathcal{T}_{\mathfrak{g},\Gamma} = \{\mathcal{O}_{\mathfrak{g}} \cap \Gamma : \mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g},\Omega}\}$ , the  $\mathcal{T}_{\mathfrak{g}}$ -subspace  $\mathfrak{T}_{\mathfrak{g},\Gamma} = (\Gamma, \mathcal{T}_{\mathfrak{g},\Gamma})$  is pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected. Then, for every  $(\xi, \zeta) \in \Gamma \times \Gamma$ ,

$$(\exists \mathcal{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Gamma}]) (\exists \varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g},\Gamma}) [\Gamma \supseteq \mathcal{Q}_{\mathfrak{g},\gamma} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta}|_{[0,1]})].$$

Since  $\mathcal{Q}_{\mathfrak{g},\gamma} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Gamma}]$  and  $\mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Gamma}] \subseteq \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ , it follows that a  $\mathcal{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  exists such that  $\mathcal{Q}_{\mathfrak{g},\omega} \cap \Gamma = \mathcal{Q}_{\mathfrak{g},\gamma}$ . Furthermore, since  $\mathcal{Q}_{\mathfrak{g},\omega} \subseteq \Gamma$  and  $\mathcal{Q}_{\mathfrak{g},\omega} \supseteq \mathcal{Q}_{\mathfrak{g},\gamma}$ , it follows that  $\Gamma \supseteq \mathcal{Q}_{\mathfrak{g},\omega} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta}|_{[0,1]})$ . Therefore, for every  $(\xi, \zeta) \in \Gamma \times \Gamma \subseteq \Omega \times \Omega$ ,

$$(\exists \mathcal{Q}_{\mathfrak{g},\omega} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]) (\exists \varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g},\Omega}) [\Gamma \supseteq \mathcal{Q}_{\mathfrak{g},\omega} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta}|_{[0,1]})].$$

Hence, the subset  $\Gamma \subseteq \Omega$ , with the absolute  $\mathfrak{g}$ -topology  $\mathcal{T}_{\mathfrak{g},\Omega} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ , is pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected. Q.E.D.

The relationship between the notions of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness and pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness follows.

THEOREM 3.30. Let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . If  $\mathcal{S}_{\mathfrak{g}}$  is pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected, then  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ .

PROOF. Let  $\mathcal{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$  be an arbitrary pathwise  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . If  $\mathcal{S}_{\mathfrak{g}} = \emptyset$ , then  $\mathcal{S}_{\mathfrak{g}} \notin \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g}}]$  and, therefore,  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ . Suppose  $\mathcal{S}_{\mathfrak{g}} \neq \emptyset$ , consider any point  $\xi \in \mathcal{S}_{\mathfrak{g}}$ . Since  $\mathcal{S}_{\mathfrak{g}}$  is pathwise  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected, for every  $\zeta \in \mathcal{S}_{\mathfrak{g}}$ , there is a path  $\varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$  from the initial point  $\xi \in \mathcal{S}_{\mathfrak{g}}$  to the terminal point  $\zeta \in \mathcal{S}_{\mathfrak{g}}$ , and a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set  $\mathcal{Q}_{\mathfrak{g},(\xi,\zeta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , containing  $\xi, \zeta \in \mathcal{S}_{\mathfrak{g}}$ , such that  $\mathcal{Q}_{\mathfrak{g},(\xi,\zeta)} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta|_{[0,1]}})$ . Clearly,  $\text{im}(\varphi_{\mathfrak{g},\zeta|_{[0,1]}}) \in \mathcal{C}[[0, 1]; \mathfrak{T}_{\mathfrak{g}}]$ . Moreover,  $\mathcal{S}_{\mathfrak{g}} \supseteq \mathcal{Q}_{\mathfrak{g},(\xi,\zeta)} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta|_{[0,1]}})$  and, consequently,  $\mathcal{S}_{\mathfrak{g}} \supseteq \bigcup_{\zeta \in \mathcal{S}_{\mathfrak{g}}} \mathcal{Q}_{\mathfrak{g},(\xi,\zeta)} \supseteq \bigcup_{\zeta \in \mathcal{S}_{\mathfrak{g}}} \text{im}(\varphi_{\mathfrak{g},\zeta|_{[0,1]}}) = \mathcal{S}_{\mathfrak{g}}$ , since  $\xi \in \mathcal{S}_{\mathfrak{g}}$ . But,  $\xi \in \text{im}(\varphi_{\mathfrak{g},\zeta|_{[0,1]}})$  for every  $\zeta \in \mathcal{S}_{\mathfrak{g}}$  and, hence,  $\bigcap_{\zeta \in \mathcal{S}_{\mathfrak{g}}} \text{im}(\varphi_{\mathfrak{g},\zeta|_{[0,1]}}) \neq \emptyset$ . Furthermore,  $\text{im}(\varphi_{\mathfrak{g},\zeta|_{[0,1]}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  for every  $\zeta \in \mathcal{S}_{\mathfrak{g}}$ , and by the relation  $\mathcal{S}_{\mathfrak{g}} = \bigcup_{\zeta \in \mathcal{S}_{\mathfrak{g}}} \text{im}(\varphi_{\mathfrak{g},\zeta|_{[0,1]}})$ , it follows, then, that  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ . This proves the theorem. Q.E.D.

Thus, pathwise  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness is a stronger form of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness. For this reason, we stated that pathwise  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness is a further refinement of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected. An immediate consequence of such a statement is the following proposition.

PROPOSITION 3.31. *If  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a pathwise  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected  $\mathcal{T}_{\mathfrak{g}}$ -space, then it is also  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected:*

$$(3.24) \quad \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\text{PC})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\text{PC})}) \Rightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(\text{C})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(\text{C})}).$$

PROOF. Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a  $\mathcal{T}_{\mathfrak{g}}$ -space, and suppose it be  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated. Then,  $\mathfrak{T}_{\mathfrak{g}}$  has a nonempty proper  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -open-closed set  $\mathcal{S}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \cap \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ . There exists, then,  $(\xi, \zeta) \in \mathcal{S}_{\mathfrak{g}} \times \mathcal{C}(\mathcal{S}_{\mathfrak{g}})$ . Let  $\varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$  be a path from  $\xi$  to  $\zeta$ . Clearly,  $[0, 1] \supset \text{im}(\varphi_{\mathfrak{g},\zeta}^{-1}|_{\mathcal{S}_{\mathfrak{g}}})$  for  $0 \in \text{im}(\varphi_{\mathfrak{g},\zeta}^{-1}|_{\mathcal{S}_{\mathfrak{g}}})$  and  $1 \notin \text{im}(\varphi_{\mathfrak{g},\zeta}^{-1}|_{\mathcal{S}_{\mathfrak{g}}})$ , or for  $0 \notin \text{im}(\varphi_{\mathfrak{g},\zeta}^{-1}|_{\mathcal{S}_{\mathfrak{g}}})$  and  $1 \in \text{im}(\varphi_{\mathfrak{g},\zeta}^{-1}|_{\mathcal{S}_{\mathfrak{g}}})$ . Since  $\varphi_{\mathfrak{g},\zeta} \in \mathcal{C}[[0, 1]; \mathfrak{T}_{\mathfrak{g}}]$ , it follows that  $\text{im}(\varphi_{\mathfrak{g},\zeta}^{-1}|_{\mathcal{S}_{\mathfrak{g}}})$  is both open and closed. But, this contradicts the fact that  $[0, 1]$  is connected. Hence, the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected. Q.E.D.

DEFINITION 3.32. Let  $\phi_{\mathfrak{g},\zeta}, \varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$  be two paths in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  satisfying  $(\phi_{\mathfrak{g},\zeta}(0), \phi_{\mathfrak{g},\zeta}(1)) = (\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)) = (\xi, \zeta)$ . Then,  $\phi_{\mathfrak{g},\zeta}$  is said to be "homotopic" to  $\varphi_{\mathfrak{g},\zeta}$ , written  $\phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$ , if there exists a  $([0, 1]^2, \mathfrak{T}_{\mathfrak{g}})$ -continuous map  $\mathfrak{h}_{\mathfrak{g}} : [0, 1]^2 \rightarrow \mathfrak{T}_{\mathfrak{g}}$ , called a "homotopy" from  $\phi_{\mathfrak{g},\zeta}$  to  $\varphi_{\mathfrak{g},\zeta}$ , written  $\mathfrak{h}_{\mathfrak{g}} : \phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$ , satisfying,

$$(3.25) \quad \begin{aligned} \mathfrak{h}_{\mathfrak{g}}(\lambda, \mu) &= (1 - \mu)\phi_{\mathfrak{g},\zeta}(\lambda) + \mu\varphi_{\mathfrak{g},\zeta}(\lambda) \quad \forall \mu \in \{0, 1\}, \\ \mathfrak{h}_{\mathfrak{g}}(\lambda, \mu) &= (1 - \lambda)\xi + \lambda\zeta \quad \forall \lambda \in \{0, 1\}. \end{aligned}$$

The homotopy  $\mathfrak{h}_{\mathfrak{g}} : \phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$  is said to establish a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -homotopy from  $\phi_{\mathfrak{g},\zeta}$  to  $\varphi_{\mathfrak{g},\zeta}$  in a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected set  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  if it belongs to the class:

$$(3.26) \quad \mathfrak{g}\text{-H}[[0, 1]^2; \mathcal{R}_{\mathfrak{g}}] \stackrel{\text{def}}{=} \{\mathfrak{h}_{\mathfrak{g}} : (\exists \phi_{\mathfrak{g},\zeta}, \varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathcal{R}_{\mathfrak{g}}) [\mathfrak{h}_{\mathfrak{g}} : \phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}]\}.$$

For any  $\phi_{\mathfrak{g},\zeta}, \varphi_{\mathfrak{g},\zeta}, \psi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , the statements  $\mathfrak{h}_{\mathfrak{g}} : \phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$ ,  $\mathfrak{h}_{\mathfrak{g}} : \phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$  implies  $\mathfrak{h}_{\mathfrak{g}} : \varphi_{\mathfrak{g},\zeta} \simeq \phi_{\mathfrak{g},\zeta}$  and,  $\mathfrak{h}_{\mathfrak{g}} : \phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$  and  $\mathfrak{h}_{\mathfrak{g}} : \varphi_{\mathfrak{g},\zeta} \simeq \psi_{\mathfrak{g},\zeta}$  implies  $\mathfrak{h}_{\mathfrak{g}} : \phi_{\mathfrak{g},\zeta} \simeq \psi_{\mathfrak{g},\zeta}$  hold, as shown in the following theorem.

**THEOREM 3.33.** *The  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -homotopy relation is an equivalence relation in the collection of all paths in any  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected set  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  of a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ .*

**PROOF.** *Reflexivity.* Let  $\phi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathcal{R}_{\mathfrak{g}}$  be any path, where  $\mathcal{R}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  is any  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected set in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . Then the  $([0, 1]^2, \mathcal{R}_{\mathfrak{g}})$ -continuous map  $\mathfrak{h}_{\mathfrak{g}} : [0, 1]^2 \rightarrow \mathcal{R}_{\mathfrak{g}}$  defined, for every  $(\lambda, \mu) \in [0, 1]^2$ , by  $\mathfrak{h}_{\mathfrak{g}}(\lambda, \mu) = \phi_{\mathfrak{g},\zeta}(\lambda)$  is a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -homotopy from  $\phi_{\mathfrak{g},\zeta}$  to  $\phi_{\mathfrak{g},\zeta}$ , and that defined, for every  $(\lambda, \mu) \in [0, 1]^2$ , by  $\mathfrak{h}_{\mathfrak{g}}(\lambda, \mu) = \varphi_{\mathfrak{g},\zeta}(\mu)$  is a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -homotopy from  $\varphi_{\mathfrak{g},\zeta}$  to  $\varphi_{\mathfrak{g},\zeta}$ . Hence,  $\mathfrak{h}_{\mathfrak{g}} \in \mathfrak{g}\text{-H}[[0, 1]^2; \mathcal{R}_{\mathfrak{g}}]$ , and  $\simeq$  is reflexive.

*Symmetry.* Let  $\mathfrak{h}_{\mathfrak{g}} \in \mathfrak{g}\text{-H}[[0, 1]^2; \mathcal{R}_{\mathfrak{g}}]$  be a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -homotopy  $\mathfrak{h}_{\mathfrak{g}} : \phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$ . Then the  $([0, 1]^2, \mathcal{R}_{\mathfrak{g}})$ -continuous map  $\hat{\mathfrak{h}}_{\mathfrak{g}} : [0, 1]^2 \rightarrow \mathcal{R}_{\mathfrak{g}}$  defined, for every  $(\lambda, \mu) \in [0, 1]^2$ , by  $\hat{\mathfrak{h}}_{\mathfrak{g}}(\lambda, \mu) = \mathfrak{h}_{\mathfrak{g}}(\lambda, 1 - \mu)$  is a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -homotopy  $\hat{\mathfrak{h}}_{\mathfrak{g}} : \varphi_{\mathfrak{g},\zeta} \simeq \phi_{\mathfrak{g},\zeta}$ . Hence,  $\hat{\mathfrak{h}}_{\mathfrak{g}} \in \mathfrak{g}\text{-H}[[0, 1]^2; \mathcal{R}_{\mathfrak{g}}]$ , and  $\simeq$  is symmetric.

*Transitivity.* Let  $\mathfrak{h}_{\mathfrak{g},\alpha}, \mathfrak{h}_{\mathfrak{g},\beta} \in \mathfrak{g}\text{-H}[[0, 1]^2; \mathcal{R}_{\mathfrak{g}}]$  be  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -homotopies  $\mathfrak{h}_{\mathfrak{g},\alpha} : \phi_{\mathfrak{g},\zeta} \simeq \varphi_{\mathfrak{g},\zeta}$  and  $\mathfrak{h}_{\mathfrak{g},\beta} : \varphi_{\mathfrak{g},\zeta} \simeq \psi_{\mathfrak{g},\zeta}$ , respectively. Consider the  $([0, 1]^2, \mathcal{R}_{\mathfrak{g}})$ -continuous map  $\mathfrak{h}_{\mathfrak{g}} : [0, 1]^2 \rightarrow \mathcal{R}_{\mathfrak{g}}$  defined, for every  $(\lambda, \mu) \in [0, 1]^2$ , by  $\mathfrak{h}_{\mathfrak{g}}(\lambda, \mu) = \mathfrak{h}_{\mathfrak{g},\alpha}(\lambda, \eta\mu)$  if  $\mu \in [0, \frac{1}{\eta}]$  and  $\mathfrak{h}_{\mathfrak{g}}(\lambda, \mu) = \mathfrak{h}_{\mathfrak{g},\beta}(\lambda, \eta\mu - 1)$  if  $\mu \in [\frac{1}{\eta}, 1]$ , where  $\eta \in (1, \infty)$ . Clearly,  $\mathfrak{h}_{\mathfrak{g}} : \phi_{\mathfrak{g},\zeta} \simeq \psi_{\mathfrak{g},\zeta}$ . Hence, it follows that,  $\mathfrak{h}_{\mathfrak{g}} \in \mathfrak{g}\text{-H}[[0, 1]^2; \mathcal{R}_{\mathfrak{g}}]$ , and  $\simeq$  is transitive. Q.E.D.

The concept of simply  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected  $\mathcal{T}_{\mathfrak{g}}$ -space is defined below.

**DEFINITION 3.34.** Let  $\varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$  be a path from  $\xi \in \mathfrak{T}_{\mathfrak{g}}$  to  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  with  $(\varphi_{\mathfrak{g},\zeta}(0), \varphi_{\mathfrak{g},\zeta}(1)) = (\xi, \zeta)$ . Then:

- I. If  $\varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \{\zeta\}$ , then  $\varphi_{\mathfrak{g},\zeta}$  is called a "constant path" at  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ :  $\varphi_{\mathfrak{g},\zeta}(\lambda) \stackrel{\text{def}}{=} \mathfrak{c}_{\mathfrak{g}}(\lambda)$  for all  $\lambda \in [0, 1]$ .
- II. If  $\varphi_{\mathfrak{g},\zeta} : \{0, 1\} \rightarrow \{\zeta\}$ , then  $\varphi_{\mathfrak{g},\zeta}$  is called a "closed path" at  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ :  $\varphi_{\mathfrak{g},\zeta}(\lambda) \stackrel{\text{def}}{=} \mathfrak{k}_{\mathfrak{g}}(\lambda)$  for all  $\lambda \in [0, 1]$ .
- III. If, for every  $\lambda \in [0, 1]$ ,  $\varphi_{\mathfrak{g},\zeta}(\lambda) = \mathfrak{k}_{\mathfrak{g}}(\lambda)$  and  $\varphi_{\mathfrak{g},\zeta}(\lambda) \simeq \mathfrak{c}_{\mathfrak{g}}(\lambda)$ , then  $\varphi_{\mathfrak{g},\zeta}$  is said to be "contractable to the point  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ ."

A  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is "simply  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected" if and only if, at each point  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ , any closed path  $\mathfrak{k}_{\mathfrak{g}} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$  is contractable to  $\zeta$ .

The necessary and sufficient conditions for a pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected  $\mathcal{T}_{\mathfrak{g}}$ -space to be simply  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected are contained in the following theorem.

**THEOREM 3.35.** *Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected  $\mathcal{T}_{\mathfrak{g}}$ -space. Then,  $\mathfrak{T}_{\mathfrak{g}}$  is simply  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected if and only if, at each  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ , any closed path  $\mathfrak{k}_{\mathfrak{g}} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$  at  $\zeta$  is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -homotopic to the constant path  $\mathfrak{c}_{\mathfrak{g}} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}$  at  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ :  $\mathfrak{h}_{\mathfrak{g}} : \mathfrak{k}_{\mathfrak{g}} \simeq \mathfrak{c}_{\mathfrak{g}}$  for each  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ .*

**PROOF.** *Necessity.* Let  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  be a pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected  $\mathcal{T}_{\mathfrak{g}}$ -space, and suppose it be simply  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected. Since  $\mathfrak{T}_{\mathfrak{g}}$  is pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected, for every  $(\xi, \zeta) \in \Omega \times \Omega$ ,

$$(\exists \mathcal{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]) (\exists \varphi_{\mathfrak{g},\zeta} : [0, 1] \rightarrow \mathfrak{T}_{\mathfrak{g}}) [\Omega \supseteq \mathcal{Q}_{\mathfrak{g}} \supseteq \text{im}(\varphi_{\mathfrak{g},\zeta}|_{[0,1]})].$$

If  $\varphi_{\mathfrak{g},\zeta} : \{0, 1\} \rightarrow \{\zeta\}$ , then  $\varphi_{\mathfrak{g},\zeta}$  is a closed path at  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ :  $\varphi_{\mathfrak{g},\zeta}(\lambda) = \mathfrak{k}_{\mathfrak{g}}(\lambda)$  for all  $\lambda \in [0, 1]$ . Since  $\mathfrak{T}_{\mathfrak{g}}$  is simply  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected, it follows that, at each point  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$ ,



the closed path  $\mathfrak{k}_g : [0, 1] \rightarrow \mathfrak{T}_g$  is contractable to  $\zeta$ . Thus, at each  $\zeta \in \mathfrak{T}_g$ , the closed path  $\mathfrak{k}_g : [0, 1] \rightarrow \mathfrak{T}_g$  at  $\zeta$  is  $\mathfrak{g}$ - $\mathfrak{T}_g$ -homotopic to the constant path  $\mathfrak{c}_g : [0, 1] \rightarrow \mathfrak{T}_g$  at  $\zeta \in \mathfrak{T}_g$ :  $\mathfrak{h}_g : \mathfrak{k}_g \simeq \mathfrak{c}_g$  for each  $\zeta \in \mathfrak{T}_g$ . The condition of the theorem is, therefore, necessary.

*Sufficiency.* Conversely, suppose that, at every point  $\zeta \in \mathfrak{T}_g$  in a pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$ , any closed path  $\mathfrak{k}_g : [0, 1] \rightarrow \mathfrak{T}_g$  at  $\zeta$  is  $\mathfrak{g}$ - $\mathfrak{T}_g$ -homotopic to the constant path  $\mathfrak{c}_g : [0, 1] \rightarrow \mathfrak{T}_g$  at  $\zeta \in \mathfrak{T}_g$ :  $\mathfrak{h}_g : \mathfrak{k}_g \simeq \mathfrak{c}_g$  for every  $\zeta \in \mathfrak{T}_g$ . Then, there exists a path  $\varphi_{g,\zeta} : [0, 1] \rightarrow \mathfrak{T}_g$  satisfying  $\varphi_{g,\zeta} : \{0, 1\} \rightarrow \{\zeta\}$  and, therefore, contractable to  $\zeta \in \mathfrak{T}_g$ . Thus, at each point  $\zeta \in \mathfrak{T}_g$ , any closed path  $\mathfrak{k}_g : [0, 1] \rightarrow \mathfrak{T}_g$  is contractable to  $\zeta$ . The  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  is, then, simply  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected. The condition of the theorem is, therefore, sufficient. Q.E.D.

The definition of local  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connectedness at a point  $\xi \in \mathfrak{T}_g$  in a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  is now given.

DEFINITION 3.36. Let  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  be a  $\mathcal{T}_g$ -space. Then:

- I.  $\mathfrak{T}_g$  is said to be "locally  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected at a point  $\xi \in \mathfrak{T}_g$ " if and only if,

$$(3.27) \quad (\forall \mathcal{U}_g \in \mathfrak{g}\text{-O}[\mathfrak{T}_g]) (\exists \mathcal{Q}_g \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]) [\xi \in \mathcal{Q}_g \subseteq \mathcal{U}_g].$$

- II.  $\mathfrak{T}_g$  is said to be "locally pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected" if and only if, given any  $(\xi, \mathcal{U}_{g,\xi}) \in \mathfrak{T}_g \times \mathfrak{g}\text{-O}[\mathfrak{T}_g]$ , there exists  $(\xi, \mathcal{Q}_{g,\xi}) \in \mathfrak{T}_g \times \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  such that  $(\zeta, \eta) \in \mathcal{Q}_g \times \mathcal{Q}_g$ , with  $\zeta \neq \eta$ , implies that,

$$(3.28) \quad (\exists \varphi_{g,\zeta} : [0, 1] \rightarrow \mathfrak{T}_g) [\{\zeta, \eta\} \subseteq \text{im}(\varphi_{g,\zeta}|_{[0,1]}) \subseteq \mathcal{Q}_{g,\xi} \subseteq \mathcal{U}_{g,\xi}].$$

The  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  is said to be "locally  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected" if and only if it is locally  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected at every point  $\xi \in \mathfrak{T}_g$ .

As an immediate consequence of the above definition, it is shown below that local pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connectedness implies locally  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected.

THEOREM 3.37. If  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  be a locally pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected  $\mathcal{T}_g$ -space, then it is locally  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected:

$$(3.29) \quad \mathfrak{g}\text{-}\mathfrak{T}_g^{(\text{LPC})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_g^{(\text{LPC})}) \Rightarrow \mathfrak{g}\text{-}\mathfrak{T}_g^{(\text{LC})} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_g^{(\text{LC})}).$$

PROOF. Let  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  be a locally pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected  $\mathcal{T}_g$ -space. Then, for any given  $(\xi, \mathcal{U}_{g,\xi}) \in \mathfrak{T}_g \times \mathfrak{g}\text{-O}[\mathfrak{T}_g]$ , there exists  $(\xi, \mathcal{Q}_{g,\xi}) \in \mathfrak{T}_g \times \mathfrak{g}\text{-Q}[\mathfrak{T}_g]$  such that  $(\zeta, \xi) \in \mathcal{Q}_g \times \mathcal{Q}_g$ , with  $\zeta \neq \xi$ , implies that,

$$(\exists \varphi_{g,\zeta} : [0, 1] \rightarrow \mathfrak{T}_g) [\{\zeta, \xi\} \subseteq \text{im}(\varphi_{g,\zeta}|_{[0,1]}) \subseteq \mathcal{Q}_{g,\xi} \subseteq \mathcal{U}_{g,\xi}].$$

Consequently,  $\xi \in \text{im}(\varphi_{g,\zeta}|_{[0,1]}) \subseteq \mathcal{Q}_{g,\xi} \subseteq \mathcal{U}_{g,\xi}$  and, therefore,  $\xi \in \mathcal{Q}_{g,\xi} \subseteq \mathcal{U}_{g,\xi}$ . Hence, it follows that

$$(\forall \mathcal{U}_g \in \mathfrak{g}\text{-O}[\mathfrak{T}_g]) (\exists \mathcal{Q}_g \in \mathfrak{g}\text{-Q}[\mathfrak{T}_g]) [\xi \in \mathcal{Q}_{g,\xi} \subseteq \mathcal{U}_{g,\xi}].$$

The  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  is therefore locally  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected. Q.E.D.

In a locally  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected  $\mathcal{T}_g$ -space, a  $\mathfrak{g}$ - $\mathfrak{T}_g$ -component is a  $\mathfrak{g}$ - $\mathfrak{T}_g$ -set, as demonstrated in the following theorem.

**THEOREM 3.38.** *If  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  be the  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -component of  $\mathcal{S}_{\mathfrak{g}}$  corresponding to  $\zeta$  in a locally  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ , then  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ :*

$$(3.30) \quad \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \subseteq \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(LC)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(LC)}) \Rightarrow \mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-O}[\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(LC)}].$$

**PROOF.** Let  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  be the  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -component of  $\mathcal{S}_{\mathfrak{g}}$  corresponding to  $\zeta$  in a locally  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$ . Then, local  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness at  $\zeta \in \mathfrak{T}_{\mathfrak{g}}$  implies

$$(\forall \mathcal{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}])(\exists \mathcal{Q}_{\mathfrak{g}} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}])[\zeta \in \mathcal{Q}_{\mathfrak{g}} \subseteq \mathcal{U}_{\mathfrak{g}}].$$

Consequently,  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] = \bigcup_{\mathcal{O}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}} \mathcal{Q}_{\mathfrak{g}} \subseteq \bigcup_{\mathcal{U}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}} \mathcal{U}_{\mathfrak{g}}$ . But, since every  $\mathcal{U}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$  satisfies  $\mathcal{U}_{\mathfrak{g}} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}})$  for some  $\mathcal{O}_{\mathfrak{g}} \in \mathcal{T}_{\mathfrak{g}}$ , it follows that the relation  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \subseteq \bigcup_{\mathcal{O}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}} \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g}}) = \text{op}_{\mathfrak{g}}(\bigcup_{\mathcal{O}_{\mathfrak{g}} \subseteq \mathcal{S}_{\mathfrak{g}}} \mathcal{O}_{\mathfrak{g}})$  holds. Thus,  $\mathfrak{g}\text{-C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ . Q.E.D.

The necessary and sufficient conditions for a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  to be locally  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected at a point  $\xi \in \mathfrak{T}_{\mathfrak{g}}$  is contained in the following theorem.

**THEOREM 3.39.** *A  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is locally  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected at a point  $\xi \in \mathfrak{T}_{\mathfrak{g}}$  if and only if,*

$$(3.31) \quad (\forall \mathcal{O}_{\mathfrak{g},\xi} \in \mathcal{T}_{\mathfrak{g}})(\exists \mathcal{Q}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}])[\xi \in \mathcal{Q}_{\mathfrak{g},\xi} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})].$$

**PROOF.** *Necessity.* Let it be assumed that the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is locally  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected at  $\xi \in \mathfrak{T}_{\mathfrak{g}}$ , and let  $\mathcal{O}_{\mathfrak{g},\xi} \in \mathcal{T}_{\mathfrak{g}}$  be an arbitrary  $\mathcal{T}_{\mathfrak{g}}$ -open neighbourhood of  $\xi$ . There exists, then, a  $\mathcal{T}_{\mathfrak{g}}$ -open neighbourhood  $\hat{\mathcal{O}}_{\mathfrak{g},\xi} \in \mathcal{T}_{\mathfrak{g}}$  of  $\xi$  such that  $\xi \in \text{op}_{\mathfrak{g}}(\hat{\mathcal{O}}_{\mathfrak{g},\xi}) \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$  and, for every  $\{\zeta, \eta\} \subseteq \hat{\mathcal{O}}_{\mathfrak{g},\xi}$ ,

$$(\exists \mathcal{Q}_{\mathfrak{g},(\zeta,\eta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}])[\{\zeta, \eta\} \subseteq \mathcal{Q}_{\mathfrak{g},(\zeta,\eta)} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})].$$

Suppose  $\eta \in \hat{\mathcal{O}}_{\mathfrak{g},\xi}$  be the arbitrary point. Then, there exists a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected set  $\mathcal{Q}_{\mathfrak{g},(\xi,\eta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  satisfying  $\{\xi, \eta\} \subseteq \mathcal{Q}_{\mathfrak{g},(\xi,\eta)} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$ . Let  $\mathcal{Q}_{\mathfrak{g},\xi} = \bigcup_{\eta \in \hat{\mathcal{O}}_{\mathfrak{g},\xi}} \mathcal{Q}_{\mathfrak{g},(\xi,\eta)} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$ . Since  $\mathcal{Q}_{\mathfrak{g},\xi} \supseteq \hat{\mathcal{O}}_{\mathfrak{g},\xi}$  and  $\bigcup_{\eta \in \hat{\mathcal{O}}_{\mathfrak{g},\xi}} \mathcal{Q}_{\mathfrak{g},(\xi,\eta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , it follows that  $\mathcal{Q}_{\mathfrak{g},\xi}$  is a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected neighbourhood of  $\xi$  contained in  $\text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$ . The condition of the theorem is, therefore, necessary.

*Sufficiency.* Conversely, suppose the following condition holds:

$$(\forall \mathcal{O}_{\mathfrak{g},\xi} \in \mathcal{T}_{\mathfrak{g}})(\exists \mathcal{Q}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}])[\xi \in \mathcal{Q}_{\mathfrak{g},\xi} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})].$$

Let  $\mathcal{O}_{\mathfrak{g},\xi} \in \mathcal{T}_{\mathfrak{g}}$  be an arbitrary  $\mathcal{T}_{\mathfrak{g}}$ -open neighbourhood of  $\xi$ . Then,  $\text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$  contains a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected neighbourhood  $\mathcal{Q}_{\mathfrak{g},\xi}$  of  $\xi$ . Since  $\mathcal{Q}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$ , for any  $\{\zeta, \eta\} \subseteq \mathcal{Q}_{\mathfrak{g},\xi}$ , there exists  $\mathcal{Q}_{\mathfrak{g},(\zeta,\eta)} \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g}}]$  such that  $\{\zeta, \eta\} \subseteq \mathcal{Q}_{\mathfrak{g},(\zeta,\eta)}$ . But  $\mathcal{Q}_{\mathfrak{g},\xi} \supseteq \mathcal{Q}_{\mathfrak{g},(\zeta,\eta)}$  and, consequently,  $\{\zeta, \eta\} \subseteq \mathcal{Q}_{\mathfrak{g},(\zeta,\eta)} \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\xi})$ . Hence, the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathcal{T}_{\mathfrak{g}})$  is locally  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected at  $\xi \in \mathfrak{T}_{\mathfrak{g}}$ . The condition of the theorem is, therefore, sufficient. Q.E.D.

The notion of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness between any  $\mathcal{T}_{\mathfrak{g}}$ -spaces  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  and the relevant basic theorems are now discussed.

**THEOREM 3.40.** *Let  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  be  $\mathcal{T}_{\mathfrak{g}}$ -spaces, let  $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \dots, \mathcal{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Lambda}]$  be  $n \geq 1$  mutually disjoint  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -sets in  $\mathfrak{T}_{\mathfrak{g},\Lambda}$ , where  $\Lambda \in \{\Omega, \Sigma\}$ , and let  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-B}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  be a  $\mathfrak{g}\text{-}(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -bijective map  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ . Then*

$$\bullet \text{ I. } \pi_{\mathfrak{g}}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}) = \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\alpha}),$$

$$\bullet \text{ II. } \pi_{\mathfrak{g}}^{-1}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}) = \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\alpha}).$$

PROOF. I. Let  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  be  $\mathcal{T}_{\mathfrak{g}}$ -spaces, let  $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \dots, \mathcal{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ , and let  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-B}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ . If  $\zeta \in \pi_{\mathfrak{g}}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha})$ , then, since  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-B}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ , there exists  $\xi \in \bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}$  such that,  $\pi_{\mathfrak{g}}^{-1}(\zeta) = \xi \in \bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}$ . Consequently,

$$\begin{aligned} \pi_{\mathfrak{g}}^{-1}(\zeta) \in \bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha} &\Rightarrow \bigvee_{\alpha \in I_n^*} (\pi_{\mathfrak{g}}^{-1}(\zeta) \in \mathcal{S}_{\mathfrak{g},\alpha}) \\ &\Rightarrow \bigvee_{\alpha \in I_n^*} (\zeta \in \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\alpha})) \Rightarrow \zeta \in \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\alpha}). \end{aligned}$$

Hence,  $\pi_{\mathfrak{g}}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}) \subseteq \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\alpha})$ . Conversely, if it be assumed that  $\zeta \in \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\alpha})$  then,

$$\begin{aligned} \bigvee_{\alpha \in I_n^*} (\zeta \in \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\alpha})) &\Rightarrow \bigvee_{\alpha \in I_n^*} (\pi_{\mathfrak{g}}^{-1}(\zeta) \in \mathcal{S}_{\mathfrak{g},\alpha}) \\ &\Rightarrow \pi_{\mathfrak{g}}^{-1}(\zeta) \in \bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha} \Rightarrow \zeta \in \pi_{\mathfrak{g}}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}). \end{aligned}$$

Hence,  $\pi_{\mathfrak{g}}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}) \supseteq \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}(\mathcal{S}_{\mathfrak{g},\alpha})$ .

II. If  $\xi \in \pi_{\mathfrak{g}}^{-1}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha})$ , where  $\mathcal{S}_{\mathfrak{g},1}, \mathcal{S}_{\mathfrak{g},2}, \dots, \mathcal{S}_{\mathfrak{g},n} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , then,

$$\begin{aligned} \pi_{\mathfrak{g}}(\xi) \in \bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha} &\Rightarrow \bigvee_{\alpha \in I_n^*} (\pi_{\mathfrak{g}}(\xi) \in \mathcal{S}_{\mathfrak{g},\alpha}) \\ &\Rightarrow \bigvee_{\alpha \in I_n^*} (\xi \in \pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\alpha})) \Rightarrow \xi \in \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\alpha}). \end{aligned}$$

Hence,  $\pi_{\mathfrak{g}}^{-1}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}) \subseteq \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\alpha})$ . Conversely, if it be supposed that  $\xi \in \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\alpha})$  then,

$$\begin{aligned} \bigvee_{\alpha \in I_n^*} (\xi \in \pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\alpha})) &\Rightarrow \bigvee_{\alpha \in I_n^*} (\pi_{\mathfrak{g}}(\xi) \in \mathcal{S}_{\mathfrak{g},\alpha}) \\ &\Rightarrow \pi_{\mathfrak{g}}(\xi) \in \bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha} \Rightarrow \xi \in \pi_{\mathfrak{g}}^{-1}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}). \end{aligned}$$

Hence,  $\pi_{\mathfrak{g}}^{-1}(\bigsqcup_{\alpha \in I_n^*} \mathcal{S}_{\mathfrak{g},\alpha}) \supseteq \bigsqcup_{\alpha \in I_n^*} \pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\alpha})$ . Q.E.D.

The following theorem shows, among others, that  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness is a  $\mathcal{T}_{\mathfrak{g}}$ -property.

**THEOREM 3.41.** *Let  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  be a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -continuous map and let  $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set. If  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected in  $\mathfrak{T}_{\mathfrak{g},\Omega}$ , then  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected in  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ :*

$$(3.32) \quad \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}] \Rightarrow \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}].$$

PROOF. Let  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ ,  $\mathcal{S}_{\mathfrak{g},\omega} = \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ , and suppose that  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , that is,  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \notin \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ . There exists, therefore,  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  such that,

$$\left( \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right) \vee \left( \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right).$$

Set  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) = \bigsqcup_{\lambda=\xi_{\omega}, \zeta_{\omega}} \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}})$ , where  $\mathcal{S}_{\mathfrak{g},\omega} \supseteq \bigcup_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{S}_{\mathfrak{g},(\omega,\lambda)}$  and, for every  $(\lambda, \mu) \in \{(\xi_{\sigma}, \xi_{\omega}), (\zeta_{\sigma}, \zeta_{\omega})\}$ , set

$$[\mathcal{U}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\mu)}}})] \vee [\mathcal{V}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\mu)}}})].$$

In other words,  $\mathcal{S}_{\mathfrak{g},(\omega,\xi_{\omega})} \subseteq \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  denotes the  $\mathfrak{T}_{\mathfrak{g}}$ -set of all  $\xi \in \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  for which  $\pi_{\mathfrak{g}}(\xi) \in \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\xi_{\omega})}}})$ , and  $\mathcal{S}_{\mathfrak{g},(\omega,\zeta_{\omega})} \subseteq \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  denotes the  $\mathfrak{T}_{\mathfrak{g}}$ -set of all  $\zeta \in \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  for which  $\pi_{\mathfrak{g}}(\zeta) \in \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\zeta_{\omega})}}})$ . Since  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}}) \neq \emptyset$  for every  $\lambda \in \{\xi_{\omega}, \zeta_{\omega}\}$ , and both the relations  $\bigsqcup_{\lambda=\xi_{\omega}, \zeta_{\omega}} \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}}) = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  and  $\bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}}) = \emptyset$  hold, it follows that,  $\mathcal{S}_{\mathfrak{g},(\omega,\lambda)} \neq \emptyset$  for every  $\lambda \in \{\xi_{\omega}, \zeta_{\omega}\}$ ,  $\bigcup_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{S}_{\mathfrak{g},(\omega,\lambda)} = \mathcal{S}_{\mathfrak{g},\omega}$  and  $\bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{S}_{\mathfrak{g},(\omega,\lambda)} = \emptyset$ . Since  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ , for any  $\lambda \in \{\xi_{\omega}, \zeta_{\omega}\}$ , there exists, for every  $(\mathcal{O}_{\mathfrak{g},(\sigma,\lambda)}, \mathcal{K}_{\mathfrak{g},(\sigma,\lambda)}) \in \mathcal{T}_{\mathfrak{g},\Sigma} \times \neg\mathcal{T}_{\mathfrak{g},\Sigma}$ ,  $(\mathcal{O}_{\mathfrak{g},(\omega,\lambda)}, \mathcal{K}_{\mathfrak{g},(\omega,\lambda)}) \in \mathcal{T}_{\mathfrak{g},\Omega} \times \neg\mathcal{T}_{\mathfrak{g},\Omega}$ , with  $\mathcal{O}_{\mathfrak{g},(\omega,\lambda)}, \mathcal{K}_{\mathfrak{g},(\omega,\lambda)} \subset \mathcal{S}_{\mathfrak{g},(\omega,\lambda)}$  and  $\mathcal{O}_{\mathfrak{g},(\sigma,\lambda)}, \mathcal{K}_{\mathfrak{g},(\sigma,\lambda)} \subset \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}})$ , such that,

$$[\pi_{\mathfrak{g}}^{-1}(\mathcal{O}_{\mathfrak{g},(\sigma,\lambda)}) \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(\omega,\lambda)})] \vee [\pi_{\mathfrak{g}}^{-1}(\mathcal{K}_{\mathfrak{g},(\sigma,\lambda)}) \supseteq \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},(\omega,\lambda)})].$$

Since  $\bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}}) = \emptyset$  implies  $\bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{S}_{\mathfrak{g},(\omega,\lambda)} = \emptyset$ , it follows, evidently, that,

$$\left( \bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \pi_{\mathfrak{g}}^{-1}(\mathcal{O}_{\mathfrak{g},(\sigma,\lambda)}) = \emptyset \right) \wedge \left( \bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \pi_{\mathfrak{g}}^{-1}(\mathcal{K}_{\mathfrak{g},(\sigma,\lambda)}) = \emptyset \right).$$

Therefore, the setting  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) = \bigsqcup_{\lambda=\xi_{\omega}, \zeta_{\omega}} \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}})$  holds. It now remains to prove that it is the case and the supposition that  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  is a contradiction. Since  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  it follows that, for all pair  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\omega}, \zeta_{\omega}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ ,

$$\begin{aligned} & \neg \left( \bigsqcup_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{U}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right) \wedge \neg \left( \bigsqcup_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{V}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right) \\ & \Leftrightarrow \neg \left( \bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{U}_{\mathfrak{g},\lambda} = \emptyset \right) \wedge \neg \left( \bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{V}_{\mathfrak{g},\lambda} = \emptyset \right) \\ & \Rightarrow \left( \bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{U}_{\mathfrak{g},\lambda} \neq \emptyset \right) \wedge \left( \bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{V}_{\mathfrak{g},\lambda} \neq \emptyset \right). \end{aligned}$$

There exists, then, a unit  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\{\eta_{\omega}\} \subset \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  such that,

$$\left( \bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{U}_{\mathfrak{g},\lambda} \supseteq \{\eta_{\omega}\} \right) \wedge \left( \bigcap_{\lambda=\xi_{\omega}, \zeta_{\omega}} \mathcal{V}_{\mathfrak{g},\lambda} \supseteq \{\eta_{\omega}\} \right).$$

Since  $\{\eta_\omega\} \subset \bigcup_{\lambda=\xi_\omega, \zeta_\omega} \mathcal{S}_{\mathfrak{g},(\omega,\lambda)} = \mathcal{S}_{\mathfrak{g},\omega}$  and  $\bigcap_{\lambda=\xi_\omega, \zeta_\omega} \mathcal{S}_{\mathfrak{g},(\omega,\lambda)} = \emptyset$ , it results that,

$$[\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \supset \mathcal{S}_{\mathfrak{g},(\omega,\xi_\omega)} \supset \{\eta_\omega\}] \vee [\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \supset \mathcal{S}_{\mathfrak{g},(\omega,\zeta_\omega)} \supset \{\eta_\omega\}].$$

On the other hand, since  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ , it follows that, for every unit  $\mathfrak{T}_{\mathfrak{g}}$ -set  $\{\eta_\sigma\} \subset \bigsqcup_{\lambda=\xi_\omega, \zeta_\omega} \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}}) = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$ ,

$$\bigvee_{\lambda=\xi_\omega, \zeta_\omega} [\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \supset \mathcal{S}_{\mathfrak{g},(\omega,\lambda)} \supset \pi_{\mathfrak{g}}^{-1}(\{\eta_\sigma\})].$$

In particular, if  $\pi_{\mathfrak{g}}^{-1}(\{\eta_\sigma\}) = \{\eta_\omega\}$ , then  $\{\eta_\sigma\} = \pi_{\mathfrak{g}}(\{\eta_\omega\})$ , leading to a contradiction. There exists, therefore,  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_\omega, \zeta_\omega} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  such that,

$$\left( \bigsqcup_{\lambda=\xi_\omega, \zeta_\omega} \mathcal{U}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right) \vee \left( \bigsqcup_{\lambda=\xi_\omega, \zeta_\omega} \mathcal{V}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right).$$

This proves that the supposition  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  is a contradiction and, hence,  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ . Q.E.D.

The following corollary is another way of saying that the  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectivity of  $\mathfrak{T}_{\mathfrak{g},\Omega}$  implies the  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectivity of  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ .

**COROLLARY 3.42.** *Let  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  be a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -continuous map and let  $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set. If  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated in  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ , then  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated in  $\mathfrak{T}_{\mathfrak{g},\Omega}$ :*

$$(3.33) \quad \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \Rightarrow \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Omega}].$$

If the image of a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -continuous map is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected, then it is also  $\mathfrak{T}_{\mathfrak{g}}$ -connected, as proved in the following proposition.

**PROPOSITION 3.43.** *Let  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  be a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -continuous map and let  $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set. If  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected in  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ , then  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{T}_{\mathfrak{g}}$ -connected in  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ :*

$$(3.34) \quad \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \Rightarrow \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \text{Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}].$$

**PROOF.** Let  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ ,  $\mathcal{S}_{\mathfrak{g},\omega} = \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  and, suppose that  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \text{D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , that is,  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \notin \text{Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ . There exists, then,  $(\mathcal{R}_{\mathfrak{g},\lambda}, \mathcal{S}_{\mathfrak{g},\lambda})_{\lambda=\xi_\sigma, \zeta_\sigma} \in \text{O}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \times \text{K}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  such that,

$$\left( \bigsqcup_{\lambda=\xi_\sigma, \zeta_\sigma} \mathcal{R}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right) \vee \left( \bigsqcup_{\lambda=\xi_\sigma, \zeta_\sigma} \mathcal{S}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right).$$

Since  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \text{D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , set  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) = \bigsqcup_{\lambda=\xi_\sigma, \zeta_\sigma} \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}})$  and, for every  $\lambda \in \{\xi_\sigma, \zeta_\sigma\}$ , let

$$[\mathcal{R}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}})] \vee [\mathcal{S}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}})].$$

On the other hand, since  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ , there exists, for any  $\lambda \in \{\xi_\sigma, \zeta_\sigma\}$ ,  $(\mathcal{O}_{\mathfrak{g},(\omega,\lambda)}, \mathcal{K}_{\mathfrak{g},(\omega,\lambda)}) \in \mathcal{T}_{\mathfrak{g},\Omega} \times \neg\mathcal{T}_{\mathfrak{g},\Omega}$ , satisfying  $\mathcal{O}_{\mathfrak{g},(\omega,\lambda)}, \mathcal{K}_{\mathfrak{g},(\omega,\lambda)} \subset \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$ ,

such that,

$$\begin{aligned} & [\pi_{\mathfrak{g}}^{-1}(\mathcal{R}_{\mathfrak{g},\lambda}) \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(\omega,\lambda)})] \vee [\pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\lambda}) \supseteq \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},(\omega,\lambda)})] \\ \Rightarrow & [\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}}) \subseteq \pi_{\mathfrak{g}}(\text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(\omega,\lambda)}))] \vee [\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}}) \\ & \supseteq \pi_{\mathfrak{g}}(\neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},(\omega,\lambda)}))]. \end{aligned}$$

Since  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}})$ , it is plain that  $\pi_{\mathfrak{g}}(\text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\omega})) = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \pi_{\mathfrak{g}}(\text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(\omega,\lambda)}))$ , and also  $\emptyset = \bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \pi_{\mathfrak{g}}(\neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},(\omega,\lambda)}))$ , implying  $\pi_{\mathfrak{g}}(\neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\omega})) = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \pi_{\mathfrak{g}}(\neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},(\omega,\lambda)}))$ , for some  $(\mathcal{O}_{\mathfrak{g},\omega}, \mathcal{K}_{\mathfrak{g},\omega}) \in \mathcal{T}_{\mathfrak{g},\Omega} \times \neg \mathcal{T}_{\mathfrak{g},\Omega}$ . But, clearly the relation  $(\pi_{\mathfrak{g}}(\text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\omega})), \pi_{\mathfrak{g}}(\neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\omega}))) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  holds. Therefore,  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  implies  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , or equivalently,  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  implies  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ . This proves the proposition. Q.E.D.

The following corollary is another way of saying that the  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectivity of  $\mathfrak{T}_{\mathfrak{g},\Omega}$  implies the  $\mathfrak{T}_{\mathfrak{g}}$ -connectivity of  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ .

**COROLLARY 3.44.** *Let  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  be a  $\mathfrak{g}\text{-}(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -continuous map and let  $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set. If  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{T}_{\mathfrak{g}}$ -separated in  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ , then  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated in  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ , then:*

$$(3.35) \quad \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{D}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \Rightarrow \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Omega}].$$

**THEOREM 3.45.** *Let  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  be a  $\mathfrak{g}\text{-}(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -irresolute map and let  $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set. If  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected in  $\mathfrak{T}_{\mathfrak{g},\Omega}$ , then  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected in  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ :*

$$(3.36) \quad \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}] \Rightarrow \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}].$$

**PROOF.** Let  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  be a  $\mathfrak{g}\text{-}(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -irresolute map  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$ , let  $\mathcal{S}_{\mathfrak{g},\omega} \subset \mathfrak{T}_{\mathfrak{g},\Omega}$  be a  $\mathfrak{T}_{\mathfrak{g}}$ -set, and suppose  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$  be  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated in  $\mathfrak{T}_{\mathfrak{g},\Sigma}$ . Since  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \notin \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , or equivalently  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , by hypothesis, there exists  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  such that,

$$\left( \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right) \vee \left( \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} = \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \right).$$

On the other hand, since  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ , there exists, for any  $\lambda \in \{\xi_{\sigma}, \zeta_{\sigma}\}$ ,  $(\mathcal{O}_{\mathfrak{g},(\omega,\lambda)}, \mathcal{K}_{\mathfrak{g},(\omega,\lambda)}) \in \mathcal{T}_{\mathfrak{g},\Omega} \times \neg \mathcal{T}_{\mathfrak{g},\Omega}$ , satisfying  $\mathcal{O}_{\mathfrak{g},(\omega,\lambda)}, \mathcal{K}_{\mathfrak{g},(\omega,\lambda)} \subset \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$ , such that,

$$[\pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\lambda}) \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},(\omega,\lambda)})] \vee [\pi_{\mathfrak{g}}^{-1}(\mathcal{V}_{\mathfrak{g},\lambda}) \supseteq \neg \text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},(\omega,\lambda)})].$$

Since both the relation  $\pi_{\mathfrak{g}}^{-1}(\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda}) = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\lambda})$  and the relation  $\pi_{\mathfrak{g}}^{-1}(\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda}) = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{V}_{\mathfrak{g},\lambda})$ . Evidently,  $\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\lambda}) \subset \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$ , and also  $\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{V}_{\mathfrak{g},\lambda}) \subset \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}})$ , and from which it follows that a  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separation  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},(\omega,\lambda)}}})$  is realised in  $\mathfrak{T}_{\mathfrak{g},\Omega}$ . Consequently,  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ . Therefore,  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  implies  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ , or equivalently,  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  implies  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g},\omega}}}) \in \mathfrak{Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ . This proves the proposition. Q.E.D.

Between any two such  $\mathcal{T}_{\mathfrak{g}}$ -spaces  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$ ,  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness, being a  $\mathcal{T}_{\mathfrak{g}}$ -property, is preserved by a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -homeomorphism  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$ .

**THEOREM 3.46.** *Let  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  be  $\mathcal{T}_{\mathfrak{g}}$ -spaces, and let  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$  be a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -homeomorphism. If  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected, then  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  is also  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected:*

$$(3.37) \quad \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma} : \quad \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{(C)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Omega}^{(C)}) \Rightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(C)} = (\Sigma, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Sigma}^{(C)}).$$

**PROOF.** Let  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  be  $\mathcal{T}_{\mathfrak{g}}$ -spaces, let  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$  be a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -homeomorphism, and suppose that  $\mathfrak{T}_{\mathfrak{g},\Sigma}$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated. There exists, then,  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Sigma}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  such that,

$$\left( \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) \right) \vee \left( \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) \right).$$

Clearly,  $\text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  and, with no loss of generality, consider the setting  $\text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) = \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}})$  so that, for every  $\lambda \in \{\xi_{\sigma}, \zeta_{\sigma}\}$ , either  $\mathcal{U}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}})$  or  $\mathcal{V}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}})$ . Since  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-Hom}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ ,  $\pi_{\mathfrak{g}}^{-1} : \mathfrak{T}_{\mathfrak{g},\Sigma} \cong \mathfrak{T}_{\mathfrak{g},\Omega}$  and, for any  $(\mathcal{S}_{\mathfrak{g},\alpha}, \mathcal{S}_{\mathfrak{g},\beta}) \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Lambda}] \times \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Lambda}]$ ,  $\pi_{\mathfrak{g}}^{-1}(\bigsqcup_{\lambda=\alpha, \beta} \mathcal{S}_{\mathfrak{g},\lambda}) = \bigsqcup_{\lambda=\alpha, \beta} \pi_{\mathfrak{g}}^{-1}(\mathcal{S}_{\mathfrak{g},\lambda})$ , where  $\Lambda \in \{\Omega, \Sigma\}$ , it results that,

$$\left( \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\lambda}) = \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) \right) \\ \vee \left( \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{V}_{\mathfrak{g},\lambda}) = \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) \right),$$

where  $\text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) = \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}})$ . On the other hand, since  $\pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\lambda}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  and  $\pi_{\mathfrak{g}}^{-1}(\mathcal{V}_{\mathfrak{g},\lambda}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  hold for every  $\lambda \in \{\xi_{\sigma}, \zeta_{\sigma}\}$ , there exist, therefore,  $(\mathcal{U}_{\mathfrak{g},\eta}, \mathcal{V}_{\mathfrak{g},\eta})_{\eta=\xi_{\omega}, \zeta_{\omega}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  such that,

$$\bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\lambda}) = \bigsqcup_{\eta=\xi_{\omega}, \zeta_{\omega}} \mathcal{U}_{\mathfrak{g},\eta}, \\ \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{V}_{\mathfrak{g},\lambda}) = \bigsqcup_{\eta=\xi_{\omega}, \zeta_{\omega}} \mathcal{V}_{\mathfrak{g},\eta}.$$

By substitution, then, it follows that,

$$\left( \bigsqcup_{\eta=\xi_{\omega}, \zeta_{\omega}} \mathcal{U}_{\mathfrak{g},\eta} = \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) \right) \\ \vee \left( \bigsqcup_{\eta=\xi_{\omega}, \zeta_{\omega}} \mathcal{V}_{\mathfrak{g},\eta} = \bigsqcup_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) \right).$$

Since  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-Hom}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  and  $\pi_{\mathfrak{g}}^{-1} \in \mathfrak{g}\text{-Hom}[\mathfrak{T}_{\mathfrak{g},\Sigma}; \mathfrak{T}_{\mathfrak{g},\Omega}]$ , for each  $\text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}})$ , there exists a unique  $\text{dom}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}})$ , with  $\text{dom}(\pi_{\mathfrak{g}}|_{\Omega}) = \bigsqcup_{\eta=\xi_{\omega}, \zeta_{\omega}} \text{dom}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}})$ . Thus,



there exists  $(\mathcal{U}_{\mathfrak{g},\eta}, \mathcal{V}_{\mathfrak{g},\eta})_{\eta=\xi_\omega, \zeta_\omega} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  such that,

$$\left( \bigsqcup_{\eta=\xi_\omega, \zeta_\omega} \mathcal{U}_{\mathfrak{g},\eta} = \text{dom}(\pi_{\mathfrak{g}|\Omega}) \right) \bigvee \left( \bigsqcup_{\eta=\xi_\omega, \zeta_\omega} \mathcal{V}_{\mathfrak{g},\eta} = \text{dom}(\pi_{\mathfrak{g}|\Omega}) \right).$$

Hence,  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated. Q.E.D.

An immediate consequence of the above theorem is the following corollary.

**COROLLARY 3.47.** *Let  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  be  $\mathcal{T}_{\mathfrak{g}}$ -spaces, and let  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma}$  be a  $\mathfrak{g}\text{-}(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -homeomorphism. If  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated, then  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  is also  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated:*

$$(3.38) \quad \mathfrak{T}_{\mathfrak{g},\Omega} \cong \mathfrak{T}_{\mathfrak{g},\Sigma} : \quad \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Omega}^{(D)} = (\Omega, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Omega}^{(D)}) \leftarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\Sigma}^{(D)} = (\Sigma, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\Sigma}^{(D)}).$$

For every  $\mu \in I_n^*$ , let  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)} = (\Omega_\mu, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)})$  stand for the shortened form of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)}(\Omega_\mu) = (\Omega_\mu, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}(\Omega_\mu))$ . In the following lemma, it is proved that the Cartesian product of two  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces is also a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space.

**LEMMA 3.48.** *If  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)} = (\Omega_\mu, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)})$ ,  $\mu \in \{\alpha, \beta\}$ , be two  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces, then  $\times_{\mu=\alpha, \beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  is also a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space.*

**PROOF.** Let  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)} = (\Omega_\mu, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)})$ ,  $\mu \in \{\alpha, \beta\}$ , be two  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces, and suppose  $\xi = (\xi_\alpha, \xi_\beta) \in \times_{\mu=\alpha, \beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  and  $\zeta = (\zeta_\alpha, \zeta_\beta) \in \times_{\mu=\alpha, \beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  be any two pairs of points in  $\times_{\mu=\alpha, \beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$ . Then,

$$[\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{(C)} \cong \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{(C)}] \wedge [\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{(C)} \times \{\zeta_\beta\} \cong \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{(C)}].$$

Consequently,  $\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{(C)}$  and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{(C)} \times \{\zeta_\beta\}$  are both  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected. But,

$$(\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{(C)}) \cap (\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{(C)} \times \{\zeta_\beta\}) = \{(\xi_\alpha, \zeta_\beta)\} \neq \emptyset.$$

Hence,  $(\{\xi_\alpha\} \times \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\beta}^{(C)}) \cup (\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\alpha}^{(C)} \times \{\zeta_\beta\})$  is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected. Accordingly,  $\xi, \zeta \in \times_{\mu=\alpha, \beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  belong to the same  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -component. That is,  $\xi, \zeta \in \mathfrak{g}\text{-}C_\Omega[\eta] \subseteq \times_{\mu=\alpha, \beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$ , the  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -component of  $\Omega = \times_{\mu=\alpha, \beta} \Omega_\mu$  corresponding to the point  $\eta \in \times_{\mu=\alpha, \beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$ . But  $\xi, \zeta \in \times_{\mu=\alpha, \beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  were arbitrary. Hence the Cartesian product  $\times_{\mu=\alpha, \beta} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  has one  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -component  $\mathfrak{g}\text{-}C_\Omega[\eta] = \times_{\mu=\alpha, \beta} \Omega_\mu$ , and is therefore a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space. Q.E.D.

More generally, the Cartesian product of  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces is also a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space; that is,  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness is a product invariant  $\mathcal{T}_{\mathfrak{g}}$ -property. The theorem follows.

**THEOREM 3.49.** *If  $\{\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)} = (\Omega_\mu, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)}) : \mu \in I_n^*\}$  be a collection of  $n \geq 1$   $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces, then  $\times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  is also a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space:*

$$(3.39) \quad \{\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)} = (\Omega_\mu, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)}) : \mu \in I_n^*\} \Rightarrow \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}^{(C)} = \times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}.$$

**PROOF.** Let  $\{\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)} = (\Omega_\mu, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)}) : \mu \in I_n^*\}$  be a collection of  $n \geq 1$   $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces, and let  $\times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  be the Cartesian product of these  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces. Moreover, let  $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$ , and let  $\mathfrak{g}\text{-}C_{S_{\mathfrak{g}}}[\zeta] \subseteq \times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  be

the  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -component of  $\mathcal{S}_{\mathfrak{g}} \subseteq \times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  corresponding to  $\zeta \in \times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$ . By hypothesis, let it be claimed that, for every  $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$ ,  $\xi \in \neg \text{op}_{\mathfrak{g}}(\mathfrak{g}\text{-}\mathcal{C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta])$  and, thus,  $\xi \in \mathfrak{g}\text{-}\mathcal{C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  since  $\mathfrak{g}\text{-}\mathcal{C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \supseteq \neg \text{op}_{\mathfrak{g}}(\mathfrak{g}\text{-}\mathcal{C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta])$ , meaning that  $\mathfrak{g}\text{-}\mathcal{C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$  must be a  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -closed set in  $\times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$ . For every  $(\mu, \sigma(\mu), \mathcal{O}_{\mathfrak{g},\sigma(\mu)}) \in \{\mu\} \times I_{\infty}^* \times \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)}$ , there exists  $I_{\sigma(\mu)} \subseteq I_{\infty}^*$  such that  $\mathcal{O}_{\mathfrak{g},\sigma(\mu)} = \bigcup_{\nu \in I_{\sigma(\mu)}^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)}$ . Thus, the class  $\mathcal{B}[\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)}] \stackrel{\text{def}}{=} \{\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} : (\nu, \mu, \sigma(\nu,\mu)) \in I_{\infty}^* \times \{\mu\} \times I_{\infty}^*\}$  is a  $\mathcal{T}_{\mathfrak{g}}$ -basis for  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)} : \mathcal{P}(\Omega_{\mu}) \rightarrow \mathcal{P}(\Omega_{\mu})$ . Therefore, for any  $\xi \in \mathcal{O}_{\mathfrak{g},\sigma(\mu)} \in \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)}$ , there exists  $\mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \in \mathcal{B}[\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)}]$  with  $\xi \in \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \subseteq \mathcal{O}_{\mathfrak{g},\sigma(\mu)} \in \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},\mu}^{(C)}$ . Now let

$$\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathcal{R}_{\mathfrak{g}} = \left( \times_{\mu \in I_n^* \setminus J_n} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)} \right) \times \left( \times_{\mu \in J_n \subset I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \right).$$

Now the following relation holds,

$$\mathcal{S}_{\mathfrak{g}} = \left( \times_{\mu \in I_n^* \setminus J_n} \{\zeta_{\mu}\} \right) \times \left( \times_{\mu \in J_n \subset I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)} \right) \cong \times_{\mu \in J_n \subset I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)},$$

and, hence,  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected. Furthermore,  $\zeta \in \mathcal{S}_{\mathfrak{g}}$  and, consequently, it follows that  $\mathfrak{g}\text{-}\mathcal{C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta] \supseteq \mathcal{S}_{\mathfrak{g}}$ . But, by the property of the intersection of Cartesian products,

$$\mathcal{R}_{\mathfrak{g}} \cap \mathcal{S}_{\mathfrak{g}} = \left( \times_{\mu \in I_n^* \setminus J_n} \{\zeta_{\mu}\} \right) \times \left( \times_{\mu \in J_n \subset I_n^*} \mathcal{O}_{\mathfrak{g},\sigma(\nu,\mu)} \right) \neq \emptyset.$$

Therefore,  $\mathcal{R}_{\mathfrak{g}} \subset \times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  contains a point of  $\mathfrak{g}\text{-}\mathcal{C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$ . Accordingly,  $\xi \in \neg \text{op}_{\mathfrak{g}}(\mathfrak{g}\text{-}\mathcal{C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]) \subseteq \mathfrak{g}\text{-}\mathcal{C}_{\mathcal{S}_{\mathfrak{g}}}[\zeta]$ . Hence the Cartesian product  $\times_{\mu \in I_n^*} \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},\mu}^{(C)}$  has one  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -component  $\mathfrak{g}\text{-}\mathcal{C}_{\Omega}[\zeta] = \times_{\mu \in I_n^*} \Omega_{\mu}$ , and is therefore a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space. Q.E.D.

The concept of  $(\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma})$ -surjective map between any such  $\mathcal{T}_{\mathfrak{g}}$ -spaces  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  is now defined.

**DEFINITION 3.50** ( $(\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma})$ -Surjective Map). A  $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -map  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  is said to be surjective if and only if it belongs the following class:

$$(3.40) \quad \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \stackrel{\text{def}}{=} \left\{ \pi_{\mathfrak{g}} : (\forall \zeta \in \mathfrak{T}_{\mathfrak{g},\Sigma}) (\exists \xi \in \mathfrak{T}_{\mathfrak{g},\Omega}) [\pi_{\mathfrak{g}}(\xi) = \zeta] \right\}.$$

If the domain of a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -irresolute surjective map is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected, then its codomain is also  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected, as demonstrated in the theorem below.

**THEOREM 3.51.** *Let  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  be  $\mathcal{T}_{\mathfrak{g}}$ -spaces. If  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  be a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -irresolute surjective map  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  and  $\mathfrak{T}_{\mathfrak{g},\Omega}$  is  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected, then  $\mathfrak{T}_{\mathfrak{g},\Sigma}$  is also  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected.*

**PROOF.** Let  $\mathfrak{T}_{\mathfrak{g},\Omega} = (\Omega, \mathcal{T}_{\mathfrak{g},\Omega})$  and  $\mathfrak{T}_{\mathfrak{g},\Sigma} = (\Sigma, \mathcal{T}_{\mathfrak{g},\Sigma})$  be  $\mathcal{T}_{\mathfrak{g}}$ -spaces of which  $\mathfrak{T}_{\mathfrak{g},\Sigma}$  is assumed to be  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated, and let  $\pi_{\mathfrak{g}} : \mathfrak{T}_{\mathfrak{g},\Omega} \rightarrow \mathfrak{T}_{\mathfrak{g},\Sigma}$  be a  $\mathfrak{g}$ - $(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -irresolute surjective map. Since  $\text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , there exists  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\sigma}, \zeta_{\sigma}} \in$

$\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  such that,

$$\left( \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{U}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) \right) \\ \vee \left( \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \mathcal{V}_{\mathfrak{g},\lambda} = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) \right),$$

where  $\bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) = \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$  so that, for every  $\lambda \in \{\xi_{\sigma}, \zeta_{\sigma}\}$ , either  $\mathcal{U}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}})$  or  $\mathcal{V}_{\mathfrak{g},\lambda} = \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}})$ . Since the relation  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  holds, there exists  $(\mathcal{O}_{\mathfrak{g},\eta}, \mathcal{K}_{\mathfrak{g},\eta}) \in \mathcal{T}_{\mathfrak{g},\Omega} \times \neg\mathcal{T}_{\mathfrak{g},\Omega}$ ,  $\eta \in \{\xi_{\omega}, \zeta_{\omega}\}$  with  $(\lambda, \eta) \in \{(\xi_{\sigma}, \xi_{\omega}), (\zeta_{\sigma}, \zeta_{\omega})\}$ , such that,

$$[\pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\lambda}) \subseteq \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\eta})] \vee [\pi_{\mathfrak{g}}^{-1}(\mathcal{V}_{\mathfrak{g},\lambda}) \supseteq \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\eta})].$$

Evidently,  $\text{dom}(\pi_{\mathfrak{g}}|_{\Omega}) = \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma})$  and  $\text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) = \text{im}(\pi_{\mathfrak{g}}|_{\Omega})$ . Since  $\text{im}(\pi_{\mathfrak{g}}|_{\Omega})$ ,  $\text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , set  $\text{im}(\pi_{\mathfrak{g}}|_{\Omega}) = \bigsqcup_{\eta=\xi_{\omega},\zeta_{\omega}} \text{im}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}})$  and  $\text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma}) = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}})$ , and for any  $(\lambda, \eta) \in \{(\xi_{\sigma}, \xi_{\omega}), (\zeta_{\sigma}, \zeta_{\omega})\}$  set

$$[\text{im}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}}) = \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) = \mathcal{U}_{\mathfrak{g},\lambda}] \\ \vee [\text{im}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}}) = \text{dom}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) = \mathcal{V}_{\mathfrak{g},\lambda}].$$

Since  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$ , for every  $(\lambda, \eta) \in \{(\xi_{\sigma}, \xi_{\omega}), (\zeta_{\sigma}, \zeta_{\omega})\}$ ,

$$\left( \bigcap_{\eta=\xi_{\omega},\zeta_{\omega}} \text{dom}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}}) = \bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) = \bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{U}_{\mathfrak{g},\lambda}) = \emptyset \right) \\ \vee \left( \bigcap_{\eta=\xi_{\omega},\zeta_{\omega}} \text{dom}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}}) = \bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) = \bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \pi_{\mathfrak{g}}^{-1}(\mathcal{V}_{\mathfrak{g},\lambda}) = \emptyset \right),$$

Thus,  $\text{dom}(\pi_{\mathfrak{g}}|_{\Omega}) = \bigsqcup_{\eta=\xi_{\omega},\zeta_{\omega}} \text{dom}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}})$ . Since the relation  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  holds, it follows that

$$\bigsqcup_{\eta=\xi_{\omega},\zeta_{\omega}} \text{dom}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}}) = \bigsqcup_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) \subseteq \bigsqcup_{\eta=\xi_{\omega},\zeta_{\omega}} \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\eta}), \\ \bigcap_{\eta=\xi_{\omega},\zeta_{\omega}} \text{dom}(\pi_{\mathfrak{g}}|_{\Omega_{\eta}}) = \bigcap_{\lambda=\xi_{\sigma},\zeta_{\sigma}} \text{im}(\pi_{\mathfrak{g}}^{-1}|_{\Sigma_{\lambda}}) \supseteq \bigcap_{\eta=\xi_{\omega},\zeta_{\omega}} \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\eta}).$$

Thus,  $\bigsqcup_{\eta=\xi_{\omega},\zeta_{\omega}} \text{op}_{\mathfrak{g}}(\mathcal{O}_{\mathfrak{g},\eta}) = \text{dom}(\pi_{\mathfrak{g}}|_{\Omega})$  and  $\bigcap_{\eta=\xi_{\omega},\zeta_{\omega}} \neg\text{op}_{\mathfrak{g}}(\mathcal{K}_{\mathfrak{g},\eta}) = \emptyset$ . There exists, then,  $(\mathcal{U}_{\mathfrak{g},\lambda}, \mathcal{V}_{\mathfrak{g},\lambda})_{\lambda=\xi_{\omega},\zeta_{\omega}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g},\Omega}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  such that,

$$\left( \bigsqcup_{\eta=\xi_{\omega},\zeta_{\omega}} \mathcal{U}_{\mathfrak{g},\eta} = \text{dom}(\pi_{\mathfrak{g}}|_{\Omega}) \right) \vee \left( \bigsqcup_{\eta=\xi_{\omega},\zeta_{\omega}} \mathcal{V}_{\mathfrak{g},\eta} = \text{dom}(\pi_{\mathfrak{g}}|_{\Omega}) \right).$$

Thus,  $\text{dom}(\pi_{\mathfrak{g}}|_{\Omega}) \in \mathfrak{g}\text{-D}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ , or equivalently  $\text{dom}(\pi_{\mathfrak{g}}|_{\Omega}) \notin \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  which contradicts the assumption that  $\text{dom}(\pi_{\mathfrak{g}}|_{\Omega}) \in \mathfrak{g}\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$ . Hence,  $\mathfrak{T}_{\mathfrak{g},\Sigma}$  must be  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}\text{-Q.E.D.}$

Pathwise  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness is also preserved under a  $\mathfrak{g}\text{-}(\mathfrak{T}_{\mathfrak{g},\Omega}, \mathfrak{T}_{\mathfrak{g},\Sigma})$ -continuous map, as proved below.

**THEOREM 3.52.** *Let  $\pi_g : \mathfrak{T}_{g,\Omega} \rightarrow \mathfrak{T}_{g,\Sigma}$  be a  $\mathfrak{g}$ - $(\mathfrak{T}_{g,\Omega}, \mathfrak{T}_{g,\Sigma})$ -continuous map and let  $\mathcal{S}_{g,\omega} \subset \mathfrak{T}_{g,\Omega}$  be a  $\mathfrak{T}_g$ -set. If  $\text{dom}(\pi_g|_{\mathcal{S}_{g,\omega}})$  is pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected in  $\mathfrak{T}_{g,\Omega}$ , then  $\text{im}(\pi_g|_{\mathcal{S}_{g,\omega}})$  is also pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected in  $\mathfrak{T}_{g,\Sigma}$ .*

**PROOF.** Let  $\pi_g \in \mathfrak{g}\text{-C}[\mathfrak{T}_{g,\Omega}; \mathfrak{T}_{g,\Sigma}]$ ,  $\mathcal{S}_{g,\omega} = \text{dom}(\pi_g|_{\mathcal{S}_{g,\omega}}) \subset \mathfrak{T}_{g,\Omega}$  be pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected in  $\mathfrak{T}_{g,\Omega}$ , and suppose  $\xi_\sigma, \zeta_\sigma \in \text{im}(\pi_g|_{\mathcal{S}_{g,\omega}})$ . Then, there exist  $\xi_\omega, \zeta_\omega \in \mathfrak{T}_{g,\Omega}$  such that  $(\pi_g(\xi_\omega), \pi_g(\zeta_\omega)) = (\xi_\sigma, \zeta_\sigma)$ . But  $\text{dom}(\pi_g|_{\mathcal{S}_{g,\omega}})$  is pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected in  $\mathfrak{T}_{g,\Omega}$  and, therefore, there exists a path  $\varphi_{g,\zeta} : [0, 1] \rightarrow \mathfrak{T}_{g,\Omega}$  such that  $\varphi_{g,\zeta}(0) = \xi_\omega$ ,  $\varphi_{g,\zeta}(1) = \zeta_\omega$ , and  $\text{im}(\varphi_{g,\zeta}|_{[0,1]}) \subseteq \text{dom}(\pi_g|_{\mathcal{S}_{g,\omega}})$ . Since  $\pi_g \in \mathfrak{g}\text{-C}[\mathfrak{T}_{g,\Omega}; \mathfrak{T}_{g,\Sigma}]$  and  $\varphi_{g,\zeta} \in \text{C}[[0, 1]; \mathfrak{T}_{g,\Omega}]$ , it follows that  $\pi_g \circ \varphi_{g,\zeta} \in \mathfrak{g}\text{-C}[[0, 1]; \mathfrak{T}_{g,\Sigma}]$ . Moreover,  $\pi_g \circ \varphi_{g,\zeta}(0) = \pi_g(\xi_\omega) = \xi_\sigma$ ,  $\pi_g \circ \varphi_{g,\zeta}(1) = \pi_g(\zeta_\omega) = \zeta_\sigma$ , and  $\text{im}(\pi_g \circ \varphi_{g,\zeta}) \subseteq \text{im}(\pi_g|_{\mathcal{S}_{g,\omega}})$ . Hence,  $\text{im}(\pi_g|_{\mathcal{S}_{g,\omega}})$  is pathwise  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connected in  $\mathfrak{T}_{g,\Sigma}$ . Q.E.D.

In the discussion section, categorical classifications of the concepts of  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connectedness and  $\mathfrak{g}$ - $\mathfrak{T}_g$ -disconnectedness are presented. Thereafter, a nice application is given and, finally, the work is terminated with a concluding remarks section.

#### 4. DISCUSSION

**4.1. CATEGORICAL CLASSIFICATIONS.** Having adopted a categorical approach in the classifications of the  $\mathcal{T}_g$ -properties called  $\mathfrak{g}$ - $\mathfrak{T}_g$ -connectedness, and also  $\mathfrak{g}$ - $\mathfrak{T}_g$ -disconnectedness, in the  $\mathcal{T}_g$ -space  $\mathfrak{T}_g$ , the dual aims of the present section are, to establish the various relations amongst the elements of the sequence  $\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_g^{(C)} \rangle = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_g^{(C)})_{\nu \in I_3^0}$  of  $\mathfrak{g}\text{-}\mathcal{T}_g^{(C)}$ -spaces and the elements of the sequence  $\langle \mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_g^{(C)} \rangle = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}^{(C)})_{\nu \in I_3^0}$  of  $\mathfrak{g}\text{-}\mathcal{T}^{(C)}$ -spaces, and to illustrate them through diagrams.

If a  $\mathcal{T}_g$ -space  $\mathfrak{T}_g = (\Omega, \mathcal{T}_g)$  is  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_g$ -separated, then  $\mathfrak{T}_g$  has a nonempty proper  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_g$ -open-closed set  $\mathcal{S}_g \in \mathfrak{g}\text{-}\nu\text{-}\text{O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-}\nu\text{-}\text{K}[\mathfrak{T}_g]$ , where  $\nu \in I_3^0$ . But, for every  $\mathfrak{T}_g$ -set  $\mathcal{S}_g \subset \mathfrak{T}_g$ , the relation  $\text{int}_g(\mathcal{S}_g) \subseteq \text{cl}_g \circ \text{int}_g(\mathcal{S}_g) \subseteq \text{cl}_g \circ \text{int}_g \circ \text{cl}_g(\mathcal{S}_g) \supseteq \text{int}_g \circ \text{cl}_g(\mathcal{S}_g)$  holds; for every  $\mathfrak{T}_g$ -set  $\mathcal{S}_g \subset \mathfrak{T}_g$ , the relation given by  $\text{cl}_g(\mathcal{S}_g) \supseteq \text{int}_g \circ \text{cl}_g(\mathcal{S}_g) \supseteq \text{int}_g \circ \text{cl}_g \circ \text{int}_g(\mathcal{S}_g) \subseteq \text{cl}_g \circ \text{int}_g(\mathcal{S}_g)$  holds. Consequently,

$$\begin{aligned} \text{op}_{g,0}(\mathcal{S}_g) \subseteq \text{op}_{g,1}(\mathcal{S}_g) \subseteq \text{op}_{g,3}(\mathcal{S}_g) \supseteq \text{op}_{g,2}(\mathcal{S}_g) \quad \forall \mathcal{S}_g \subset \mathfrak{T}_g; \\ \neg \text{op}_{g,0}(\mathcal{S}_g) \supseteq \neg \text{op}_{g,1}(\mathcal{S}_g) \supseteq \neg \text{op}_{g,3}(\mathcal{S}_g) \subseteq \neg \text{op}_{g,2}(\mathcal{S}_g) \quad \forall \mathcal{S}_g \subset \mathfrak{T}_g. \end{aligned}$$

Therefore, if  $\mathcal{S}_g \subset \mathfrak{T}_g$  is a nonempty proper  $\mathfrak{g}$ - $\mathfrak{T}_g$ -open-closed set then,

$$\begin{aligned} \mathcal{S}_g \in \mathfrak{g}\text{-}0\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-}0\text{-K}[\mathfrak{T}_g] &\implies \mathcal{S}_g \in \mathfrak{g}\text{-}1\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-}1\text{-K}[\mathfrak{T}_g] \\ &\Downarrow \\ \mathcal{S}_g \in \mathfrak{g}\text{-}2\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-}2\text{-K}[\mathfrak{T}_g] &\implies \mathcal{S}_g \in \mathfrak{g}\text{-}3\text{-O}[\mathfrak{T}_g] \cap \mathfrak{g}\text{-}3\text{-K}[\mathfrak{T}_g]. \end{aligned}$$

In other words,  $\mathfrak{g}$ -3- $\mathfrak{T}_g$ -separation implies  $\mathfrak{g}$ -1- $\mathfrak{T}_g$ -separation and the latter in turn implies  $\mathfrak{g}$ -0- $\mathfrak{T}_g$ -separation. On the other hand,  $\mathfrak{g}$ -2- $\mathfrak{T}_g$ -separation is implied by  $\mathfrak{g}$ -3- $\mathfrak{T}_g$ -separation. Similar implications also hold for  $\mathfrak{g}$ - $\mathfrak{T}_g$ -separateness in a  $\mathcal{T}$ -space

$\mathfrak{T} = (\Omega, \mathcal{T})$ . For, if  $\mathcal{S} \subset \mathfrak{T}$  is a nonempty proper  $\mathfrak{g}$ - $\mathfrak{T}$ -open-closed set then,

$$\begin{aligned} \mathcal{S} \in \mathfrak{g}\text{-}0\text{-}O[\mathfrak{T}] \cap \mathfrak{g}\text{-}0\text{-}K[\mathfrak{T}] &\implies \mathcal{S} \in \mathfrak{g}\text{-}1\text{-}O[\mathfrak{T}] \cap \mathfrak{g}\text{-}1\text{-}K[\mathfrak{T}] \\ &\downarrow \\ \mathcal{S} \in \mathfrak{g}\text{-}2\text{-}O[\mathfrak{T}] \cap \mathfrak{g}\text{-}2\text{-}K[\mathfrak{T}] &\implies \mathcal{S} \in \mathfrak{g}\text{-}3\text{-}O[\mathfrak{T}] \cap \mathfrak{g}\text{-}3\text{-}K[\mathfrak{T}]. \end{aligned}$$

As above,  $\mathfrak{g}\text{-}3\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separation implies  $\mathfrak{g}\text{-}1\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separation and the latter in turn implies  $\mathfrak{g}\text{-}0\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separation. On the other hand,  $\mathfrak{g}\text{-}2\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separation is implied by  $\mathfrak{g}\text{-}3\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separation. For visualization, a so-called *categorical connectedness diagram*, ex-

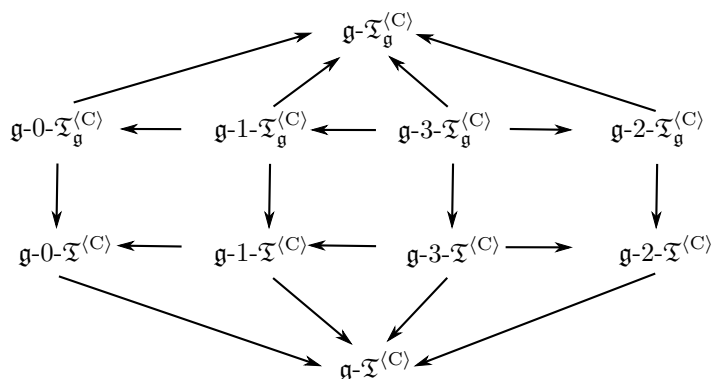


FIGURE 1. Relationships:  $\mathfrak{g}$ - $\mathfrak{T}$ -connected and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected spaces.

pressing the various relations amongst  $\mathfrak{g}$ - $\mathfrak{T}$ -connected and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected spaces, is presented in FIG. 1 and that, expressing the various relationships amongst  $\mathfrak{g}$ - $\mathfrak{T}$ -connected and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected spaces, so-called *categorical disconnectedness diagram*, is presented in FIG. 2. The categorical classifications of  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(LC)} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(LC)})$ ,  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(PC)} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(PC)})$ ,  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(LPC)} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(LPC)})$ , and  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(SC)} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(SC)})$  called, respectively, *locally*, *pathwise*, *locally pathwise*, and *simply*  $\mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces can be diagrammed in an analogous manner. The following implications concern the transformations of  $\mathfrak{g}$ - $\mathfrak{T}$ -connected sets under some types of  $\mathfrak{g}$ - $\mathfrak{T}$ -maps.

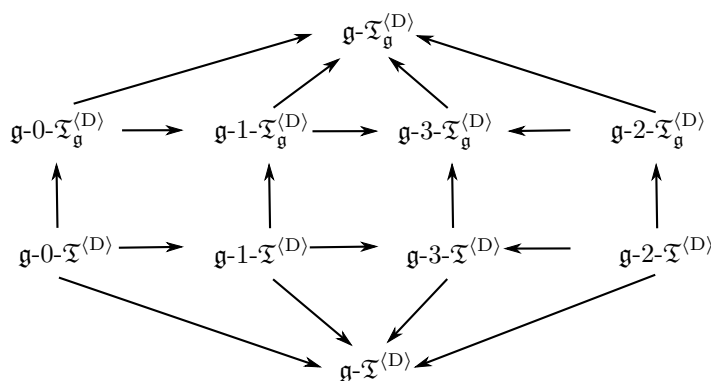


FIGURE 2. Relationships:  $\mathfrak{g}$ - $\mathfrak{T}$ -separated and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -separated spaces.

For every  $\nu \in I_3^0$ , if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  holds, then  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g}}}}) \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  implies  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g}}}}) \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , and hence the following implication:

$$(4.1) \quad \begin{aligned} & (\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g}}}}) \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]) \wedge (\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-I}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}] \cap \mathfrak{g}\text{-}\nu\text{-S}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]) \\ & \quad \Downarrow \\ & \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g}}}}) \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]. \end{aligned}$$

For every  $\nu \in I_3^0$ , if  $\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]$  holds, then  $\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g}}}}) \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]$  implies  $\text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g}}}}) \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]$ , and hence the following implication:

$$(4.2) \quad \begin{aligned} & (\text{dom}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g}}}}) \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Omega}]) \wedge (\pi_{\mathfrak{g}} \in \mathfrak{g}\text{-}\nu\text{-C}[\mathfrak{T}_{\mathfrak{g},\Omega}; \mathfrak{T}_{\mathfrak{g},\Sigma}]) \\ & \quad \Downarrow \\ & \text{im}(\pi_{\mathfrak{g}|_{\mathcal{S}_{\mathfrak{g}}}}) \in \mathfrak{g}\text{-}\nu\text{-Q}[\mathfrak{T}_{\mathfrak{g},\Sigma}]. \end{aligned}$$

In the following section a nice application comprising of some interesting cases is discussed.

**4.2. A NICE APPLICATION.** Focusing on basic concepts from the point of view of the theory of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness, we shall now present a nice application comprising of some interesting cases. Let  $\Omega_{\sigma} = \{\xi_{\nu} : \nu \in I_{\sigma}^*\}$  denote the underlying set, conditioned by the parameter  $\sigma \in I_{\infty}^*$ , and consider the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g},\sigma} = (\Omega_{\sigma}, \mathcal{T}_{\mathfrak{g},\sigma})$ , where  $\mathcal{T}_{\mathfrak{g},\sigma} : \mathcal{P}(\Omega_{\sigma}) \rightarrow \mathcal{P}(\Omega_{\sigma})$  will be defined in the following cases. Set  $\sigma = 1$ . Then,  $\Omega_1 = \{\xi_1\}$ ,  $\mathcal{T}_{\mathfrak{g},1} = \{\emptyset, \Omega_1\} = \{\mathcal{O}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}\}$ ,  $\neg\mathcal{T}_{\mathfrak{g},1} = \{\Omega_1, \emptyset\} = \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}\}$ , and, for every  $(\mu, \nu) \in I_2^* \times I_3^0$  it results that  $\text{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},\mu}), \neg\text{op}_{\mathfrak{g},\nu}(\mathcal{K}_{\mathfrak{g},\mu}) \in \{\mathcal{O}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}, \mathcal{K}_{\mathfrak{g},2}\} = \{\emptyset, \Omega_1\}$ . Therefore, for every  $\nu \in I_3^0$ ,  $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g},1}] = \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g},1}] = \{\emptyset, \Omega_1\}$ . Thus, for every  $\nu \in I_3^0$ , there exists neither a pair  $(\mathcal{U}_{\mathfrak{g},\xi}, \mathcal{U}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g},1}] \times \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g},1}]$  of nonempty  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open sets nor a pair  $(\mathcal{V}_{\mathfrak{g},\xi}, \mathcal{V}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g},1}] \times \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g},1}]$  of nonempty  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -closed sets such that:

$$\left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \Omega \right) \vee \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \Omega \right).$$

Evidently, the  $\mathfrak{T}_{\mathfrak{g}}$ -sets  $\emptyset, \Omega_1 \subseteq \mathcal{U}$  are the only  $\mathfrak{T}_{\mathfrak{g}}$ -open-closed sets, and  $\mathfrak{g}\text{-C}_{\Omega_1}[\xi_1] = \{\xi_1\} = \Omega_1$  is the unique  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -component in  $\mathfrak{T}_{\mathfrak{g},1}$ . Thus, the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g},1} = (\Omega_1, \mathcal{T}_{\mathfrak{g},1})$  is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},1}^{(C)} = (\Omega_1, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},1}^{(C)})$ , and the latter in turn implies that it is also a  $\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space  $\mathfrak{T}_{\mathfrak{g},1}^{(C)} = (\Omega_1, \mathcal{T}_{\mathfrak{g},1}^{(C)})$ . Hence, *every indiscrete  $\mathcal{T}_{\mathfrak{g}}$ -space which is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected is also  $\mathfrak{T}_{\mathfrak{g}}$ -connected*. Furthermore, the underlying set  $\Omega_1 = \{\xi_1\}$  being a 1-point set, it also follows that, *every discrete  $\mathcal{T}_{\mathfrak{g}}$ -space that has at most one point is both  $\mathfrak{T}_{\mathfrak{g}}$ -connected and  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connected*. Set  $\sigma = 2$ . Then,  $\Omega_2 = \{\xi_1, \xi_2\}$ . Choose  $\mathcal{T}_{\mathfrak{g},2} = \{\emptyset, \Omega_2\} = \{\mathcal{O}_{\mathfrak{g},1}, \mathcal{O}_{\mathfrak{g},2}\}$  so that,  $\neg\mathcal{T}_{\mathfrak{g},2} = \{\Omega_2, \emptyset\} = \{\mathcal{K}_{\mathfrak{g},1}, \mathcal{K}_{\mathfrak{g},2}\}$ . Then, the collection of  $\mathfrak{T}_{\mathfrak{g}}$ -open sets is  $\text{O}[\mathfrak{T}_{\mathfrak{g},2}] = \{\emptyset, \Omega_2\}$ , and  $\text{K}[\mathfrak{T}_{\mathfrak{g},2}] = \text{O}[\mathfrak{T}_{\mathfrak{g},2}]$  stands for the collection of  $\mathfrak{T}_{\mathfrak{g}}$ -closed sets. On the other hand, for every  $\nu \in I_3^0$ ,  $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g},2}] = \text{O}[\mathfrak{T}_{\mathfrak{g},2}] \cup \{\{\xi_1\}, \{\xi_2\}\} = \text{K}[\mathfrak{T}_{\mathfrak{g},2}] \cup \{\{\xi_1\}, \{\xi_2\}\} = \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g},2}]$ . Clearly, there exists a pair  $(\mathcal{U}_{\mathfrak{g},\xi}, \mathcal{U}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g},2}] \times \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g},2}]$  of nonempty  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -open sets or a pair  $(\mathcal{V}_{\mathfrak{g},\xi}, \mathcal{V}_{\mathfrak{g},\zeta}) \in \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g},2}] \times \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g},2}]$  of

nonempty  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -closed sets such that:

$$\left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{U}_{\mathfrak{g},\lambda} = \Omega \right) \vee \left( \bigsqcup_{\lambda=\xi,\zeta} \mathcal{V}_{\mathfrak{g},\lambda} = \Omega \right).$$

This is realised when either  $(\mathcal{U}_{\mathfrak{g},\xi}, \mathcal{U}_{\mathfrak{g},\zeta}) = (\{\xi_1\}, \{\xi_2\})$  or  $(\mathcal{V}_{\mathfrak{g},\xi}, \mathcal{V}_{\mathfrak{g},\zeta}) = (\{\xi_2\}, \{\xi_1\})$ . On the other hand, there exists neither a pair  $(\mathcal{U}_{\mathfrak{g},\xi}, \mathcal{U}_{\mathfrak{g},\zeta}) \in \mathcal{O}[\mathfrak{T}_{\mathfrak{g},2}] \times \mathcal{O}[\mathfrak{T}_{\mathfrak{g},2}]$  of nonempty  $\mathfrak{T}_{\mathfrak{g}}$ -open sets nor a pair  $(\mathcal{V}_{\mathfrak{g},\xi}, \mathcal{V}_{\mathfrak{g},\zeta}) \in \mathcal{K}[\mathfrak{T}_{\mathfrak{g},2}] \times \mathcal{K}[\mathfrak{T}_{\mathfrak{g},2}]$  of nonempty  $\mathfrak{T}_{\mathfrak{g}}$ -closed sets such that the above statement holds. Thus, the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g},2} = (\Omega_2, \mathcal{T}_{\mathfrak{g},2})$  is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},2}^{(D)} = (\Omega_2, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},2}^{(D)})$  but not a  $\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space  $\mathfrak{T}_{\mathfrak{g},2}^{(D)} = (\Omega_2, \mathcal{T}_{\mathfrak{g},2}^{(D)})$ . Alternatively said, *every  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space is a  $\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space but the converse need not be true in general.* Moreover, the underlying set  $\Omega_2 = \{\xi_1, \xi_2\}$  being a 2-point set, it follows that *every discrete  $\mathcal{T}_{\mathfrak{g}}$ -space that has at least two points is  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated.* It is plain that every  $(\mathcal{R}_{\mathfrak{g}}, \mathcal{S}_{\mathfrak{g}}) \in \{(\{\xi_1\}, \{\xi_1, \xi_2\}), (\{\xi_2\}, \{\xi_1, \xi_2\})\}$  is a pair of nonempty  $\mathfrak{T}_{\mathfrak{g}}$ -sets which are not  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -separated, for,  $\{\xi_1\} = \{\xi_1\} \cap \{\xi_1, \xi_2\} = \mathcal{C}(\{\xi_2\}) = \mathcal{C}(\{\xi_2\} \cap \{\xi_1, \xi_2\}) \neq \emptyset$ , and  $\mathcal{S}_{\mathfrak{g}} = \{\xi_1, \xi_2\}$  is the only  $\mathfrak{T}_{\mathfrak{g}}$ -set satisfying  $\mathcal{S}_{\mathfrak{g}} = \{\xi_1\} \sqcup \{\xi_2\}$ . Accordingly,  $\mathfrak{g}\text{-}\mathcal{Q}[\mathfrak{T}_{\mathfrak{g},2}] = \{\{\xi_1\}, \{\xi_2\}\}$  and  $\mathfrak{g}\text{-}\mathcal{D}[\mathfrak{T}_{\mathfrak{g},2}] = \{\{\xi_1, \xi_2\}\}$ . Observe in passing that,  $\Omega_2 = \bigsqcup_{\zeta=\xi_1, \xi_2} \mathfrak{g}\text{-}\mathcal{C}_{\Omega_2}[\zeta]$ . Thus, *if a  $\mathcal{T}_{\mathfrak{g}}$ -space has more than one  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -component, then it is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(D)}$ -space.* Set  $\sigma > 2$ . Then,  $\Omega_{>2} = \{\xi_{\alpha} : \alpha \in I_{\sigma>2}^*\}$ . Let  $\mathcal{T}_{\mathfrak{g},>2} : \mathcal{P}(\Omega_{>2}) \rightarrow \mathcal{P}(\Omega_{>2})$  generate the elements of  $\mathcal{T}_{\mathfrak{g},>2} = \{\mathcal{O}_{\mathfrak{g},(\alpha,\beta)} : (\alpha, \beta) \in I_{\infty}^0 \times I_{\infty}^0\}$  and  $\neg\mathcal{T}_{\mathfrak{g},>2} = \{\mathcal{K}_{\mathfrak{g},(\alpha,\beta)} = \mathcal{C}(\mathcal{O}_{\mathfrak{g},(\alpha,\beta)}) : (\alpha, \beta) \in I_{\infty}^0 \times I_{\infty}^0\}$  as thus:

$$(4.3) \quad \mathcal{O}_{\mathfrak{g},(\alpha,\beta)} = \begin{cases} \emptyset & \forall (\alpha, \beta) \in \{0\} \times \{0\}; \\ \{\xi_{\alpha+\mu} : \mu \in I_{\beta}^0\} & \forall (\alpha, \beta) \in I_{\infty}^* \times I_{\infty}^0; \\ \Omega_{>2} & \forall (\alpha, \beta) \in \{1\} \times \{\infty\}. \end{cases}$$

Clearly,  $\Omega_{>2} \subseteq \mathfrak{U}$  is an  $\infty$ -point set. Furthermore, it is easily verified that,  $\mathcal{T}_{\mathfrak{g},>2}(\emptyset) = \emptyset$ ,  $\mathcal{T}_{\mathfrak{g},>2}(\mathcal{O}_{\mathfrak{g},(\alpha,\beta)}) \subseteq \mathcal{O}_{\mathfrak{g},(\alpha,\beta)}$  for every  $(\alpha, \beta) \in I_{\infty}^0 \times I_{\infty}^0$ , and, finally,  $\mathcal{T}_{\mathfrak{g},>2}(\bigcup_{(\alpha,\beta) \in I_{\infty}^0 \times I_{\infty}^0} \mathcal{O}_{\mathfrak{g},(\alpha,\beta)}) = \bigcup_{(\alpha,\beta) \in I_{\infty}^0 \times I_{\infty}^0} \mathcal{T}_{\mathfrak{g},>2}(\mathcal{O}_{\mathfrak{g},(\alpha,\beta)})$ . Hence, it follows that  $\mathcal{T}_{\mathfrak{g},>2} : \mathcal{P}(\Omega_{>2}) \rightarrow \mathcal{P}(\Omega_{>2})$  is a  $\mathfrak{g}$ -topology on the  $\infty$ -point set  $\Omega_{>2}$ . On the other hand, it can be shown that, for every  $(\alpha, \beta, \nu) \in I_{\infty}^0 \times I_{\infty}^0 \times I_{\infty}^0$ ,

$$\mathcal{O}_{\mathfrak{g},(\alpha,0)} \cap \mathcal{O}_{\mathfrak{g},(\alpha,\beta)} \subseteq \text{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},(\alpha,0)}) \cap \text{op}_{\mathfrak{g},\nu}(\mathcal{O}_{\mathfrak{g},(\alpha,\beta)}) = \{\xi_{\alpha}\} \in \mathfrak{g}\text{-}\mathcal{Q}[\mathfrak{T}_{\mathfrak{g},>2}].$$

This implies that the  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g},>2} = (\Omega_{>2}, \mathcal{T}_{\mathfrak{g},>2})$  is a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(LC)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},>2}^{(LC)} = (\Omega_{>2}, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},>2}^{(LC)})$ , and hence, it is also a  $\mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -space  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g},>2}^{(C)} = (\Omega_{>2}, \mathfrak{g}\text{-}\mathcal{T}_{\mathfrak{g},>2}^{(C)})$ .

Moreover,  $\mathcal{T}_{\mathfrak{g}}$ -properties relative to such  $\mathcal{T}_{\mathfrak{g}}$ -spaces  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(LC)} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(LC)})$ ,  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(PC)} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(PC)})$ ,  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(LPC)} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(LPC)})$ , and, also,  $\mathfrak{g}\text{-}\nu\text{-}\mathfrak{T}_{\mathfrak{g}}^{(SC)} = (\Omega, \mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(SC)})$  called, respectively, *locally*, *pathwise*, *locally pathwise*, and *simply*  $\mathfrak{g}\text{-}\nu\text{-}\mathcal{T}_{\mathfrak{g}}^{(C)}$ -spaces can be discussed in an analogous manner by slight modifications of some  $\mathcal{T}_{\mathfrak{g}}$ -properties found in those cases. The next section provides concluding remarks and future directions of the theory of  $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$ -connectedness discussed in the preceding sections.



4.3. CONCLUDING REMARKS. In this chapter, a new theory called *Theory of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -Connectedness* has been developed, the foundation of which was based on the theories of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -sets and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -maps. A careful perusal of the Mathematical developments of the earlier sections will show that the proposed theory has, in its own rights, several advantages. The very first advantage is that the theory holds equally well when  $(\Lambda, \mathcal{T}_{\mathfrak{g},\Lambda}) = (\Lambda, \mathcal{T}_{\Lambda})$ , where  $\Lambda \in \{\Omega, \Sigma\}$ , and other characteristics adapted on this basis, in which case it might be called *Theory of  $\mathfrak{g}$ - $\mathfrak{T}$ -Connectedness*.

Hence, in a  $\mathcal{T}_{\mathfrak{g}}$ -space the theoretical framework categorises such pairs of concepts as  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected open and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected closed,  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected semi-open and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected semi-closed,  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected preopen and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected preclosed, and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected semi-preopen and  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected semi-preclosed as  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connected of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way; in a  $\mathcal{T}$ -space it categorises such pairs of concepts as  $\mathfrak{g}$ - $\mathfrak{T}$ -connected open and  $\mathfrak{g}$ - $\mathfrak{T}$ -connected closed,  $\mathfrak{g}$ - $\mathfrak{T}$ -connected semi-open and  $\mathfrak{g}$ - $\mathfrak{T}$ -connected semi-closed,  $\mathfrak{g}$ - $\mathfrak{T}$ -connected preopen and  $\mathfrak{g}$ - $\mathfrak{T}$ -connected preclosed, and  $\mathfrak{g}$ - $\mathfrak{T}$ -connected semi-preopen and  $\mathfrak{g}$ - $\mathfrak{T}$ -connected semi-preclosed as  $\mathfrak{g}$ - $\mathfrak{T}$ -connected of categories 0, 1, 2, and 3, respectively, and theorises the concepts in a unified way.

It is an interesting topic for future research to develop the theory of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness of mixed categories. More precisely, for some pair  $(\nu, \mu) \in I_3^0 \times I_3^0$  such that  $\nu \neq \mu$ , to develop the theory of  $\mathfrak{g}$ - $\mathfrak{T}_{\mathfrak{g}}$ -connectedness with respect to the elements of the classes  $\{(\mathcal{U}_{\mathfrak{g},\nu}, \mathcal{U}_{\mathfrak{g},\mu}) : (\mathcal{U}_{\mathfrak{g},\nu}, \mathcal{U}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mu\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\}$  and  $\{(\mathcal{V}_{\mathfrak{g},\nu}, \mathcal{V}_{\mathfrak{g},\mu}) : (\mathcal{V}_{\mathfrak{g},\nu}, \mathcal{V}_{\mathfrak{g},\mu}) \in \mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-}\mu\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\}$  in a  $\mathcal{T}_{\mathfrak{g}}$ -space  $\mathfrak{T}_{\mathfrak{g}}$ , as [2] and [9] developed the theory of b-open and b-closed sets in a  $\mathcal{T}$ -space  $\mathfrak{T}$ . Such a theory is what we thought would certainly be worth considering, and the discussion of this chapter terminates here.

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