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Communication

UNIFICATION THEORIES. EXAMPLES AND APPLICATIONS

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Abstract: We consider several unification problems in mathematics. We refer to transcendental numbers. Also, we present some ways to unify the main non-associative algebras (Lie algebras and Jordan algebras) and associative algebras.

Keywords: transcendental numbers, Euler formula, Yang-Baxter equation, Jordan algebras, Lie algebras, associative algebras, coalgebras

Classification: MSC 17C05, 17C50, 16T15, 16T25, 17B01, 03B05, 51A05, 68-04

1. Introduction

The purpose of this paper is twofold: (i) we present new results on transcendantal numbers; (ii) we obtain new results in the theory of the unification of non-associative structures. Also, at the end of this paper one could ask what could be the definition(s) of unification(s) in mathematics.

In the next section we will give several examples of unification problems.This section is related to the paper [1] on transcendantal numbers. Some knowledge of Hopf algebra theory are needed in order to understant some results from this section.

The third section presents structures which unify (non-)associative structures. The main non-associative structures are Lie algebras and Jordan algebras. Arguable less studied, Jordan algebras have applications in physics, differential geometry, ring geometries, quantum groups, analysis, biology, etc (see [2]). There are several ways to unify Lie algebras, Jordan algebras and associative algebras. We

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will also refer to cases when the unification of (non-)associative structures could be realised just in the conclusions of teorems.

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All tensor products will be defined over the field k.

2. Examples of unification problems

In this section we will give several examples of unification problems related to transcental numbers.

The following two identities whith transcendental numbers

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \,, \tag{1}$$

$$\int_{-\infty}^{+\infty} e^{-ix^2} dx = \sqrt{\frac{\pi}{2}}(1-i) \,, \tag{2}$$

were unified and proved (by contour integration) in [9].

The next formulas (from [1]) can be also unified:

$$e^{\pi i} + 1 = 0 \,, \tag{3}$$

$$|e^i - \pi| > e \tag{4}$$

and

$$|\, e^{1-z} + e^{\bar{z}} \,| > \pi \qquad \forall z \in \mathbb{C} \,. \tag{5}$$

Remark 2.1 *Let us consider the two variable complex function* $f : \mathbb{C} \times \mathbb{C} \to \mathbb{R}, \quad f(z, w) = |e^z + e^w|$*, which gives the lenght of the sum of the vectors* e^z *and* e^w*. The formulas (3), (4) and (5) can be unified using the function* $f(z, w)$*:*

$f(\pi i, \ 0) = 0, \ \ f(1 - z, \bar{z}) > \pi \ \ \forall z \in \mathbb{C}, \quad f(i, \ \pi i + \ln \pi) > e.$

Remark 2.2 *The function* $f(z, w)$ *can be expressed in another form using the formula* $|e^{x+i\alpha} + e^{y+i\beta}| = \rho\sqrt{1 + \sin(2\theta)\cos(\alpha - \beta)}$*, where* $\rho = \sqrt{e^{2x} + e^{2y}}$ *and* $\theta = \cos^{-1}\left(\frac{e^x}{\rho}\right)$*. The relations from Remark 2.1 can be reinterpretated using this formula. For example, in the first formula:* $\rho = \sqrt{2}, \ \theta = \frac{\pi}{4}, \ \alpha = \pi, \ \beta = 0$*; so,* $f(\pi i, \ 0) = \sqrt{2}\sqrt{1 + \sin(\frac{\pi}{2})\cos(\pi)} = 0.$

Remark 2.3 *While properties about the image of the function* $f(z, w)$ *unify the formulas (3), (4) and (5), the formula* $e^{x+iy} = e^x(\cos y + i \sin y)$ *could be considered a common part (or an essential part) of all of them.*

This latest formula can be related to a certain subcoalgebra of the trigonometric coalgebra (see [11]). Indeed, the properties of the trigonometric function cos *and* sin *lead to the trigonometric coalgebra, given by the maps:* $\Delta(c) = c \otimes c - s \otimes s, \ \ \Delta(s) = s \otimes c + c \otimes s, \ \ \varepsilon(c) = 1, \ \ \varepsilon(s) = 0$. *The Euler's relation, leads to the subcoalgebra generated by* $c + is$.

The dual case states that $1 + ix$ *generates an ideal in the* $\mathbb{C}$ *algebra* $\frac{\mathbb{C}[X]}{X^2+1} = \mathbb{C}[x]$, *where* $x^2 = -1$. *In other words, the Euler's relation implies that* $\forall a, b \in \mathbb{C}$ *there exists* $c \in \mathbb{C}$ *such that* $(a + bx)(1 + ix) = c(1 + ix)$. *(This can be checked directly.)*

Acording to [11], the role of such objects in number theory is unexplored at the moment.

Remark 2.4 *The properties of the hyperbolic function* cosh *and* sinh *lead to the following coalgebra, given by the maps:* $\Delta(c) = c \otimes c + s \otimes s, \ \ \Delta(s) = s \otimes c + c \otimes s, \ \ \varepsilon(c) = 1, \ \ \varepsilon(s) = 0$. *There exists a subcoalgebra generated by* $c + s$, *which can be related to Theorem 1 of [1], leading to some kind of "Euler formula" for hyperbolic functions.*

Remark 2.5 *The coalgebras from 2.3 and 2.4 can be unified as follows. For* $a \in k$, *we consider the coalgebra generated by* c *and* s, $\Delta(c) = c \otimes c + a^2 s \otimes s, \ \ \Delta(s) = s \otimes c + c \otimes s, \ \ \varepsilon(c) = 1, \ \ \varepsilon(s) = 0$. *There exists a subcoalgebra generated by* $c + as$.

Remark 2.6 *The following inequalities hold (see also [1]):* $\pi > |e^i - \pi| > e$.

The last inequality was proved in [1]: $|e^i - \pi| > e \iff (\pi + e)(\pi - e) + 1 > 2\pi \cos 1$. *Now,* $\pi > 3.141, \ \ e < 2.719$; *so,* $(\pi + e)(\pi - e) + 1 > 3.47292$ *and* $\cos 1 < 1 - \frac{1}{2} + \frac{1}{4!} = \frac{13}{24}$, $2\pi \cos 1 < 3.142 \times \frac{13}{12} = 3.4038(3)$.

The inequality $\pi > |e^i - \pi|$ *is equivalent to* $2\pi \cos 1 > 1$, *which follows from* $\cos 1 < \cos 60^o = \frac{1}{2}$.

Remark 2.7 *The next formula generalises the Basel problem* $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6}$:

$$\sum_{1}^{n} \frac{1}{k^2} < \frac{2}{3} \left(\frac{n+1}{n}\right)^n \quad \forall n \in \mathbb{N}^*. \tag{6}$$

It could be an interesting problem to prove a similar formula for non-associative algebras. Likewise, one could try to generalize it for q-shifted factorials (see, for example, [10]).

3. The unification of non-associative structures

3.1. UJLA structures

The UJLA structures could be seen as structures which comprise the information encapsulated in associative algebras, Lie algebras and Jordan algebras.

Definition 3.1 *For a* k*-space* V*, let* $\eta : V \otimes V \to V, \ \ a \otimes b \mapsto ab$, *be a linear map such that:*

$$(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab), \tag{7}$$

$$(a^2b)a = a^2(ba), \quad (ab)a^2 = a(ba^2), \quad (ba^2)a = (ba)a^2, \quad a^2(ab) = a(a^2b), \tag{8}$$

$\forall\, a, b, c \in V$. *Then,* (V, η) *is called a* **UJLA structure**.

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Remark 3.2 *The UJLA structures unify Jordan, Lie and (non-unital) associative algebras.*

Remark 3.3 *If* $(A,\ \theta)$*, where* $\theta : A \otimes A \longrightarrow A,\ \ \theta(a \otimes b) = ab$*, is a (non-unital) associative algebra, then we define a UJLA structure* $(A,\ \theta')$*, where* $\theta'(a \otimes b) = \alpha ab + \beta ba$*, for some* $\alpha,\ \beta \in k$*. For* $\alpha = \beta = \frac{1}{2}$*,* $(A,\ \theta')$ *is a Jordan algebra, and for* $\alpha = 1 = -\beta$*,* $(A,\ \theta')$ *is a Lie algebra.*

Theorem 3.4 **(Nichita [3])** *Let* (V, η) *be a UJLA structure. Then,* $(V, \eta'),\ \ \eta'(a \otimes b) = [a, b] = ab - ba$ *is a Lie algebra.*

Theorem 3.5 **(Nichita [3])** *Let* (V, η) *be a UJLA structure. Then,* $(V, \eta'),\ \ \eta'(a \otimes b) = a \circ b = \frac{1}{2}(ab + ba)$ *is a Jordan algebra.*

Remark 3.6 *The structures from the two above theorems are related by the relation:*
$[a, b \circ c] + [b, c \circ a] + [c, a \circ b] = 0.$

Remark 3.7 *The classification of UJLA structures is an open problem.*

Remark 3.8 *If the characteristic of* k *is 2, then a Lie algebra is also a Jordan algebra.*

Proof. Because the characteristic of k *is 2, the Lie algebra* L *is also commutative.*

It follows easily that $[[x, x], x] = 0\ \ \forall x \in L$*.*

Now, in the Jacobi identity we take $z = x^2$*:* $[[x, y], x^2] + [[y, x^2], x] + [[x^2, x], y] = 0.$

From the above observations it follows that $[[x, y], x^2] = [x, [y, x^2]]$*. Therefore* L *is also a Jordan algebra.*

3.2. Yang–Baxter equations

The authors of [4] argued that the Yang–Baxter equation leads to another unification of (non-)associative structures.

For V a k-space, we denote by $\tau : V \otimes V \longrightarrow V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$, and by $I : V \longrightarrow V$ the identity map of the space V; for $R : V \otimes V \longrightarrow V \otimes V$ a k-linear map, let $R^{12} = R \otimes I,\ R^{23} = I \otimes R,\ R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

Definition 3.9 *A Yang-Baxter operator is an invertible* k*-linear map,* $R : V \otimes V \longrightarrow V \otimes V$*, which satisfies the braid condition (sometimes called the* Yang-Baxter equation*):*

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \tag{9}$$

If R *satisfies (9) then both* $R \circ \tau$ *and* $\tau \circ R$ *satisfy the* quantum Yang-Baxter equation *(QYBE):*

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \tag{10}$$

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Therefore, the equations (9) and (10) are equivalent.

For A be a (unitary) associative k-algebra, and $\alpha, \beta, \gamma \in k$, the authors of [5] defined the k-linear map $R^A_{\alpha,\beta,\gamma} : A \otimes A \to A \otimes A$,

$$a \otimes b \mapsto \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b \tag{11}$$

which is a Yang-Baxter operator if and only if one of the following cases holds:
(i) $\alpha = \gamma \neq 0, \quad \beta \neq 0$; (ii) $\beta = \gamma \neq 0, \quad \alpha \neq 0$; (iii) $\alpha = \beta = 0, \quad \gamma \neq 0$.

An interesting property of (11), can be visualized in knot theory, where the link invariant associated to $R^A_{\alpha,\beta,\gamma}$ is the Alexander polynomial.

For $(L, [,])$ a Lie algebra over k, $z \in Z(L) = \{z \in L : [z, x] = 0 \ \forall\, x \in L\}$, and $\alpha \in k$, the authors of the papers [6] and [7] defined the following Yang-Baxter operator: $\phi^L_\alpha \ : \ L \otimes L \ \longrightarrow \ L \otimes L$,

$$x \otimes y \mapsto \alpha[x, y] \otimes z + y \otimes x \ . \tag{12}$$

Remark 3.10 *The formulas (11) and (12) lead to the unification of associative algebras and Lie algebras in the framework of Yang-Baxter structures. At this moment, we do not have a satisfactory answer to the question how Jordan algebras fit in this framework (several partial answers were given).*

3.3. Unification of the conclusions of theorems

Sometimes it is not easy to find structures which unify theorems for (non-)associative structures, but we could unify just the conclusions of teorems, as we will see in the next theorems.

Theorem 3.11 *If A is a Jordan algebra, a Lie algebra or an associative algebra, and if $a, b \in A$, then*

$$D : A \to A, \ \ D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b$$

is a derivation.

Proof. We consider three cases.

If A is a Jordan algebra, then $D(x) = a(bx)+b(ax)+(ax)b-a(xb)-(xb)a-(xa)b = a(bx)-(xa)b = a(bx) - b(ax)$. According to [8], D is a derivation.

If A is a Lie algebra, then $D(x) = a(bx)+b(ax)+(ax)b-a(xb)-(xb)a-(xa)b = a(bx)-b(ax) = a(bx) + b(xa) = (ab)x$. So, D is a derivation.

If A is an associative algebra, then $D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b = (ab + ba)x - x(ab + ba)$. So, D is a derivation.

Theorem 3.12 *If A is a Jordan algebra, a Lie algebra or an associative algebra, and if $a, b \in A$, then $D : A \to A, \ \ D(x) = a(bx) - (xa)b$ is a derivation.*

Proof. We consider three cases, and follow similar steps as in the previous proof.

Remark 3.13 *At the end of this paper one could ask what could be the definition(s) of unification(s) in mathematics. There exist unifications of structures (and categories) and unification of theorems. Could these unifications be formalised?*

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