

The quantum black hole as a gravitational hydrogen atom

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Abstract

In this paper only one basic assumption has been made: if we try to describe black holes, their behavior should be understood in the same language as the one we use for particles; black holes should be treated just like atoms. They must be quantum forms of matter, moving in accordance with Schrödinger equations just like everything else. In particular, Rosen's quantization approach to the gravitational collapse is applied in the simple case of a pressureless “*star of dust*” by finding the gravitational potential, the Schrödinger equation and the solution for the collapse's energy levels. By applying the constraints for a Schwarzschild black hole (BH) and by using the concept of *BH effective state*, previously introduced by one of the authors (CC), the BH quantum gravitational potential, Schrödinger equation and the BH energy spectrum are found. Remarkably, such an energy spectrum is in agreement (in its absolute value) with the one which was conjectured by Bekenstein in 1974 and consistent with other ones in the literature. This approach also permits to find an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale. Moreover, two fundamental issues about black hole quantum physics are addressed by this model: the area quantization and the singularity resolution. As regards the former, a result similar to the one obtained by Bekenstein, but with a different coefficient, has been found. About the latter, it is shown that the traditional classical singularity in the core of the Schwarzschild BH is replaced, in a full quantum treatment, by

a two-particle system where the two components strongly interact with each other via a quantum gravitational potential. The two-particle system seems to be non-singular from the quantum point of view and is analogous to the hydrogen atom because it consists of a “nucleus” and an “electron”.

1 Introduction

It is a general conviction that, in the search of a quantum gravity theory, a black hole should play a role similar to that of the hydrogen atom in quantum mechanics [9]. It should be a “theoretical laboratory” where one discusses and tries to understand conceptual problems and potential contradictions in the attempt to unify Einstein’s general theory of relativity with quantum mechanics. This analogy suggests that black holes should be regular quantum systems with a discrete mass spectrum [9]. In this paper, the authors attempt to contribute to the above by finding the Schrödinger equation and the wave function of the Schwarzschild BH. The knowledge of such quantities could, in principle, also play a role in the solution of the famous BH information paradox [10], because in the proposed approach black holes seem to be well defined quantum mechanical systems, having ordered and discrete quantum spectra. This issue appears consistent with the unitarity of the underlying quantum gravity theory and with the idea that information should come out in BH evaporation.

A quantization approach proposed 25 years ago by the historical collaborator of Einstein, Nathan Rosen [5], will be applied to the gravitational collapse in the simple case of a pressureless “*star of dust*”. Therefore, the gravitational potential, the Schrödinger equation and the solution for the collapse’s energy levels will be found. After that, the constraints for a BH will be applied and this will permit to find the BH quantum gravitational potential, Schrödinger equation and energy spectrum. Such an energy spectrum, in its absolute value, is in agreement with both the one conjectured by Bekenstein in 1974 [7] and that found by Maggiore’s description of BH in terms of quantum membranes [8]. This approach also permits to find an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale.

Furthermore, the authors try to clarify two important issues such as Bekenstein area law and singularity resolution. Regarding the former, a result similar to that obtained by Bekenstein, but with a different coefficient, has been found. About the latter, it is shown that the traditional classical singularity in the core of the Schwarzschild BH is replaced, in a full quantum treatment, by a two-particle system where the two components strongly interact with each other via a quantum gravitational potential. The two-particle system seems to be non-singular from the quantum point of view and is analogous to the hydrogen atom because it consists of a “core” and an “electron”.

Returning to the Rosen’s quantization approach, it has also been recently applied to a cosmological framework by one of the authors (FF) and collaborators in [11] and to the famous Hartle-Hawking initial state by both of the authors and an additional collaborator in [26].

For the sake of completeness, one stresses that this quantization approach is only applicable to homogeneous space-times where the Weyl tensor vanishes. As soon as one introduces inhomogeneities (for example in the Lemaître-Tolman-Bondi (LTB) models [23 - 25]), there exists open sets of initial data where the collapse ends to a BH absolutely similar to Oppenheimer-Snyder-Datt (OSD) collapse. Hence, the exterior space-time remains the same whereas the interior is absolutely different. A further generalization of the proposed approach, which could be the object of future works, should include also a cosmological term or other sources of dark energy.

2 Application of Rosen's quantization approach to the gravitational collapse

Classically, the gravitational collapse in the simple case of a pressureless “*star of dust*” with uniform density is well known [1]. Historically, it was originally analysed in the famous paper of Oppenheimer and Snyder [2]. Instead, a different approach has been developed by Beckerdoff and Misner [3]. Furthermore, a non-linear electrodynamics Lagrangian has been recently added in this collapse's framework by one of the authors (CC) and a collaborator in [4]. This different approach permits to find a way to remove the black hole singularity at the classical level [4]. The traditional, classical framework of this kind of gravitational collapse is well known [1 - 3]. In regard to the interior of the collapsing star, one indeed uses the well-known Friedmann-Lemaître-Robertson-Walker (FLRW) line-element with comoving hyper-spherical coordinates χ , θ , φ [1]. Therefore, one writes down [1] (hereafter Planck units will be used, i.e. $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$)

$$ds^2 = d\tau^2 + a(\tau)(-d\chi^2 - \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where the origin of coordinates is set at the centre of the star, and $a(\tau)$ is the scale factor given by the familiar cycloidal relation [1]

$$\begin{aligned} a &= \frac{1}{2}a_m(1 + \cos\eta), \\ \tau &= \frac{1}{2}a_m(\eta + \sin\eta), \end{aligned} \quad (2)$$

while the density is given by [1]

$$\rho = \left(\frac{3a_m}{8\pi}\right) a^{-3} = \left(\frac{3}{8\pi a_m^2}\right) \left[\frac{1}{2}(1 + \cos\eta)\right]^{-3}. \quad (3)$$

Setting $\sin^2\chi$ one chooses the case of positive curvature, which corresponds to a gas sphere whose dynamics begins at rest with a finite radius, and, in turn, it is the only one of interest [1]. Thus, the choice $k = 1$ is made for dynamical reasons (the initial rate of change of density is null, that means “momentum of maximum expansion” [1]), but the dynamics also depends on the field equations. As it has

been stressed in the paper's Introduction, for isotropic models, a cosmological term - or other sources of dark energy - can be in principle included in future works, in order to obtain a more realistic physical framework for the collapse.

In order to discuss the simplest model of a "star of dust", that is the case of zero pressure, one sets the stress-energy tensor as [1]

$$T = \rho u \otimes u, \quad (4)$$

where ρ is the density of the collapsing star and u the four-vector velocity of the matter. On the other hand, the external geometry is given by the Schwarzschild line-element [1]

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - r^2 (\sin^2 \theta d\varphi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{2M}{r}}, \quad (5)$$

where M is the total mass of the collapsing star. Since there are no pressure gradients, which can deflect the particles motion, the particles on the surface of any ball of dust move along radial geodesics in the exterior Schwarzschild space-time [1]. Considering a ball which begins at rest with finite radius (in terms of the Schwarzschild radial coordinate) $r = r_i$ at the (Schwarzschild) time $t = 0$, the geodesics motion of its surface is given by the following equations [1]:

$$r = \frac{1}{2} r_i (1 + \cos \eta), \quad (6)$$

$$t = 2M \ln \left[\frac{\sqrt{\frac{r_i}{2M} - 1} + \tan\left(\frac{\eta}{2}\right)}{\sqrt{\frac{r_i}{2M} - 1} - \tan\left(\frac{\eta}{2}\right)} \right] + 2M \sqrt{\frac{r_i}{2M} - 1} \left[\eta + \left(\frac{r_i}{4M}\right) (\eta + \sin \eta) \right]. \quad (7)$$

The proper time measured by a clock put on the surface of the collapsing star is [1]

$$\tau = \sqrt{\frac{r_i^3}{8M}} (\eta + \sin \eta). \quad (8)$$

The collapse begins at $r = r_i$, $\eta = \tau = t = 0$, and ends at the singularity $r = 0$, $\eta = \pi$ after a duration of proper time measured by the falling particles [1]

$$\Delta\tau = \pi \sqrt{\frac{r_i^3}{8M}}, \quad (9)$$

which coincidentally corresponds, as it is well known, to the interval of Newtonian time for free-fall collapse in Newtonian theory. Different from the cosmological case, where the solution is homogeneous and isotropic everywhere, here the internal homogeneity and isotropy of the FLRW line-element are broken at the star's surface, that is, at some radius $\chi = \chi_0$ [1]. At that surface, which is a 3-dimensional world tube enclosing the star's fluid [1], the interior FLRW geometry must smoothly match the exterior Schwarzschild geometry [1]. One

considers a range of χ given by $0 \leq \chi \leq \chi_0$, with $\chi_0 < \frac{\pi}{2}$ during the collapse [1]. For the pressureless case the match is possible [1]. The external Schwarzschild solution indeed predicts a cycloidal relation for the star's circumference [1]

$$C = 2\pi r = 2\pi \left[\frac{1}{2} r_i (1 + \cos \eta) \right], \quad (10)$$

$$\tau = \sqrt{\frac{r_i^3}{8M}} (\eta + \sin \eta).$$

The interior FLRW predicts a similar cycloidal relation [1]

$$C = 2\pi r = 2\pi a \sin \chi_0 = \pi \sin \chi_0 a_m (1 + \cos \eta), \quad (11)$$

$$\tau = \frac{1}{2} a_m (\eta + \sin \eta).$$

Therefore, the two predictions agree perfectly for all time if and only if [1]

$$r_i = a_0 \sin \chi_0, \quad (12)$$

$$M = \frac{1}{2} a_0 \sin^3 \chi_0,$$

where r_i and a_0 are the values of the Schwarzschild radial coordinate in Eq. (5) and of the scale factor in Eq. (1) at the beginning of the collapse, respectively. Thus, Eqs. (12) represent the requested match, while the Schwarzschild radial coordinate, in the case of the matching between the internal and external geometries, is [1]

$$r = a \sin \chi_0. \quad (13)$$

The attentive reader notes that the initial conditions on the matching are the simplest possible that could be relaxed, still having a continuous matching without extra surface terms. In fact, taking the interior solution to be homogeneous requires very fine tuned initial conditions for the collapse and the dynamics of the edge. So, on one hand, further analyses for a better characterization of the initial conditions on the matching between the internal and external geometries could be the object of future works. On the other hand, despite the analysis of this paper is not as general as possible, one stresses that the BH quantization is one of the most important problems of modern theoretical physics which has not yet been solved. Thus, in order to attempt to solve such a fundamental problem, one must start from the simplest case rather than from more complicated ones. This is in complete analogy with the history of general relativity. In fact, the first solution of Einstein field equations was the Schwarzschild solution, but it was not a general, rotating solution which included cosmological term or other sources of dark energy, as well the corresponding gravitational collapse developed by Oppenheimer and Snyder did not include a class of non-homogeneous models. Thus, as this is a new approach to the BH quantization, here one starts from the simplest conditions rather than from more complicated ones. So, the initial conditions on the matching that are applied here are exactly the ones proposed by Oppenheimer and Snyder in their historical paper on the gravitational collapse [2]. It is well known that the final result of the gravitational collapse studied by Oppenheimer and Snyder is the Schwarzschild BH [1].

Thus, let us see what happens when the star is completely collapsed, i.e. when the star is a BH. One sees that, inserting $r_i = 2M = r_g$, where r_g is the gravitational radius (the Schwarzschild radius), in Eqs. (12), one obtains $\sin^2 \chi_0 = 1$. Therefore, as the range $\chi > \frac{\pi}{2}$ must be discarded [1], one concludes that it is $\chi_0 = \frac{\pi}{2}$ for a BH.

In the following, the quantization approach derived by Rosen in [5] will be applied to the above case. Some differences will be found, because here one analyses the case of a collapsing star, while Rosen analysed a closed homogeneous and isotropic universe [5]. Let us start by rewriting the FLRW line-element (1) in spherical coordinates and comoving time as [1, 5]

$$ds^2 = d\tau^2 - a^2(\tau) \left(\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right). \quad (14)$$

The Einstein field equation [1, 5]

$$G_{\mu\nu} = -8\pi T_{\mu\nu} \quad (15)$$

gives the relations (we are assuming zero pressure) [5]

$$\begin{aligned} \dot{a}^2 &= \frac{8}{3}\pi a^2 \rho - 1, \\ \ddot{a} &= -\frac{4}{3}\pi a \rho, \end{aligned} \quad (16)$$

with $\dot{a} = \frac{da}{d\tau}$. For consistency, one gets [5]

$$\frac{d\rho}{da} = -\frac{3\rho}{a}, \quad (17)$$

which, when integrated gives [5]

$$\rho = \frac{C}{a^3}. \quad (18)$$

In the collapse case, C is determined by the initial conditions as [1]

$$C = \frac{3a_0}{8\pi}. \quad (19)$$

By analysing a closed homogeneous isotropic universe rather than a collapsing object, Rosen obtained a different value of C [5]. So, one rewrites Eq. (18) as

$$\rho = \frac{3a_0}{8\pi a^3}. \quad (20)$$

By multiplying the first of (16) for $M/2$ one gets [5]

$$\frac{1}{2}M\dot{a}^2 - \frac{4}{3}\pi M a^2 \rho = -\frac{M}{2}, \quad (21)$$

which seems like the energy equation for a particle in one-dimensional motion having coordinate a [5]

$$E = T + V, \quad (22)$$

where the kinetic energy is [5]

$$T = \frac{1}{2}M\dot{a}^2, \quad (23)$$

and the potential energy is [5]

$$V(a) = -\frac{4}{3}\pi Ma^2\rho. \quad (24)$$

Thus, the total energy is [5]

$$E = -\frac{M}{2}. \quad (25)$$

From the second of Eqs. (16), one gets the equation of motion of this particle:

$$M\ddot{a} = -\frac{4}{3}M\pi a\rho. \quad (26)$$

The momentum of the particle is [5]

$$P = M\dot{a}, \quad (27)$$

with an associated Hamiltonian [5]

$$\mathcal{H} = \frac{P^2}{2M} + V. \quad (28)$$

Till now, the problem has been discussed from the classical point of view. In order to discuss it from the quantum point of view, one needs to define a wavefunction as [5]

$$\Psi \equiv \Psi(a, \tau). \quad (29)$$

Thus, in correspondence of the classical equation (28), one gets the traditional Schrödinger equation [5]

$$i\frac{\partial\Psi}{\partial\tau} = -\frac{1}{2M}\frac{\partial^2\Psi}{\partial a^2} + V\Psi. \quad (30)$$

For a stationary state with energy E one obtains [5]

$$\Psi = \Psi(a) \exp(-iE\tau), \quad (31)$$

and Eq. (29) becomes [5]

$$-\frac{1}{2M}\frac{\partial^2\Psi}{\partial a^2} + V\Psi = E\Psi. \quad (32)$$

Inserting Eq. (20) into Eq. (24) one obtains

$$V(a) = -\frac{Ma_0}{2a}. \quad (33)$$

Based on the different value of the constant C , this equation is different from the one which was found by Rosen in [5]. Setting [5]

$$\Psi = aX, \quad (34)$$

Eq. (32) becomes [5]

$$-\frac{1}{2M} \left(\frac{\partial^2 X}{\partial a^2} + \frac{2}{a} \frac{\partial X}{\partial a} \right) + VX = EX. \quad (35)$$

With V given by Eq. (33), Eq. (35) is analogous to the Schrödinger equation in polar coordinates for the s states ($l = 0$) of a hydrogen-like atom [5, 6] in which the squared electron charge e^2 is replaced by $\frac{Ma_0}{2}$. Thus, for the bound states ($E < 0$) the energy spectrum is

$$E_n = -\frac{a_0^2 M^3}{8n^2}, \quad (36)$$

where n is the principal quantum number. Following [5], one inserts Eq. (25) into Eq. (36), obtaining the mass spectrum of the gravitational collapse as

$$M_n = \frac{a_0^2 M_n^3}{4n^2} \Rightarrow M_n = \frac{2n}{a_0}. \quad (37)$$

On the other hand, by using Eq. (25) one finds the energy levels of the collapsing star as

$$E_n = -\frac{n}{a_0}. \quad (38)$$

In fact, Eq. (37) represents the spectrum of the total mass of the collapsing star, while Eq. (38) represents the energy spectrum of the collapsing star where the gravitational energy, which is given by Eq. (33), is included. The total energy of a quantum system with bound states is indeed negative. It is also important to clarify the issue concerning the gravitational energy. It is well known that, in the framework of the general theory of relativity, the gravitational energy cannot be localized [1]. This is a consequence of Einstein's equivalence principle (EEP) [1], which implies that one can always find in any given locality a reference's frame (the local Lorentz reference's frame) in which all local gravitational fields are null. No local gravitational fields means no local gravitational energy-momentum and, in turn, no stress-energy tensor for the gravitational field [1]. In any case, this general situation admits an important exception [1], given by the case of a spherical star [1], which is exactly the case analysed in this paper. In fact, in this case the gravitational energy is localized not by mathematical conventions, but by the circumstance that transfer of energy is detectable by local measures, see Box 23.1 of [1] for details. Therefore, one can surely consider Eq. (33) as the gravitational potential energy of the collapsing star.

3 Black hole energy spectrum, ground state and singularity resolution

Now, let us consider the case of a completely collapsed star, i.e. a BH, which means $\chi_0 = \frac{\pi}{2}$, $r = a$ and $r_i = a_0 = 2M = r_g$ in Eqs. (12), see the discussion below Eq. (13). Then, Eqs. from (33) to (38) become

$$V(r) = -\frac{M^2}{r}, \quad (39)$$

$$\Psi = rX, \quad (40)$$

$$-\frac{1}{2M} \left(\frac{\partial^2 X}{\partial r^2} + \frac{2}{r} \frac{\partial X}{\partial r} \right) + VX = EX, \quad (41)$$

$$E_n = -\frac{r_g^2 M^3}{8n^2}, \quad (42)$$

$$M_n = \sqrt{n}, \quad (43)$$

$$E_n = -\sqrt{\frac{n}{4}}. \quad (44)$$

Equations (39), (41), (43) and (44) should be the exact gravitation potential energy, Schrödinger equation, mass spectrum and energy spectrum for the Schwarzschild BH interpreted as “gravitational hydrogen atom”, respectively. Actually, a further final correction is needed. To clarify this point, let us compare Eq. (39) with the analogous potential energy of an hydrogen atom, which is [6]

$$V(r) = -\frac{e^2}{r}. \quad (45)$$

Equations (39) and (45) are formally identical, but there is an important physical difference. In the case of Eq. (45) the electron’s charge is constant for all the energy levels of the hydrogen atom. Instead, in the case of Eq. (39), based on the emissions of Hawking quanta or on the absorptions of external particles, the BH mass changes during the jumps from one energy level to another. In fact, such a BH mass decreases for emissions and increases for absorptions. Therefore, one must also consider this dynamical behavior of quantum BHs. A way to take into account this dynamical behavior is by introducing the *BH effective state* [13, 14]. Let us start from the emissions of Hawking quanta. If one neglects the above mentioned BH dynamical behavior, the probability of emission of Hawking quanta is the one originally found by Hawking, which represents a strictly thermal spectrum [16]

$$\Gamma \sim \exp\left(-\frac{\omega}{T_H}\right), \quad (46)$$

where ω is the energy-frequency of the emitted particle and $T_H \equiv \frac{1}{8\pi M}$ is the Hawking temperature. Taking into account the BH dynamical behavior, i.e. the

BH contraction enabling a varying BH geometry, one gets the famous correction of Parikh and Wilczek [17]

$$\Gamma \sim \exp \left[-\frac{\omega}{T_H} \left(1 - \frac{\omega}{2M} \right) \right] \implies \Gamma = \alpha \exp \left[-\frac{\omega}{T_H} \left(1 - \frac{\omega}{2M} \right) \right], \quad (47)$$

where $\alpha \sim 1$ and the additional term $\frac{\omega}{2M}$ is present. By introducing the *effective temperature* [13, 14]

$$T_E(\omega) \equiv \frac{2M}{2M - \omega} T_H = \frac{1}{4\pi(2M - \omega)}, \quad (48)$$

Eq. (47) can be rewritten in a Boltzmann-like form similar to Eq. (46) [13, 14]

$$\Gamma = \alpha \exp[-\beta_E(\omega)\omega] = \alpha \exp \left(-\frac{\omega}{T_E(\omega)} \right), \quad (49)$$

where $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor*, with [13, 14]

$$\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}. \quad (50)$$

Therefore, the effective temperature replaces the Hawking temperature in the equation of the probability of emission as dynamical quantity. There are indeed various fields of science where one can take into account the deviation from the thermal spectrum of an emitting body by introducing an effective temperature which represents the temperature of a black body that would emit the same total amount of radiation [13, 14]. The effective temperature depends on the energy-frequency of the emitted radiation and the ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M - \omega}$ represents the deviation of the BH radiation spectrum from the strictly thermal feature due to the BH dynamical behavior [13, 14]. Besides, one can introduce other *effective quantities*. In particular, if M is the initial BH mass *before* the emission, and $M - \omega$ is the final BH mass *after* the emission, the *BH effective mass* and the *BH effective horizon* can be introduced as [13, 14]

$$M_E \equiv M - \frac{\omega}{2}, \quad r_E \equiv 2M_E. \quad (51)$$

They represent the BH mass and horizon *during* the BH contraction, i.e. *during* the emission of the particle [13, 14], respectively. These are average quantities [13, 14]. The variable r_E is indeed the average of the initial and final horizons while M_E is the average of the initial and final masses [13, 14]. In regard to the effective temperature, it is the inverse of the average value of the inverses of the initial and final Hawking temperatures; *before* the emission we have $T_H^i = \frac{1}{8\pi M}$, *after* the emission $T_H^f = \frac{1}{8\pi(M - \omega)}$ [13, 14]. To show that the effective mass is the correct quantity which characterizes the BH dynamical behavior, one can use Hawking's periodicity argument [18 - 20]. One rewrites Eq. (50) as [20]

$$\beta_E(\omega) \equiv \frac{1}{T_E(\omega)} = \beta_H \left(1 - \frac{\omega}{2M} \right), \quad (52)$$

where $\beta_H \equiv \frac{1}{T_H}$. Following Hawking's arguments [18 - 20], the Euclidean form of the metric is given by [20]

$$ds_E^2 = x^2 \left[\frac{d\tau}{4M \left(1 - \frac{\omega}{2M}\right)} \right]^2 + \left(\frac{r}{r_E} \right)^2 dx^2 + r^2 (\sin^2 \theta d\varphi^2 + d\theta^2). \quad (53)$$

This equation is regular at $x = 0$ and $r = r_E$. One also treats τ as an angular variable with period $\beta_E(\omega)$ [18 - 20]. Following [20], one replaces the quantity $\sum_i \beta_i \frac{\hbar^i}{M^{2i}}$ in [18] with the quantity $-\frac{\omega}{2M}$. Then, following step by step the detailed analysis in [18] one obtains [20]

$$ds_E^2 \equiv \left(1 - \frac{2M_E}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M_E}{r}} - r^2 (\sin^2 \theta d\varphi^2 + d\theta^2), \quad (54)$$

and one also easily shows that r_E in Eq. (53) is the same as in Eq. (51).

Despite the above analysis has been realized for emissions of particles, one immediately argues for symmetry that the same analysis works also in the case of absorptions of external particles, which can be considered as emissions having opposite sign. Thus, the effective quantities (51) become

$$M_E \equiv M + \frac{\omega}{2}, \quad r_E \equiv 2M_E. \quad (55)$$

Now they represents the BH mass and horizon *during* the BH expansion, i.e. *during* the absorption of the particle, respectively. Hence, Eq. (54) implies that, in order to take the BH dynamical behavior into due account, one must replace the BH mass M with the BH effective mass M_E in Eqs. (39), (41), (42) and (25), obtaining

$$V(r) = -\frac{M_E^2}{r}, \quad (56)$$

$$-\frac{1}{2M_E} \left(\frac{\partial^2 X}{\partial r^2} + \frac{2}{r} \frac{\partial X}{\partial r} \right) + VX = EX, \quad (57)$$

$$E_n = -\frac{r_E^2 M_E^3}{8n^2}, \quad (58)$$

$$E = -\frac{M_E}{2}. \quad (59)$$

Now, from the quantum point of view, we want to obtain the energy eigenvalues as being absorptions starting from the BH formation, that is from the BH having null mass. This implies that we must replace $M \rightarrow 0$ and $\omega \rightarrow M$ in Eq. (55). Thus, we obtain

$$M_E \equiv \frac{M}{2}, \quad r_E \equiv 2M_E = M. \quad (60)$$

Following again [5], one inserts Eqs. (59) and (60) into Eq. (58), obtaining the BH mass spectrum as

$$M_n = 2\sqrt{n}, \quad (61)$$

and by using again Eq. (25) one finds the BH energy levels as

$$E_n = -\sqrt{n}. \quad (62)$$

Remarkably, in its absolute value, this final result is consistent with the BH energy spectrum which was conjectured by Bekenstein in 1974 [7]. Bekenstein indeed obtained $E_n \sim \sqrt{n}$ by using the Bohr-Sommerfeld quantization condition because he argued that the Schwarzschild BH behaves as an adiabatic invariant. Besides, Maggiore [8] conjectured a quantum description of BH in terms of quantum membranes. He obtained the energy spectrum

$$E_n = \sqrt{\frac{A_0 n}{16\pi}}. \quad (63)$$

One sees that, in its absolute value, the result of Eq. (62) is consistent also with Maggiore's result. On the other hand, it should be noted that both Bekenstein and Maggiore used heuristic analyses, approximations and/or conjectures. Instead, Eq. (43) has been obtained through an exact quantization process. In addition, neither Bekenstein nor Maggiore realized that the BH energy spectrum must have negative eigenvalues because the "gravitational hydrogen atom" is a quantum system composed by bound states.

Let us again consider the analogy between the potential energy of a hydrogen atom, given by Eq. (45), and the effective potential energy of the "gravitational hydrogen atom" given by Eq. (56). Eq. (45) represents the interaction between the nucleus of the hydrogen atom, having a charge e and the electron, having a charge $-e$. Equation (56) represents the interaction between the nucleus of the "gravitational hydrogen atom", i.e. the BH, having an effective, dynamical mass M_E , and another, mysterious, particle, i.e. the "electron" of the "gravitational hydrogen atom" having again an effective, dynamical mass M_E . Therefore, let us ask: what is the "electron" of the BH? An intriguing answer to this question has been given by one of the authors (CC), who recently developed a semi-classical Bohr-like approach to BH quantum physics where, for large values of the principal quantum number n , the BH quasi-normal modes (QNMs), "triggered" by emissions (Hawking radiation) and absorption of external particles, represent the "electron" which jumps from a level to another one; the absolute values of the QNMs frequencies represent the energy "shells" of the "gravitational hydrogen atom". In this context, the QNM jumping from a level to another one has been indeed interpreted in terms of a particle quantized on a circle [13, 14], which is analogous to the electron travelling in circular orbits around the hydrogen nucleus, similar in structure to the solar system, of Bohr's semi-classical model of the hydrogen atom [21, 22]. Therefore, the results in the present paper seem consistent with the above mentioned works [13, 14].

For the BH ground state ($n = 1$), from Eq. (61) one gets the mass as

$$M_1 = 2, \quad (64)$$

in Planck units. Thus, in standard units one gets $M_1 = 2m_P$, where m_P is the Planck mass, $m_P = 2,17645 \times 10^{-8} \text{Kg}$. To this mass is associated a total

negative energy arising from Eq. (62), which is

$$E_1 = -1, \quad (65)$$

and a Schwarzschild radius

$$r_{g1} = 4. \quad (66)$$

Hence, this is the state having minimum mass and minimum energy (the energy of this state is minimum in absolute value; in its real value, being negative, it is maximum). In other words, this ground state represents the smallest possible BH. In the case of Bohr's semi-classical model of hydrogen atom, the Bohr radius, which represents the classical radius of the electron at the ground state, is [6]

$$\text{Bohr radius} = b_1 = \frac{1}{m_e e^2}, \quad (67)$$

where m_e is the electron mass. To obtain the correspondent "Bohr radius" for the "gravitational hydrogen atom", one needs to replace both the electron mass m_e and the charge e in Eq. (67) with the effective mass of the BH ground state, which is $\frac{M_1}{2} = 1$. Thus, now the "Bohr radius" becomes

$$b_1 = 1, \quad (68)$$

which in standard units reads $b_1 = l_P$, where $l_P = 1,61625 \times 10^{-35} m$ is the Planck length. Hence, we have found that the "Bohr radius" associated to the smallest possible black hole is equal to the Planck length. Now, following [5], the wave-function associated to the BH ground state is

$$\Psi_1 = 2b_1^{-\frac{3}{2}} r \exp\left(-\frac{r}{b_1}\right) = 2r \exp(-r), \quad (69)$$

where Ψ_1 is normalized as

$$\int_0^\infty \Psi_1^2 dr = 1. \quad (70)$$

The size of this BH is of the order of

$$\bar{r}_1 = \int_0^\infty \Psi_1^2 r dr = \frac{3}{2} b_1 = \frac{3}{2}. \quad (71)$$

The issue that the size of the BH ground state is, on average, shorter than the gravitational radius could appear surprising, but one recalls again that one interprets the "BH electron states" in terms of BH QNMs [13, 14]. Thus, the BH size which is, on average, shorter than the gravitational radius, seems consistent with the issue that the BH horizon oscillates with damped oscillations when the BH energy state jumps from a quantum level to another one through emissions of Hawking quanta and/or absorption of external particles.

This seems an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale. This Schwarzschild BH ground state represents the BH minimum energy level which is compatible with the generalized

uncertainty principle (GUP) [12]. The GUP indeed prevents a BH from its total evaporation by stopping Hawking's evaporation process in exactly the same way that the usual uncertainty principle prevents the hydrogen atom from total collapse [12].

Now, let us discuss a fundamental issue. Can one say that the quantum BH expressed by the system of equations from (56) to (59) is non-singular? It seems that the correct answer is yes. It is well known that, in the classical general relativistic framework, in the internal geometry all time-like radial geodesics of the collapsing star terminate after a lapse of finite proper time in the termination point $r = 0$ and it is impossible to extend the internal space-time manifold beyond that termination point [1]. Thus, the point $r = 0$ represents a singularity based on the rigorous definition by Schmidt [37]. But what happens in the quantum framework that has been analysed in this paper is completely different. By inserting the constraints for a Schwarzschild BH in Rosen's quantization process applied to the gravitational collapse, it has been shown that the completely collapsed object has been split in a two-particle system where the two components strongly interact with each other through a quantum gravitational interaction. In concrete terms, the system that has been analysed is indeed formally equal to the well known system of two quantum particles having finite distance with the mutual attraction of the form $1/r$ [6]. These two particles are the "nucleus" and the "electron" of the "gravitational hydrogen atom". Thus, the key point is the meaning of the word "particle" in a quantum framework. Quantum particles remain in an uncertain, non-deterministic, smeared, probabilistic wave-particle orbital state [6]. Then, they cannot be localized in a particular "termination point where all time-like radial geodesics terminate". As it is well known, such a localization is also in contrast with the Heisenberg uncertainty principle (HUP). The HUP says indeed that either the location or the momentum of a quantum particle such as the BH "electron" can be known as precisely as desired, but as one of these quantities is specified more precisely, the value of the other becomes increasingly indeterminate. This is not simply a matter of observational difficulty, but rather a fundamental property of nature. This means that, within the tiny confines of the "gravitational atom", the "electron" cannot really be regarded as a "point-like particle" having a definite energy and location. Thus, it is somewhat misleading to talk about the BH "electron" "falling into" the BH "nucleus". In other words, the Schwarzschild radial coordinate in the system of equations from (56) to (59) cannot become equal to zero. The GUP makes even stronger this last statement: as we can notice from its general expression [38]

$$\Delta x \Delta p \geq \frac{1}{2} \left[1 + \eta (\Delta p)^2 + \dots \right], \quad (72)$$

it implies a non-zero lower bound on the minimum value of the uncertainty on the particle's position (Δx) which is of order of the Planck length [38]. In other words, the GUP implies the existence of a minimal length in quantum gravity.

One notes also another important difference between the hydrogen atom of quantum mechanics [6] and the "gravitational hydrogen atom" discussed in this paper. In the standard hydrogen atom the nucleus and the electron are different

particles. In the quantum BH analysed here they are equal particles instead, as one easily checks in the system of equations from (56) to (59). Thus, the “nucleus” and the “electron” can be mutually exchanged without varying the physical properties of the system. Hence, the quantum state of the two particles seems even more uncertain, more non-deterministic, more smeared and more probabilistic than the corresponding quantum states of the particles of the hydrogen atom. These quantum argumentations seem to be strong argumentations in favour of the non-singular behavior of the Schwarzschild BH in a quantum framework. Notice that the results in this paper are also in agreement with the general conviction that quantum gravity effects become fundamental in the presence of strong gravitational fields. In a certain sense, the results in this paper permit to “see into” the Schwarzschild BH. The authors hope to further deepen these fundamental issues in future works.

4 Area quantization

Bekenstein proposed that the area of the BH horizon is quantized in units of the Planck length in quantum gravity (let us remember that the Planck length is equal to one in Planck units) [7]. His result was that the Schwarzschild BH area quantum is $\Delta A = 8\pi$ [7].

In the Schwarzschild BH the *horizon area* A is related to the mass by the relation $A = 16\pi M^2$. Thus, a variation ΔM of the mass implies a variation

$$\Delta A = 32\pi M \Delta M \quad (73)$$

of the area. Let us consider a BH which is excited at the level n . The corresponding BH mass is given by Eq. (61), that is

$$M_n = 2\sqrt{n}. \quad (74)$$

Now, let us assume that a neighboring particle is captured by the BH causing a transition from n to $n + 1$. Then, the variation of the BH mass is

$$M_{n+1} - M_n = \Delta M_{n \rightarrow n+1}, \quad (75)$$

where

$$M_{n+1} = 2\sqrt{n+1}. \quad (76)$$

Therefore, using Eqs. (73) and (75) one gets

$$\Delta A_n \equiv 32\pi M_n \Delta M_{n \rightarrow n+1}. \quad (77)$$

Equation (77) should give the area quantum of an excited BH when one considers an absorption from the level n to the level $n + 1$ in function of the principal quantum number n . But, let us consider the following problem. An emission from the level $n + 1$ to the level n is now possible due to the potential emission of a Hawking quantum. Then, the correspondent mass lost by the BH will be

$$M_{n+1} - M_n = -\Delta M_{n \rightarrow n+1} \equiv \Delta M_{n+1 \rightarrow n}. \quad (78)$$

Hence, the area quantum for the transition (78) should be

$$\Delta A_n \equiv 32\pi M_{n+1} \Delta M_{n+1 \rightarrow n}, \quad (79)$$

and one gets the strange result that the absolute value of the area quantum for an emission from the level $n+1$ to the level n is different from the absolute value of the area quantum for an absorption from the level n to the level $n+1$ because it is $M_{n+1} \neq M_n$. One expects the area spectrum to be the same for absorption and emission instead. In order to resolve this inconsistency, one considers the *effective mass*, which has been introduced in Section 3, corresponding to the transitions between the two levels n and $n+1$. In fact, the effective mass is the same for emission and absorption

$$M_{E(n, n+1)} \equiv \frac{1}{2} (M_n + M_{n+1}) = \sqrt{n} + \sqrt{n+1}. \quad (80)$$

By replacing M_{n+1} with $M_{E(n, n+1)}$ in equation (79) and M_n with $M_{E(n, n+1)}$ in equation (77) one obtains

$$\begin{aligned} \Delta A_{n+1} &\equiv 32\pi M_{E(n, n+1)} \Delta M_{n+1 \rightarrow n} && \text{emission} \\ \Delta A_n &\equiv 32\pi M_{E(n, n+1)} \Delta M_{n \rightarrow n+1} && \text{absorption} \end{aligned} \quad (81)$$

and now it is $|\Delta A_n| = |\Delta A_{n-1}|$. By using Eqs. (75) and (80) one finds

$$|\Delta A_n| = |\Delta A_{n+1}| = 64\pi, \quad (82)$$

which is similar to the original result found by Bekenstein in 1974 [7], but with a different coefficient. This is not surprising because there is no general consensus on the area quantum. In fact, in [27, 28] Hod had the intriguing idea to consider the black hole QNMs like quantum levels for absorption of particles, obtaining a different numerical coefficient. On the other hand, the Hod's rule

$$\Delta A = 4 \ln 3 \quad (83)$$

is actually a special case of the one suggested by Mukhanov in [29], who proposed

$$\Delta A = 4 \ln k, \quad k = 2, 3, \dots \quad (84)$$

This entered into the joint paper of Bekenstein and Mukhanov [30] and then into the review of Bekenstein [9].

Thus, the approach in this paper seems consistent with the Bekenstein area law.

5 Discussion and conclusion remarks

Rosen's quantization approach has been applied to the gravitational collapse in the simple case of a pressureless "*star of dust*". In this way, the gravitational

potential, the Schrödinger equation and the solution for the collapse's energy levels have been found. After that, by applying the constraints for a BH and by using the concept of BH effective state [13, 14], the analogous results and the energy spectrum have been found for the Schwarzschild BH. Remarkably, such an energy spectrum is consistent (in its absolute value) with both the one which was found by Bekenstein in 1974 [7] and that found by Maggiore [8]. The discussed approach also permitted to find an interesting quantum representation of the Schwarzschild BH ground state at the Planck scale; in other words, the smallest BH has been found, by also showing that it has a mass of two Planck masses and a "Bohr radius" equal to the Planck length. Furthermore, two fundamental issues such as Bekenstein area law and singularity resolution have been discussed. Thus, despite the gravitational collapse analysed in this paper is the simplest possible, the analysis that it has been performed permitted to obtain important results in BH quantum physics.

Finally, for the sake of completeness, it is necessary to discuss how the results found in this paper are related to those appearing in the literature. The results of Bekenstein and Maggiore concerning the BH energy spectrum have been previously cited. In general, such an energy spectrum has been discussed and derived in many different ways, see for example [14], [29 - 32]. In the so called reduced phase space quantisation method [31, 32] the BH energy spectrum gets augmented by an additional zero-point energy; this becomes important if one attempts to address the ultimate fate of BH evaporation, but, otherwise, it can safely be ignored for macroscopic BHs for which the principal quantum number n will be extremely large. It is important to stress that the discreteness of the energy spectrum needs a drastic departure from the thermal behavior of the Hawking radiation spectrum. A popular way to realize this is through the popular tunnelling framework arising from the paper of Parikh and Wilczek [17]. In that case, the energy conservation forces the BH to contract during the emission of the particle; the horizon recedes from its original radius and becomes smaller at the end of the emission process [17]. Therefore, BHs do not exactly emit like perfect black bodies [17], see also the discussion on this issue in Section 3. Moreover, Loop Quantum Gravity predicts a discrete energy spectrum which indicates a physical Planck scale cutoff of the Hawking temperature law [33]. In the framework of String-Theory one can identify microscopic BHs with long chains living on the worldvolume of two dual Euclidean brane pairs [34]. This leads to a discrete Bekenstein-like energy spectrum for the Schwarzschild BH [35]. The Bekenstein-like energy spectrum is present also in canonical quantization [36]. This approach yields a BH picture that is shown to be equivalent to a collection of oscillators whose density of levels is corresponding to that of the statistical bootstrap model [36].

Thus, it seems that the results of this paper are consistent with the previous literature on BH quantum physics.

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References

- [1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Feeman and Co., 1973).
- [2] J. R. Oppenheimer and H. Snyder, *Phys. Rev.* 56, 455 (1939).
- [3] D. L. Beckerdoff and C. W. Misner, D. L. Beckerdoff's A. B. Senior Thesis, Princeton Univeristy (1962).
- [4] C. Corda and H. J. Mosquera Cuesta, *Mod. Phys. Lett. A* 25, 2423 (2010).
- [5] N. Rosen, *Int. Journ. Theor. Phys.*, 32, 8, (1993).
- [6] A. Messiah, *Quantum Mechanics, Vol. 1*, North-Holland, Amsterdam (1961).
- [7] J. Bekenstein, *Lett. Nuovo Cimento*, 11, 467 (1974).
- [8] M. Maggiore, *Nucl. Phys. B* 429, 205 (1994).
- [9] J. D. Bekenstein, in *Prodeedings of the Eight Marcel Grossmann Meeting*, T. Piran and R. Ruffini, eds., pp. 92-111 (World Scientific Singapore 1999).
- [10] S. W. Hawking, *Phys. Rev. D* 14, 2460 (1976).
- [11] A. Feoli, E. Benedetto, M. Capriolo and F. Feleppa, *Eur. Phys. J. Plus* 132, 211 (2017).
- [12] R. J. Adler, P. Chen and D. I. Santiago, *Gen. Rel. Grav.* 33, 2101-2108 (2001, Third Award in the Gravity Research Foundation Essay Competition).
- [13] C. Corda, *Class. Quantum Grav.* 32, 195007 (2015).
- [14] C. Corda, *Adv. High En. Phys.* 2015, 867601 (2015).
- [16] S. W. Hawking, *Commun. Math. Phys.* 43, 199 (1975).
- [17] M. K. Parikh and F. Wilczek, *Phys. Rev. Lett.* 85, 5042 (2000).
- [18] R. Banerjee and B. R. Majhi, *Phys. Lett. B* 674, 218 (2009).
- [19] S. W. Hawking, "The Path Integral Approach to Quantum Gravity", in *General Relativity: An Einstein Centenary Survey*, eds. S. W. Hawking and W. Israel, (Cambridge University Press, 1979).

- [20] C. Corda, *Ann. Phys.* **337**, 49 (2013), final version corrected by typos in arXiv:1305.4529.
- [21] N. Bohr, *Phil. Magaz.* **6**, 26, 151, 1–25 (1913).
- [22] N. Bohr, *Phil. Mag.* **6**, 26, 153, 476– 502, (1913).
- [23] G. Lemaitre, *Ann. Soc. Sci. Bruxelles*, A53, 51 (1933).
- [24] R. C. Tolman, *Proc. Natl. Acad. Sci.* **20** (3), 169 (1934).
- [25] H. Bondi, *MNRAS* **107** (5), 410 (1947).
- [26] F. Feleppa, I. Licata, C. Corda, *Phys. Dark Un.* **26**, 100381 (2019).
- [27] S. Hod, *Phys. Rev. Lett.* **81**, 4293 (1998).
- [28] S. Hod, *Gen. Rel. Grav.* **31**, 1639 (1999, Fifth Award in the Gravity Research Foundation Essay Competition).
- [29] V. Mukhanov, *JETP Letters* **44**, 63 (1986).
- [30] J. D. Bekenstein and V. Mukhanov, *Phys. Lett. B* **360**, 7 (1995).
- [31] S. Das, P. Ramadevi and U.A. Yajnik, *Mod. Phys. Lett. A* **17**, 993 (2002).
- [32] A. Barvinski, S. Das, G. Kunstatter, *Phys.Lett. B* **517**, 415 (2001).
- [33] C. Röken, *Class. Quantum Grav.* **30**, 015005 (2013).
- [34] A. Krause, *Int. J. Mod. Phys. A* **20**, 4055 (2005).
- [35] A. Krause, *Int. J. Mod. Phys. A* **20**, 2813 (2005).
- [36] C. Vaz, *Phys. Rev. D* **61**, 064017 (2000).
- [37] B. G. Schmidt, *Gen. Rel. Grav.* **1**, 269 (1971).
- [38] A. Kempf, G. Mangano, R. B. Mann, *Phys. Rev. D* **52**, 1108 (1995).