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Black hole as gravitational hydrogen atom by Rosen's quantization approach

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Abstract

We apply Rosen's approach to the quantization of the gravitational collapse in the simple case of a pressureless "*star of dust*" and we find the gravitational potential, the Schroedinger equation and the solution for the collapse's energy levels without any approximation. By applying the constrains for a black hole (BH), we found the analogous quantum quantities and the BH mass spectrum, again without any approximation. Remarkably, such a mass spectrum is the same which was found by Bekenstein in 1974. Finally, our approach permits to find the exact quantum representation of the Schwarzschild BH ground state at the Planck scale.

1 Introduction

It is a general conviction that, in the search of a quantum gravity theory, a BH should play a role similar to the hydrogen atom in quantum mechanics [9]. It should be a "theoretical laboratory" where one discusses and tries to understand conceptual problems and potential contradictions in the attempts to unify Einstein's general theory of relativity with quantum mechanics. This analogy suggested that BHs should be regular quantum systems with a discrete mass spectrum [9]. The biggest problems in the above picture are that, till now, in our knowledge, nobody has found the BH Schroedinger equation and nobody knows if BHs can be described by a wave function. The knowledge of such

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quantities could also play an important role in the solution of the famous BH information paradox [10]. In this work, a solution for both of these fundamental problems will be found for the Schwarzschild BH. A quantization approach proposed 25 years ago by the historical collaborator of Einstein, Nathan Rosen [5], to the quantization of the gravitational collapse in the simple case of a pressureless "*star of dust*" will be applied. Thus, the gravitational potential, the Schroedinger equation and the solution for the collapse's energy levels will be found without any approximation. After that, the constrains for a BH will be applied and this will permit to find the analogous quantum quantities and the BH mass spectrum, again without any approximation. It is quite intriguing that such a mass spectrum is the same which was found by Bekenstein in 1974 [7]. Finally, our approach permits to find the exact quantum representation of the Schwarzschild BH ground state at the Planck scale and the results presented in this paper seem consistent with a Bohr-like approach to BH quantum physics recently developed by one of us (CC) [13, 14]. For the sake of completeness, we remark that Rosen's quantization approach has been recently applied also to a cosmological framework by one of us (FF) and collaborators in [11].

2 Application of Rosen's quantization approach to the gravitational collapse

Classically, the gravitational collapse in the simple case of a pressureless "star of dust" is well known [1]. From the historical point of view, it was originally analysed in the famous paper of Oppenheimer and Snyder [2]. A different approach has been instead developed by Beckerdoff and Misner [3]. More recently, a non-linear electrodynamics Lagrangian has been added in this collapse's framework by one of us (CC) and a collaborator in [4]. This different approach permitted to obtain a way to remove the BH singularity at the classical level [4]. The traditional, classical framework of this kind of gravitational collapse is well known [1 - 3]. For the interior of the collapsing star, one indeed uses the well-known Friedmann-Lemaitre-Robertson-Walker (FLRW) line-element which represents comoving hyper-spherical coordinates for the interior of the star [1]. Thus, in terms of the conformal time η, one writes down [1] (hereafter we will use Planck units, i.e. $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$)

$$ds^2 = a(\eta)(-d\eta^2 + d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

where $a(\eta)$ is the scale factor of a conformal space-time. Setting $\sin^2\chi$ one chooses the case of positive curvature, which corresponds to a gas sphere whose dynamics begins at rest with a finite radius, and, in turn, it is the only one of interest [1]. In order to discuss the simplest model of a *"star of dust"*, that is, the case of zero pressure, one sets the stress-energy tensor as [1]

$$T = \rho u \otimes u, 2 \tag{2}$$

where ρ is the density of the collapsing star and u the four-vector velocity of the matter.

On the other hand, the external geometry is given by the Schwarzschild line-element [1]

$$ds^2 = (1 - \frac{2M}{r})dt^2 - r^2(\sin^2\theta d\varphi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{2M}{r}}, \tag{3}$$

where M is the total mass of the collapsing star. The internal homogeneity and isotropy of the FLRW line-element are broken at the star's surface, that is, a some radius $\chi = \chi_0$. Thus, one considers a range of χ given by $0 \leq \chi \leq \chi_0$, with $\chi_0 < \frac{\pi}{2}$ during the collapse [1]. Hence, the interior FLRW geometry must match the exterior Schwarzschild geometry. Such a matching is given by [1]

$$\begin{aligned} r_i &= a_0 \sin\chi_0 \\ M &= \tfrac{1}{2} a_0 \sin^3\chi_0, \end{aligned} \tag{4}$$

where r_i and a_0 are the values of the Schwarzschild radial coordinate in Eq. (3) and of the scale factor in Eq. (1) at the beginning of the collapse, respectively. Thus, the Schwarzschild radial coordinate, in the case of the matching between the internal and external geometries , is [1]

$$r = a \sin\chi_0. \tag{5}$$

Let us see what happens when the star is completely collapsed, i.e. when the star is a BH. On sees that, inserting $r_i = 2M = r_g$, where r_g is the gravitational radius (the Schwarzschild radius), in Eqs. (4), one gets $\sin^2\chi_0 = 1$. Thus, as the range $\chi > \frac{\pi}{2}$ must be discarded [1], one concludes that it is $\chi_0 = \frac{\pi}{2}$ for a BH.

In the following, we will apply the quantization approach derived by Rosen in [5] to the above case. We will find some thin difference, because we analyse the case of a collapsing star, while Rosen analysed a closed homogeneous and isotropic universe [5] . Let us start by rewriting the FLRW line-element (1) in spherical coordinates and comoving time as [1, 5]

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1 - r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right). \tag{6}$$

The Einstein field equation [1, 5]

$$G_{\mu\nu} = -8\pi T_{\mu\nu} \tag{7}$$

gives the relations (we are assuming zero pressure) [5]

$$\begin{aligned} \dot{a}^2 &= \tfrac{8}{3}\pi a^2 \rho - 1 \\ \ddot{a} &= -\tfrac{4}{3}\pi a \rho \end{aligned} \tag{8}$$

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with $\dot{a} = \frac{da}{dt}$. For consistency, one gets [5]

$$\frac{d\rho}{da} = -\frac{3\rho}{a}, \tag{9}$$

which, when integrated gives [5]

$$\rho = \frac{C}{a^3}. \tag{10}$$

In the collapse case, C is determined by the initial conditions as [1]

$$C = \frac{3a_0}{8\pi}. \tag{11}$$

By analysing a closed homogeneous isotropic universe rather than a collapsing object, in [5] Rosen obtained a different value of C. Thus, one rewrites Eq. (10) as

$$\rho = \frac{3a_0}{8\pi a^3}. \tag{12}$$

By multiplying the first of (8) for $M/2$ one gets [5]

$$\frac{M\dot{a}^2}{2} - \frac{4}{3}\pi M a^2 \rho = \frac{M}{2}, \tag{13}$$

which can be interpreted as an energy equation for a particle in one-dimensional motion having coordinate a [5] as

$$E = T + V, \tag{14}$$

where the kinetic energy is [5]

$$T = \frac{M\dot{a}^2}{2} \tag{15}$$

and the potential energy is [5]

$$V(a) = -\frac{4}{3}\pi M a^2 \rho. \tag{16}$$

Thus, the total energy is [5]

$$E = -\frac{M}{2}. \tag{17}$$

From the second of Eqs. (8), one gets the equation of motion of this particle as

$$M\ddot{a} = -\frac{4}{3}M\pi a\rho. \tag{18}$$

The momentum of the particle is [5]

$$P = M\dot{a}, \tag{19}$$

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with an associated Hamiltonian [5]

$$H = \frac{P^2}{2M} + V. \tag{20}$$

Till now, we discussed the problem from the classical point of view. In order to discuss it from the quantum point of view, we need to define a wave-function as [5]

$$\Psi \equiv \Psi\left(a, t\right). \tag{21}$$

Thus, in correspondence of the classical equation (20), one gets the traditional Schrodinger equation [5]

$$i\frac{\partial \Psi}{\partial t} = -\frac{1}{2M}\frac{\partial^2 \Psi}{\partial a^2} + V\Psi. \tag{22}$$

For a stationary state with energy E one obtains [5]

$$\Psi = \Psi\left(a\right)\exp-\left(iEt\right) \tag{23}$$

and Eq. (21) becomes [5]

$$-\frac{1}{2M}\frac{\partial^2 \Psi}{\partial a^2} + V\Psi = E\Psi. \tag{24}$$

Inserting Eq. (12) into Eq. (16) one obtains

$$V(a) = -\frac{Ma_0}{2a^2}. \tag{25}$$

Based on the different value of the constant C, this equation is different from the one which was found by Rosen in [5]. Setting [5]

$$\Psi = aX, \tag{26}$$

Eq. (24) becomes [5]

$$-\frac{1}{2M}\left(\frac{\partial^2 X}{\partial a^2} + \frac{2}{a}\frac{\partial X}{\partial a}\right) + VX = EX. \tag{27}$$

With V given by Eq. (25), Eq. (27), is analogous to the Schrodinger equation in polar coordinates for the s states ($l = 0$) of a hydrogen-like atom [5, 6] in which e^2 is replaced by Ma_0. Thus, for the bound states ($E < 0$) the energy spectrum is

$$E_n = -\frac{a_0^2 M^3}{2n^2}, \tag{28}$$

where n is the principal quantum number. Following [5], one inserts Eq. (17) into Eq. (28), obtaining the mass spectrum of the gravitational collapse as

$$M_n = \frac{a_0^2 M_n^3}{n^2} \Rightarrow M_n = \frac{n}{a_0}. \tag{29}$$

Rosen's discussion in [5] can be followed. For the ground state ($n = 1$) the mass is $M_1 = \frac{1}{a_0}$ (in Planck units the Planck mass is equal to 1). The wave-function associated to this ground state is given by

$$\Psi_1 = 2b_1^{-\frac{3}{2}} a \exp - \left(\frac{a}{b_1}\right), \tag{30}$$

where the "Bohr radius" is given by

$$b_1 = \frac{1}{a_0^2} a_0^3 = a_0. \tag{31}$$

Thus, the "Bohr radius" is the product of the initial scale factor and the Planck length ($l_P \sim 10^{-33} cm$ in standard units) and Ψ_1 is normalized as

$$\int_0^\infty \Psi_1^2 da = 1. \tag{32}$$

The size of the collapsing star is of the order of

$$\bar{a}_1 = \int_0^\infty \Psi_1^2 a da = \frac{3}{2} a_0. \tag{33}$$

For an arbitrary value of n one gets

$$\bar{a}_n = \frac{3}{2} \frac{a_0}{\sqrt{n}} \sqrt{n} = \frac{3}{2} M_n \frac{a_0^2}{n} = \frac{3}{2} a_0, \tag{34}$$

which means that the size of the collapsing star does not depend on its quantum excited state.

3 Black hole mass spectrum and ground state

Now, let us consider the case of a completely collapsed star, i.e. a BH, which means $\chi_0 = \frac{\pi}{2}$ and $r_i = a_0 = 2M = r_g$, in Eqs. (4), see the discussion below Eq. (5). Then, Eq. (29) becomes

$$M_n = \sqrt{\frac{n}{2}}. \tag{35}$$

Remarkably, this is the same BH mass spectrum which was found by Bekenstein in 1974 [7]. Bekenstein indeed used the Bohr-Sommerfeld quantization condition because he argued that the Schwarzschild BH behaves as an adiabatic invariant. Maggiore [8] conjectured a quantum description of BH in terms of quantum membranes. He obtained the mass spectrum

$$M_n = \sqrt{\frac{A_0 n}{16\pi}}. \tag{36}$$

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Thus, he was forced to set $A_0 = 8\pi$ in order to find Bekenstein's result in [7]. In addition, we stress that both Bekenstein and Maggiore used approximations and/or conjectures. Instead, we obtained Eq. (35) through an exact quantization process. Further, if we use again the condition$\chi_0 = \frac{\pi}{2}$ and $r_i = a_0 = 2M = r_g$, we obtain the following remarkable results. Eq. (25) becomes

$$V(a) = -\frac{M^2}{r^2}, \tag{37}$$

which results the *exact potential energy,* i.e. without any approximation, for the Schwarzschild BH interpreted as "gravitational hydrogen atom". Hence, Eqs. (22), (24) and (27) represent the *exact Schrodinger equation* for a Schwarzschild BH if one uses the potential energy (37). For the BH ground state ($n = 1$), one gets the mass as

$$M_1 = \frac{\sqrt{2}}{2}, \tag{38}$$

with an associated wave-function

$$\Psi_1 = 2r(n=1)_g r \exp - \left(\frac{r}{r(n=1)_g} \right), \tag{39}$$

and now the "Bohr radius" coincides with the gravitational radius. Ψ_1 is normalized as

$$\int_0^\infty \Psi_1^2 dr = 1. \tag{40}$$

The size of this BH is of the order of

$$\bar{r}_1 = \int_0^\infty \Psi_1^2 r dr = \frac{3}{2} r(n=1)_g. \tag{41}$$

We wrote $r(n=1)_g$ in Eqs. (39) and (41) because now the gravitational radius is function of the BH quantum principal number. Thus, we have remarkably found the exact quantum representation of the Schwarzschild BH ground state at the Planck scale. In particular, Eq. (38) represents the mass of the smallest Schwarzschild BH, while the correspondent Schwarzschild BH ground state represents the BH minimum energy level which is compatible with the generalized uncertainty principle (GUP) [12]. The GUP indeed prevents a BH from its total evaporation by stopping Hawking's evaporation process in exactly the same way that the usual uncertainty principle prevents the hydrogen atom from total collapse [12]. In standard units one gets $M_1 = \frac{\sqrt{2}}{2} m_P$, where m_P is the Planck mass, $m_P = 2,17645x10^{-8} Kg$.

For an arbitrary value of n one gets

$$\bar{r}_n = \frac{3}{2} \frac{r(n)_g}{\sqrt{n}} \sqrt{n} = \frac{3}{2} M_n \frac{r^2(n)_g}{n} = \frac{3}{2} r(n)_g. \tag{42}$$

which means that the Schwarzschild BH size does not depend on its quantum excited state. The issue that the BH size is, on average, larger than the gravitational radius could appear surprising, but we recall that one of us, (CC),

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recently developed a Bohr-like approach to BH quantum physics where the BH quasi-normal modes (QNMs), "triggered" by emissions (Hawking radiation) and absorption of external particles, represent the "electron" which jumps from a level to another one, and the absolute values of the QNMs frequencies, represent the energy "shells" of the "gravitational atom", see for example [13] and the complete review [14]. Thus, the BH size which is, on average, larger than the gravitational radius, seem consistent with the issue that the BH horizon oscillates with damped oscillations when the BH energy state jumps from a quantum level to another one through emissions of Hawking quanta and/or absorption of external particles.

4 Conclusion remarks

Rosen's approach has been applied to the quantization of the gravitational collapse in the simple case of a pressureless "*star of dust*". In that way, the gravitational potential, the Schroedinger equation and the solution for the collapse's energy levels have been found without any approximation. After that, by applying the constrains for a BH, it has been found the analogous quantum quantities and the BH mass spectrum, again without any approximation. Remarkably, such a mass spectrum coincides with the one which was found by Bekenstein in 1974. Finally, the discussed approach permitted to find the exact quantum representation of the Schwarzschild BH ground state at the Planck scale. Our results seem consistent with the recent Bohr-like approach to BH quantum physics developed by one of us (CC).

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