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Black hole as gravitational hydrogen atom by Rosen's quantization approach

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Abstract

We apply Rosen's approach to the quantization of the gravitational collapse in the simple case of a pressureless "*star of dust*" and we find the gravitational potential, the Schrodinger equation and the solution for the collapse's energy levels without any approximation. By applying the constrains for a Schwarzschild black hole (BH), and by using the concept of *BH effective state*, previously introduced by one of us (CC) we found the analogous quantum quantities and the BH energy spectrum, again without any approximation. Remarkably, such a energy spectrum is equal (in absolute value) to the one which was found by Bekenstein in 1974 and consistent with the one found by Maggiore's description of BH in terms of quantum membranes. Finally, our approach permits to find the exact quantum representation of the Schwarzschild BH ground state at the Planck scale.

1 Introduction

It is a general conviction that, in the search of a quantum gravity theory, a BH should play a role similar to the hydrogen atom in quantum mechanics [9]. It should be a "theoretical laboratory" where one discusses and tries to understand conceptual problems and potential contradictions in the attempts to unify Einstein's general theory of relativity with quantum mechanics. This

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analogy suggested that BHs should be regular quantum systems with a discrete mass spectrum [9]. The biggest problems in the above picture are that, till now, in our knowledge, nobody has found the BH Schrodinger equation and nobody knows if BHs can be described by a wave function. The knowledge of such quantities could also play an important role in the solution of the famous BH information paradox [10]. In this work, a solution for both of these fundamental problems will be found for the Schwarzschild BH. A quantization approach proposed 25 years ago by the historical collaborator of Einstein, Nathan Rosen [5], to the quantization of the gravitational collapse in the simple case of a pressureless "*star of dust*" will be applied. Thus, the gravitational potential, the Schrodinger equation and the solution for the collapse's energy levels will be found without any approximation. After that, the constrains for a BH will be applied and this will permit to find the analogous quantum quantities and the BH mass spectrum, again without any approximation. It is quite intriguing that such a mass spectrum is similar to the mass spectrum which was found by Bekenstein in 1974 [7] and consistent with the mass spectrum found by Maggiore's description of BH in terms of quantum membranes. Finally, our approach permits to find the exact quantum representation of the Schwarzschild BH ground state at the Planck scale and the results presented in this paper seem consistent with a Bohr-like approach to BH quantum physics recently developed by one of us (CC) [13, 14]. For the sake of completeness, we remark that Rosen's quantization approach has been recently applied also to a cosmological framework by one of us (FF) and collaborators in [11].

2 Application of Rosen's quantization approach to the gravitational collapse

Classically, the gravitational collapse in the simple case of a pressureless "star of dust" is well known [1]. From the historical point of view, it was originally analysed in the famous paper of Oppenheimer and Snyder [2]. A different approach has been instead developed by Beckerdoff and Misner [3]. More recently, a non-linear electrodynamics Lagrangian has been added in this collapse's framework by one of us (CC) and a collaborator in [4]. This different approach permitted to obtain a way to remove the BH singularity at the classical level [4]. The traditional, classical framework of this kind of gravitational collapse is well known [1 - 3]. For the interior of the collapsing star, one indeed uses the well-known Friedmann-Lemaitre-Robertson-Walker (FLRW) line-element which represents comoving hyper-spherical coordinates for the interior of the star [1]. Thus, in terms of the conformal time η, one writes down [1] (hereafter we will use Planck units, i.e. $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$)

$$ds^2 = a(\eta)(-d\eta^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

where $a(\eta)$ is the scale factor of a conformal space-time. Setting $\sin^2\chi$ one chooses the case of positive curvature, which corresponds to a gas sphere whose dynamics begins at rest with a finite radius, and, in turn, it is the only one of interest [1]. In order to discuss the simplest model of a *"star of dust"* , that is, the case of zero pressure, one sets the stress-energy tensor as [1]

$$T = \rho u \otimes u, 2 \tag{2}$$

where ρ is the density of the collapsing star and u the four-vector velocity of the matter.

On the other hand, the external geometry is given by the Schwarzschild line-element [1]

$$ds^2 = (1 - \frac{2M}{r})dt^2 - r^2(\sin^2\theta d\varphi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{2M}{r}}, \tag{3}$$

where M is the total mass of the collapsing star. The internal homogeneity and isotropy of the FLRW line-element are broken at the star's surface, that is, a some radius $\chi = \chi_0$. Thus, one considers a range of χ given by $0 \leq \chi \leq \chi_0$, with $\chi_0 < \frac{\pi}{2}$ during the collapse [1]. Hence, the interior FLRW geometry must match the exterior Schwarzschild geometry. Such a matching is given by [1]

$$\begin{aligned} r_i &= a_0 \sin\chi_0 \\ M &= \tfrac{1}{2} a_0 \sin^3\chi_0, \end{aligned} \tag{4}$$

where r_i and a_0 are the values of the Schwarzschild radial coordinate in Eq. (3) and of the scale factor in Eq. (1) at the beginning of the collapse, respectively. Thus, the Schwarzschild radial coordinate, in the case of the matching between the internal and external geometries , is [1]

$$r = a \sin\chi_0. \tag{5}$$

Let us see what happens when the star is completely collapsed, i.e. when the star is a BH. On sees that, inserting $r_i = 2M = r_g$, where r_g is the gravitational radius (the Schwarzschild radius), in Eqs. (4), one gets $\sin^2\chi_0 = 1$. Thus, as the range $\chi > \frac{\pi}{2}$ must be discarded [1], one concludes that it is $\chi_0 = \frac{\pi}{2}$ for a BH.

In the following, we will apply the quantization approach derived by Rosen in [5] to the above case. We will find some thin difference, because we analyse the case of a collapsing star, while Rosen analysed a closed homogeneous and isotropic universe [5]. Let us start by rewriting the FLRW line-element (1) in spherical coordinates and comoving time as [1, 5]

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right). \tag{6}$$

The Einstein field equation [1, 5]

$$G_{\mu\nu} = -8\pi T_{\mu\nu} \tag{7}$$

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gives the relations (we are assuming zero pressure) [5]

$$\begin{aligned}\dot{a}^2 &= \tfrac{8}{3}\pi a^2\rho - 1 \\ \ddot{a} &= -\tfrac{4}{3}\pi a\rho\end{aligned} \tag{8}$$

with $\dot{a} = \frac{da}{dt}$. For consistency, one gets [5]

$$\frac{d\rho}{da} = -\frac{3\rho}{a}, \tag{9}$$

which, when integrated gives [5]

$$\rho = \frac{C}{a^3}. \tag{10}$$

In the collapse case, C is determined by the initial conditions as [1]

$$C = \frac{3a_0}{8\pi}. \tag{11}$$

By analysing a closed homogeneous isotropic universe rather than a collapsing object, in [5] Rosen obtained a different value of C. Thus, one rewrites Eq. (10) as

$$\rho = \frac{3a_0}{8\pi a^3}. \tag{12}$$

By multiplying the first of (8) for $M/2$ one gets [5]

$$\frac{M\dot{a}^2}{2} - \frac{4}{3}\pi M a^2\rho = \frac{M}{2}, \tag{13}$$

which can be interpreted as an energy equation for a particle in one-dimensional motion having coordinate a [5] as

$$E = T + V, \tag{14}$$

where the kinetic energy is [5]

$$T = \frac{M\dot{a}^2}{2} \tag{15}$$

and the potential energy is [5]

$$V(a) = -\frac{4}{3}\pi M a^2\rho. \tag{16}$$

Thus, the total energy is [5]

$$E = -\frac{M}{2}. \tag{17}$$

From the second of Eqs. (8), one gets the equation of motion of this particle as

$$M\ddot{a} = -\frac{4}{3}M\pi a\rho. \tag{18}$$

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The momentum of the particle is [5]

$$P = M\dot{a}, \tag{19}$$

with an associated Hamiltonian [5]

$$H = \frac{P^2}{2M} + V. \tag{20}$$

Till now, we discussed the problem from the classical point of view. In order to discuss it from the quantum point of view, we need to define a wave-function as [5]

$$\Psi \equiv \Psi\left(a, t\right). \tag{21}$$

Thus, in correspondence of the classical equation (20), one gets the traditional Schrodinger equation [5]

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2M}\frac{\partial^2 \Psi}{\partial a^2} + V\Psi. \tag{22}$$

For a stationary state with energy E one obtains [5]

$$\Psi = \Psi\left(a\right)\exp - \left(iEt\right) \tag{23}$$

and Eq. (21) becomes [5]

$$-\frac{1}{2M}\frac{\partial^2 \Psi}{\partial a^2} + V\Psi = E\Psi. \tag{24}$$

Inserting Eq. (12) into Eq. (16) one obtains

$$V(a) = -\frac{Ma_0}{2a}. \tag{25}$$

Based on the different value of the constant C, this equation is different from the one which was found by Rosen in [5]. Setting [5]

$$\Psi = aX, \tag{26}$$

Eq. (24) becomes [5]

$$-\frac{1}{2M}\left(\frac{\partial^2 X}{\partial a^2} + \frac{2}{a}\frac{\partial X}{\partial a}\right) + VX = EX. \tag{27}$$

With V given by Eq. (25), Eq. (27), is analogous to the Schrodinger equation in polar coordinates for the s states ($l = 0$) of a hydrogen-like atom [5, 6] in which the squared electron charge e^2 is replaced by $\frac{Ma_0}{2}$. Thus, for the bound states ($E < 0$) the energy spectrum is

$$E_n = -\frac{a_0^2 M^3}{8n^2}, \tag{28}$$

where n is the principal quantum number. Following [5], one inserts Eq. (17) into Eq. (28), obtaining the mass spectrum of the gravitational collapse as

$$M_n = \frac{a_0^2 M_n^3}{4n^2} \Rightarrow M_n = \frac{2n}{a_0}. \tag{29}$$

On the other hand, by using Eq. (17) one finds the energy levels of the collapsing star as

$$E_n = -\frac{n}{a_0}. \tag{30}$$

In fact, Eq. (29) represents the spectrum of the "inert" mass of the collapsing star, while Eq. (30) represents the spectrum of the "dynamical" energy of the collapsing star where the gravitational energy, which is given by Eq. (25) is included. We indeed recall that the total energy of a quantum system with bound states is negative. It is also important to clarify the issue of the gravitational energy. It is well known that, in the framework of general relativity, the gravitational energy cannot be localized [1]. This is a consequence of Einstein's equivalence principle (EEP) [1]. EEP implies indeed that one can always find in any given locality a reference's frame (the local Lorentz reference's frame) in which ALL local gravitational fields are null. No local gravitational fields means no local gravitational energy-momentum and, in turn, no stress-energy tensor for the gravitational field [1]. In any case, this general situation admits an important exception [1], given by the case of a spherical star [1], which is exactly the case that we analyze in this paper. In fact, in this case the gravitational energy is localized not by mathematical conventions, but by the circumstance that transfer of energy is detectable by local measures, see Box 23.1 of [1] for details. Thus, we can surely consider Eq. (25) as the gravitational potential energy of the collapsing star.

3 Black hole energy spectrum and ground state

Now, let us consider the case of a completely collapsed star, i.e. a BH, which means $\chi_0 = \frac{\pi}{2}$, $r = a$ and $r_i = a_0 = 2M = r_g$, in Eqs. (4), see the discussion below Eq. (5). Then, Eqs. from (25) to (30) become

$$V(r) = -\frac{M^2}{r}, \tag{31}$$

$$\Psi = rX, \tag{32}$$

$$-\frac{1}{2M}\left(\frac{\partial^2 X}{\partial r^2} + \frac{2}{r}\frac{\partial X}{\partial r}\right) + VX = EX, \tag{33}$$

$$E_n = -\frac{r_g^2 M^3}{8n^2}, \tag{34}$$

$$M_n = \sqrt{n}, \tag{35}$$

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$$E_n = -\sqrt{\frac{n}{4}}. \tag{36}$$

Eqs. (31), (33), (35) and (36) should be the exact (i.e. without any approximation) gravitation potential energy, Schrodinger equation, inert mass spectrum and energy spectrum for the Schwarzschild BH interpreted as "gravitational hydrogen atom", respectively.

Actually, a further final correction is needed. To clarify this point, let us compare our Eq. (31) with the analogous potential energy of an hydrogen atom which is [6]

$$V(r) = -\frac{e^2}{r}. \tag{37}$$

Eqs. (31) and (37) are formally identical, but there is an important difference. In the case of Eq. (37) the electron's charge is constant for all the energy levels of the hydrogen atom. Instead, in the case of Eq. (31), based on the emissions of Hawking quanta or on the absorptions of external particles, the BH mass changes from the jumps from an energy level to another. In fact, such a BH mass decreases for emissions and increases for absorptions. Thus, one must also consider this dynamical behavior of quantum BHs. A good way to take into account this dynamical behavior is by introducing the *BH effective state* [13, 14]. Let us start from the emissions of Hawking quanta. If one neglects the BH dynamical behavior, the probability of emission of Hawking quanta is the one originally found by Hawking, which represents a strictly thermal spectrum [16]

$$\Gamma \sim \exp(-\frac{\omega}{T_H}), \tag{38}$$

where ω is the energy-frequency of the emitted particle and $T_H \equiv \frac{1}{8\pi M}$ is the Hawking temperature. Taking into account the BH dynamical behavior i.e. the BH contraction enabling a varying BH geometry, one gets the famous correction of Parikh and Wilczek [17]

$$\Gamma \sim \exp[-\frac{\omega}{T_H}(1-\frac{\omega}{2M})] \quad \Longrightarrow \quad \Gamma = \alpha \exp[-\frac{\omega}{T_H}(1-\frac{\omega}{2M})], \tag{39}$$

where $\alpha \sim 1$ and the additional term $\frac{\omega}{2M}$ is present. By introducing the *effective temperature* [13, 14]

$$T_E(\omega) \equiv \frac{2M}{2M-\omega} T_H = \frac{1}{4\pi(2M-\omega)}, \tag{40}$$

Eq. (40) can be rewritten in a Boltzmann-like form similar to Eq. (38) [13, 14]

$$\Gamma = \alpha \exp[-\beta_E(\omega)\omega] = \alpha \exp(-\frac{\omega}{T_E(\omega)}), \tag{41}$$

where $\exp[-\beta_E(\omega)\omega]$ is the *effective Boltzmann factor,* with [13, 14]

$$\beta_E(\omega) \equiv \frac{1}{T_E(\omega)}. \tag{42}$$

Hence, the effective temperature replaces the Hawking temperature in the equation of the probability of emission as dynamical quantity. We recall that there are various fields of science where we can take into account the deviation from the thermal spectrum of an emitting body by introducing an effective temperature which represents the temperature of a black body that would emit the same total amount of radiation [13, 14]. The effective temperature depends on the energy-frequency of the emitted radiation and the ratio $\frac{T_E(\omega)}{T_H} = \frac{2M}{2M-\omega}$ represents the deviation of the BH radiation spectrum from the strictly thermal feature due to the BH dynamical behavior [13, 14]. By introducing the effective temperature, one can introduce other *effective quantities*. In particular, if M is the initial BH mass *before* the emission, and $M - \omega$ is the final BH mass *after* the emission, the *BH effective mass* and the *BH effective horizon* can be introduced as [13, 14]

$$M_E \equiv M - \frac{\omega}{2}, \ r_E \equiv 2M_E. \tag{43}$$

They represent the BH mass and the BH horizon *during* the BH contraction, i.e. *during* the emission of the particle [13, 14]. These are average quantities [13, 14]. r_E is indeed the average of the initial and final horizons while M_E is the average of the initial and final masses [13, 14]. Instead, the effective temperature is the inverse of the average value of the inverses of the initial and final Hawking temperatures (*before* the emission $T_{H \text{ initial}} = \frac{1}{8\pi M}$, *after* the emission $T_{H \text{ final}} = \frac{1}{8\pi(M-\omega)}$) [13, 14]. In order to rigorously show that the effective mass is the correct quantity which characterizes the BH dynamical behavior, one can use Hawking's periodicity argument [18 - 20]. One rewrites Eq. (42) as [20]

$$\beta_E(\omega) \equiv \frac{1}{T_E(\omega)} = \beta_H \left(1 - \frac{\omega}{2M}\right), \tag{44}$$

where $\beta_H \equiv \frac{1}{T_H}$. Following Hawking' s arguments [18 - 20], the Euclidean form of the metric is given by [20]

$$ds_E^2 = x^2 \left[\frac{d\tau}{4M\left(1 - \frac{\omega}{2M}\right)}\right]^2 + \left(\frac{r}{r_E}\right)^2 dx^2 + r^2(\sin^2\theta d\varphi^2 + d\theta^2). \tag{45}$$

This equation is regular at $x = 0$ and $r = r_E$. One also treats τ as an angular variable with period $\beta_E(\omega)$ [18 - 20]. Following [20], one replaces the quantity $\sum_i \beta_i \frac{\hbar^i}{M^{2i}}$ in [18] with the quantity $-\frac{\omega}{2M}$. Then, if one follows step by step the detailed analysis in [18] one obtains [20]

$$ds_E^2 \equiv (1 - \frac{2M_E}{r})dt^2 - \frac{dr^2}{1 - \frac{2M_E}{r}} - r^2(\sin^2\theta d\varphi^2 + d\theta^2), \tag{46}$$

and one also easily shows that r_E in Eq. (45) is the same as in Eq. (43).

Despite we realized the above analysis for emissions of particles, one immediately argues for symmetry that the same analysis works also in the case of absorptions of external particles, which can be considered as emissions having opposite sign. Thus, the effective quantities (43) become

$$M_E \equiv M + \frac{\omega}{2}, \; r_E \equiv 2M_E, \tag{47}$$

and now they represents the BH mass and the BH horizon *during* the BH expansion, i.e. *during* the absorption of the particle. Hence, Eq. (46) implies that, in order to take the BH dynamical behavior into due account, one must replace the BH mass M with the BH effective mass M_E in Eqs. (31), (33), (34), and (17) obtaining

$$V(r) = -\frac{M_E^2}{r}, \tag{48}$$

$$-\frac{1}{2M_E}\left(\frac{\partial^2 X}{\partial r^2} + \frac{2}{r}\frac{\partial X}{\partial r}\right) + VX = EX, \tag{49}$$

$$E_n = -\frac{r_E^2 M_E^3}{8n^2}, \tag{50}$$

$$E = -\frac{M_E}{2}. \tag{51}$$

Now, from the quantum point of view, we want to obtain the energy eigenvalues as being absorptions starting from the BH formation, that is from the BH having null mass. This implies that we must replace $M \to 0$ and $\omega \to M$ in Eq. (47). Thus, we obtain

$$M_E \equiv \frac{M}{2}, \; r_E \equiv 2M_E = M. \tag{52}$$

Following again [5], one inserts Eqs. (51) and (52) into Eq. (50), obtaining the BH inert mass spectrum as

$$M_n = \sqrt{2n}, \tag{53}$$

and by using again Eq. (17) one finds the BH energy levels as

$$E_n = -\sqrt{\frac{n}{2}}. \tag{54}$$

Remarkably, in its absolute value this final result is equal to the BH enery spectrum which was found by Bekenstein in 1974 [7]. Bekenstein indeed obtained $E_n = \sqrt{\frac{n}{2}}$ by using the Bohr-Sommerfeld quantization condition because he argued that the Schwarzschild BH behaves as an adiabatic invariant. Maggiore [8] conjectured a quantum description of BH in terms of quantum membranes. He obtained the energy spectrum

$$E_n = \sqrt{\frac{A_0 n}{16\pi}}. \tag{55}$$

Thus, he was forced to set $A_0 = 8\pi$ in order to find Bekenstein's result in [7]. We see that, in its absolute value, our result is consistent also with Maggiore's result. On the other hand, we stress that both Bekenstein and Maggiore used heuristic analyses, approximations and/or conjectures. Instead, we obtained Eq. (35) through an exact quantization process. In addition, neither Bekenstein nor Maggiore realized that the BH energy spectrum must have negative eigenvalues because the "gravitational hydrogen atom" is a quantum system composed by bound states.

Let us again consider the analogy between the potential energy of an hydrogen atom, given by Eq. (37) and the effective potential energy of our "gravitational hydrogen atom" given by Eq. (48). Eq. (37) represents the interaction between the nucleus of the hydrogen atom, having a charge e and the electron, having a charge $-e$. Eq. (48) represents the interaction between the nucleus of the "gravitational hydrogen atom", i.e. the BH, having an effective, dynamical mass M_E, and another, mysterious, particle, i.e. the "electron" of the "gravitational hydrogen atom" having again an effective, dynamical mass M_E. Thus, let us ask: what is the "electron" of the BH? An intriguing answer to this question has been given by one of us (CC), who recently developed a semi-classical Bohr-like approach to BH quantum physics where, for large values of the principal quantum number n, the BH quasi-normal modes (QNMs), "triggered" by emissions (Hawking radiation) and absorption of external particles, represent the "electron" which jumps from a level to another one, and the absolute values of the QNMs frequencies, represent the energy "shells" of the "gravitational hydrogen atom", see for example [13] and the complete review [14]. In that case, the QNM jumping from a level to another one has been indeed interpreted in terms of a particle quantized on a circle [13, 14] which is analogous to the electron travelling in circular orbits around the hydrogen nucleus, similar in structure to the solar system, of Bohr's semi-classical model of the hydrogen atom [21, 22]. Thus, the results in the present paper seem consistent with the works [13, 14].

For the BH ground state ($n = 1$), from Eq. (53) one gets the mass as

$$M_1 = \sqrt{2} \tag{56}$$

in Planck units. Thus, in standard units one gets $M_1 = \sqrt{2}m_P$, where m_P is the Planck mass, $m_P = 2,17645\times10^{-8}Kg$. To this mass is associated a total negative energy arising from Eq. (54) which is

$$E_1 = -\frac{\sqrt{2}}{2}. \tag{57}$$

Hence this is the state having minimum mass and minimum energy (the energy of this state is minimum in absolute value; in its real value, being negative, it is maximum). In other words, this ground state represents the smallest possible BH. We recall that, in the case of Bohr's semi-classical model of hydrogen atom, the Bohr radius, which represents the classical radius of the electron at the ground state, is [6]

$$Bohr\ radius = \frac{1}{m_e e^2}, \tag{58}$$

where m_e is the electron mass. In order to obtain the correspondent "Bohr radius" for the "gravitational hydrogen atom", one needs to replace both m_e

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and e in Eq. (58) with the effective mass of the BH ground state, which is $\frac{M_1}{2}=\frac{\sqrt{2}}{2}$. Thus, now the "Bohr radius" becomes

$$b_1 = 2\sqrt{2}, \tag{59}$$

which in standard units reads $b_1 = 2\sqrt{2}l_P$, where $l_P = 1,61625\times10^{-35}m$ is the Planck length. It also corresponds to the Schwarzschild radius associated to M_1 and to two times the effective, dynamical, Schwarzschild radius associated to the effective mass of the BH ground state $\frac{M_1}{2}$. Following [5], the wave-function associated to the BH ground state is

$$\Psi_1 = 2b_1^{-\frac{3}{2}} r \exp - \left(\frac{r}{b_1}\right), \tag{60}$$

where Ψ_1 is normalized as

$$\int_0^\infty \Psi_1^2 dr = 1. \tag{61}$$

The size of this BH is of the order of

$$\bar{r}_1 = \int_0^\infty \Psi_1^2 r dr = \frac{3}{2}b_1 = 3\sqrt{2}. \tag{62}$$

The issue that the size of the BH ground state is, on average, longer than the gravitational radius could appear surprising, but we recall again that one interprets the "BH electron states" in terms of BH QNMs [13, 14]. Thus, the BH size which is, on average, longer than the gravitational radius, seem consistent with the issue that the BH horizon oscillates with damped oscillations when the BH energy state jumps from a quantum level to another one through emissions of Hawking quanta and/or absorption of external particles.

Thus, we have remarkably found the exact quantum representation of the Schwarzschild BH ground state at the Planck scale. This Schwarzschild BH ground state represents the BH minimum energy level which is compatible with the generalized uncertainty principle (GUP) [12]. The GUP indeed prevents a BH from its total evaporation by stopping Hawking's evaporation process in exactly the same way that the usual uncertainty principle prevents the hydrogen atom from total collapse [12].

4 Conclusion remarks

Rosen's approach has been applied to the quantization of the gravitational collapse in the simple case of a pressureless "*star of dust*". In that way, the gravitational potential, the Schrodinger equation and the solution for the collapse's energy levels have been found without any approximation. After that, by applying the constrains for a BH and by using the concept of BH effective state [13, 14], it has been found the analogous quantum quantities and the BH energy spectrum, again without any approximation. Remarkably, such a energy spectrum is equal (in its absolute value) to the mass spectrum which was found by

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Bekenstein in 1974 [7] and consistent with the one found by Maggiore's description of BH in terms of quantum membranes [8]. Finally, the discussed approach permitted to find the exact quantum representation of the Schwarzschild BH ground state at the Planck scale. In other words, we have found the smallest BH and we have shown that it has a mass of order of the Planck mass, a "Bohr radius" of order of the Planck length and a gravitational radius equal to the "Bohr radius". Our results seem consistent with the recent Bohr-like approach to BH quantum physics developed by one of us (CC) [13, 14].

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