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Article

Target matrix estimators in risk-based portfolios

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Abstract: Portfolio weights solely based on risk avoid estimation error from the sample mean, but they are still affected from the misspecification in the sample covariance matrix. To solve this problem, we shrink the covariance matrix towards the Identity, the Variance Identity, the Single-index model, the Common Covariance, the Constant Correlation and the Exponential Weighted Moving Average target matrices. By an extensive Monte Carlo simulation, we offer a comparative study of these target estimators, testing their ability in reproducing the *true* portfolio weights. We control for the dataset dimensionality and the shrinkage intensity in the Minimum Variance, Inverse Volatility, Equal-risk-contribution and Maximum Diversification portfolios. We find out that the Identity and Variance Identity have very good statistical properties, being well-conditioned also in high-dimensional dataset. In addition, the these two models are the best target towards to shrink: they minimise the misspecification in risk-based portfolio weights, generating estimates very close to the population values. Overall, shrinking the sample covariance matrix helps reducing weights misspecification, especially in the Minimum Variance and the Maximum Diversification portfolios. The Inverse Volatility and the Equal-Risk-Contribution portfolios are less sensitive to covariance misspecification, hence they benefit less from shrinkage.

Keywords: Estimation Error; Shrinkage; Target Matrix; Risk-Based Portfolios.

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1. Introduction

The seminal contributions of Markowitz (Markowitz 1952, 1956) lay the foundations for his well-known portfolio building technique. Albeit elegant in its formulation and easy to be implemented also in real-world applications, the Markowitz model relies on securities returns sample mean and sample covariance as inputs to estimate the optimal allocation. However, there is large consensus on the fact that sample estimators carry on large estimation error; this directly affects portfolio weights that often exhibit extreme values, fluctuating over time with very poor performance out-of-sample (DeMiguel, Garlappi, and Uppal 2009).

This problem has been tackled from different perspectives: (Jorion 1986) and (Michaud 2014) suggest Bayesian alternatives to the sample estimators; (Jagannathan and Ma 2003) add constraints to the Markowitz model limiting the estimation error; (Black and Litterman 1992) derive an alternative portfolio construction technique exclusively based on the covariance matrix among asset, avoiding to estimate the mean value for each security and converging to the Markowitz Minimum Variance portfolio with no short-sales. This latter technique is supported by results in (Merton 1980) and (Chopra and Ziemba 1993) who clearly demonstrated how the mean estimation process can lead to more severe distortions than those in the case of the covariance matrix.

Following this perspective, estimation error can be reduced by considering risk-based portfolios: findings suggest they have good out-of-sample performance without much turnover (DeMiguel, Garlappi, and Uppal 2009). There is a recent research strand focused on deriving risk-based portfolios other than the Minimum Variance one. In this context, (Qian E. 2006) designs a way to select assets assigning to each of them the same contribution to the overall portfolio risk; (Choueifaty and

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Coignard 2008) propose a portfolio where diversification is the key criterion in asset selection; (Maillard, Roncalli, and Teïletche 2010) offer a novel portfolio construction technique where weights carry on an equal risk contribution while maximising diversification. These portfolios are largely popular among practitioners[1]: they highlight the importance of diversification, risk budgeting; moreover they put risk management in a central role, offering a low computational burden to estimate weights. They are perceived as "robust" models since they do not require the explicit estimation of the mean. Unfortunately, limiting the estimation error in this way poses additional problems related to the ill-conditioning of the covariance matrix that occurs when the number of securities becomes sensitively greater than the number of observations. In this case, the sample eigenvalues become more dispersed than the population ones (Marčenko and Pastur 1967), and the sample covariance matrix directly affects weights estimation. This mean that for high-dimensional dataset the sample covariance matrix is not a reliable estimator.

To reduce misspecification effects on portfolio weights, more sophisticated estimators than the sample covariance have been proposed; the Bayes-Stein shrinkage technique (James and Stein 1961), henceforth shrinkage, stems for its practical implementation and related portfolio performance. This technique reduces the misspecification in the sample covariance matrix by shrinking it towards an alternative estimator. Here, the problem is to select a convenient target estimator as well as the shrinking intensity on the sample covariance matrix. The latter is usually derived minimising a predefined loss function, so to obtained the minimum distance between the *true* and the shrunk covariance matrices (Ledoit and Wolf 2003). A comprehensive overview on shrinkage intensity parameters can be found in (DeMiguel, Martin-Utrera, and Nogales 2013), where authors propose an alternative way of deriving the optimal intensity based on smoothed bootstrap approach. On the other hand, the target matrix is often selected among the class of structured covariance estimators (Briner and Connor 2008), especially when the matrix to shrink is the sample one. As noted in (Candelon, Hurlin, and Tokpavi 2012), the sample covariance matrix is the Maximum Likelihood Estimator (MLE) under the Normality of asset returns, hence it lets data speaks without imposing any structure. This naturally suggests it might be pulled towards a more structured alternative. Dealing with financial data, the shrinkage literature proposes six different models for the target matrix: the Single-Index market model (Ledoit and Wolf 2003), (Briner and Connor 2008), (Candelon, Hurlin, and Tokpavi 2012) and (Ardia et al. 2017); the Identity matrix (Ledoit and Wolf 2004a), (Candelon, Hurlin, and Tokpavi 2012); the Variance Identity matrix (Ledoit and Wolf 2004a); the Scaled Identity matrix (DeMiguel, Martin-Utrera, and Nogales 2013); the Constant Correlation model (Ledoit and Wolf 2004b) and (Pantaleo et al. 2011); the Common Covariance (Pantaleo et al. 2011). All these targets belong to the class of more structured covariance estimators than the sample one, thus implying the latter is the matrix to shrink.

Despite its great improvements in portfolio weights estimation under the Markowitz portfolio building framework, the shrinkage technique has been applied only in one work involving risk-based portfolios, (Ardia et al. 2017). With our work, we contribute to the existing literature filling this gap and offering a comprehensive overview about shrinkage in risk-based portfolios. In particular, we study the effect of six target matrix estimators on the weights of four risk-based portfolios. To achieve this goal, we provide an extensive Monte Carlo simulation aimed at *(i)* assessing estimators' statistical properties and similarity with the *true* target matrix; *(ii)* addressing the problem of how the selection of a specific target estimator impacts on the portfolio weights. We find out that the Identity and Variance Identity held the best statistical properties, being well-conditioned even in high-dimensional dataset. These two estimators represent also the more efficient target matrices towards which to shrink the sample one. In fact, portfolio weights derived shrinking towards the Identity and Variance Identity minimise the distance from their *true* counterparts, especially in the case of Minimum Variance and Maximum Diversification portfolios.

[1] The majority of papers on risk-based portfolios are published in journal aimed at practitioners, as the Journal of Portfolio Management.

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The rest of the paper is organised as follows. Section 2 introduces the risk-based portfolios employed in the study. Section 3 illustrates the shrinkage estimator, to move then to the six target matrix estimators and provides useful insights upon misspecification when shrinkage is applied to risk-based portfolios. In Section 4, we run an extensive Monte Carlo analysis for describing how changes in the target matrix impact on risk-based portfolio weights. Section 5 concludes.

2. Risk-Based Portfolios

Risk-based portfolios are particularly appealing since they rely only on the estimation of a proper measure of risk, i.e. the covariance matrix between asset returns. Assume an investment universe made by p assets:

$$X = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_p) \tag{1}$$

is a $n \times p$ containing an history of n log-returns for the *i-th* asset, where $i = 1, \dots, p$. The covariance matrix among asset log-returns is the symmetric square matrix Σ[2] of dimension $p \times p$, and the unknown optimal weights form the vector $\boldsymbol{\omega}$ of dimension $p \times 1$. Our working framework assume to consider four risk-based portfolios: the Minimum Variance (MV), the Inverse Volatility (IV), the Equal-Risk-Contribution (ERC) and the Maximum Diversification (MD) upon two constraints; no short-selling ($\boldsymbol{\omega} \in \Re_+^p$) and full allocation of the available wealth ($\boldsymbol{\omega}'.\boldsymbol{1}_p = 1$, where $\boldsymbol{1}_p$ is the vector of ones of length p).

The *Minimum Variance* portfolio (Markowitz 1952) derives the optimal portfolio weights by solving this minimization problem w.r.t. $\boldsymbol{\omega}$:

$$\boldsymbol{\omega}_{MV} \equiv \underset{\boldsymbol{\omega}}{\operatorname{argmin}}\{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega} \mid \boldsymbol{\omega} \in \Re_+^p, \boldsymbol{\omega}'.\boldsymbol{1}_p = 1\}, \tag{2}$$

where $\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}$ is the portfolio variance.
In the *Inverse Volatility*, also known as the equal-risk-budget (Leote de Carvalho, Lu, and Moulin 2012), is available a closed form solution. Each element of the vector $\boldsymbol{\omega}$ is given by the inverse of the i-th asset variance (denoted by $\Sigma_{i,i}^{-1}$) divided by the inverse of the sum of all asset variances:

$$\boldsymbol{\omega}_{IV} \equiv \left(\frac{\Sigma_{1,1}^{-1}}{\sum_{i=1}^{p}\Sigma_{i,i}^{-1}}, \dots, \frac{\Sigma_{p,p}^{-1}}{\sum_{i=1}^{p}\Sigma_{i,i}^{-1}}\right)'. \tag{3}$$

In the *Equal-Risk-Contribution* portfolio, as the name suggests, the optimal weights are calculated by assigning to each asset the same contribution to the whole portfolio volatility, thus originating a minimization procedure to be solved w.r.t. $\boldsymbol{\omega}$:

$$\boldsymbol{\omega}_{ERC} \equiv \underset{\boldsymbol{\omega}}{\operatorname{argmin}}\left\{\sum_{i=1}^{p}\left(\%RC_i - \frac{1}{p}\right)^2 \mid \boldsymbol{\omega} \in \Re_+^p, \boldsymbol{\omega}'.\boldsymbol{1}_p = 1\right\}, \tag{4}$$

here $\%RC_i \equiv \frac{\omega_i cov_{i,\pi}}{\sqrt{\omega'\Sigma\omega}}$ is the percentage risk contribution for the i-th asset, $\sqrt{\boldsymbol{\omega}'\Sigma\boldsymbol{\omega}}$ is the portfolio volatility as earlier defined and $\boldsymbol{\omega}_i cov_{i,\pi}$ provides a measure of the covariance of the i-th exposure to the total portfolio π, weighted by the corresponding $\boldsymbol{\omega}_i$.

Turning to the *Maximum Diversification*, as in (Choueifaty and Coignard 2008) we preliminary define $DR(\boldsymbol{\omega})$ as the portfolio's diversification ratio:

$$DR(\boldsymbol{\omega}) \equiv \frac{\omega'\sqrt{\operatorname{diag}(\Sigma)}}{\sqrt{\omega'\Sigma\omega}},$$

where $\operatorname{diag}(\Sigma)$ is a $p \times 1$ vector which takes all the asset variances $\Sigma_{i,i}$ and $\boldsymbol{\omega}'\sqrt{\operatorname{diag}(\Sigma)}$ is the weighted average volatility. By construction it is $DR(\boldsymbol{\omega}) \geq 1$, since the portfolio volatility is sub-additive (Ardia et al. 2017). Hence, the optimal allocation is the one with the highest DR:

[2] With this we refer to the population covariance matrix, which by definition is not observable and then unfeasible. Hence, Σ is estimated taking into account the observations stored in X: we will deeply treat this in the next section.

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$$\boldsymbol{\omega}_{MD} \equiv \underset{\boldsymbol{\omega}}{\operatorname{argmax}}\{DR(\boldsymbol{\omega}) |\ \boldsymbol{\omega} \in \Re^p_+, \boldsymbol{\omega}'.\mathbf{1}_p = 1\} \tag{5}$$

3. Shrinkage estimator

The shrinkage technique relies upon three ingredients: the starting covariance matrix to shrink, the target matrix towards which shrinking and the shrinkage intensity, or roughly speaking the strength at which the starting matrix must be shrunk.

In financial applications, the starting matrix to shrink is always the sample covariance matrix. This is a very convenient choice that helps in the selection of a proper shrinkage target: being the sample covariance a model-free estimator that completely reflects the relationships among data[3], it becomes natural to select a target in the class of more structured covariance estimators (Briner and Connor 2008). In addition, this strategy allows to directly control the trade-off between estimation error and model error in the resulting shrinkage estimates. In fact, the sample covariance matrix is usually affected by a large amount of estimation error. This is reduced when shrinking towards a structured target which minimizes the sampling error at the cost of adding some misspecification by imposing a specific model. At this point, the shrinkage intensity is crucial because it must be set in such a way to minimize both errors.

To define the shrinkage estimator, we start from the definition of sample covariance matrix S. Recalling Eq. [1], S is given by

$$S = \frac{1}{n-1} X' \left(I_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}'_n \right) X, \tag{6}$$

where I_n denotes the $n \times n$ identity matrix and $\mathbf{1}_n$ is the ones column vector of length n. The shrinkage methodology enhances the sample covariance matrix estimation by shrinking S towards a specific target matrix T:

$$\Sigma_s = \delta \mathrm{T} + (1-\delta) S, \tag{7}$$

where Σ_s is the shrinkage estimator; δ the shrinkage parameter and T the target matrix. In this work, we focus on the problem of selecting the target matrix. After a review of the literature on target matrices, in the following rows we present the target estimators considered in this study and we assess trough a numerical illustration the impact of misspecification in the target matrix for the considered risk-based portfolios.

2.1. Target Matrix Literature Review

The target matrix should fit a desirable number of requirements: it should be structured much enough to lower the estimation error of the sample covariance matrix while not bringing too much error from model selection. Second, it should reflect the important features of the *true* covariance matrix (Ledoit and Wolf 2004b). The crucial question is: how much structure should we impose to fill in the requirements? Table 1 shows the target matrices employed so far in the literature, summarising information about the formula for the shrinkage intensity, the wealth allocation rule and the addressed research question. Not surprisingly, all the papers shrink the sample covariance matrix. What surprises is that only six target matrices have been examined: the one relying on the Single-Index market model, the Identity matrix and the Variance Identity, the Constant Correlation model and the Common Covariance. Earlier four have been proposed by Ledoit and Wolf in separate works (Ledoit and Wolf 2003, 2004a, 2004b) and have been proposed again in subsequent works, while the Common Covariance appears only in (Pantaleo et al. 2011) and the Scaled Identity only in (DeMiguel, Martin-Utrera, and Nogales 2013).

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[3] The sample covariance matrix is the Maximum Likelihood Estimator (MLE) under Normality, therefore it lets data speaks without imposing any structure.

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Table 1. Literature Review of Target Matrices. SCVm stands for sample covariance matrix. "N.A." stands for not available.

Reference	Matrix to shrink	Target Matrix	Shrinkage Intensity	Portfolio selection rule	Research Question
(Ledoit and Wolf 2003)	SCVm	Market Model and Variance Identity	Risk-function minimisation	Classical Markowitz problem	Portfolio Performance comparison
(Ledoit and Wolf 2004a)	SCVm	Identity	Risk-function minimisation	N.A.	Theoretical paper to gauge the shrinkage asymptotic properties
(Ledoit and Wolf 2004b)	SCVm	Constant Correlation Model	Optimal shrinkage constant	Classical Markowitz problem	Portfolio Performance comparison
(Briner and Connor 2008)	SCVm	Market Model	Same as (Ledoit and Wolf, 2004b)	N.A.	Analysis of the trade-off estimation error and model specification error
(Pantaleo et al. 2011)	SCVm	Market Model, Common Covariance and Constant Correlation Model	Unbiased estimator of (Schäfer and Strimmer, 2005)	Classical Markowitz problem	Portfolio Performance comparison
(Candelon, Hurlin, and Tokpavi 2012)	SCVm	Market Model and Identity	Same as (Ledoit and Wolf,)	Black-Litterman GMVP	Portfolio Performance comparison
(DeMiguel, Martin-Utrera, and Nogales 2013)	SCVm	Scaled Identity	Expected quadratic loss	Classical Markowitz problem	Comprehensive investigation of shrinkage estimators
(Ardia et al. 2017)	SCVm	Market Model	Same as (Ledoit and Wolf, 2003)	Risk-based portfolios	Theoretical paper to assess effect on risk-based weights

In Table 1 we have listed papers taking into account their contribution to the literature, as the adoption of a novel target matrix estimator, the re-examination of a previously proposed target and the comparison among different estimators. Ledoit and Wolf popularise the shrinkage methodology in portfolio selection: in (Ledoit and Wolf 2003), they are also the first in comparing the effects of shrinking towards different targets in portfolio performance. Shrinking towards the Variance Identity and shrinking towards the Market Model are two out of eight estimators for the covariance matrix compared w.r.t. the reduction of estimation error in portfolio weights. They find significant improvements in portfolio performance when shrinking towards the Market Model. (Briner and Connor 2008) well describe the importance of selecting a target matrix among the class of structured covariance estimators, hence proposing to shrink the asset covariance matrix of demeaned returns towards the Market model as in (Ledoit and Wolf 2003). (Candelon, Hurlin, and Tokpavi 2012) compare the effect of double shrinking the sample covariance either towards the Market Model and the Identity, finding that both estimators carry on similar out-of-sample performances. (DeMiguel, Martin-Utrera, and Nogales 2013) is the first work to compare the effects of different shrinkage estimators on portfolio performance, highlighting the importance of the shrinkage intensity and proposing a scaled version of the Identity Matrix as target. Another important comparison among target matrices is due to (Pantaleo et al. 2011), who compare the Market and Constant Correlation models as in (Ledoit and Wolf 2003, 2004b) with the Common Covariance of (Schäfer and Strimmer, 2005), used as target matrix for the first time in finance. Authors assess the effects on portfolio performances while controlling for the dimensionality of the dataset, finding that the Common Covariance should not be used when the number of observations is less than the number of assets. Lastly, (Ardia et al. 2017) is the only work to implement shrinkage in risk-based portoflios. They shrink the sample covariance matrix as in (Ledoit and Wolf 2003), finding that the Minimum Variance and the Maximum Diversification portfolios are the most affected from covariance misspecification, hence they benefit the most from the shrinkage technique.

2.2. Estimators for the target matrix

We consider six estimators for the target matrix: the Identity and the Variance Identity matrix, the Single-index, the Common Covariance, the Constant Correlation and the Exponential Weighted Moving Average models. They are all structured estimator, in the sense that the number of parameters to be estimated is far less the $\frac{1}{2}p(p+1)$ required in the sample covariance case. Compared with the literature, we take into account all the previous target estimators, adding to the analysis the EWMA: this estimator well addresses the problem of heteroskedasticity in asset returns.

The identity is a matrix with ones on the diagonal and zero elsewhere. Choosing the Identity as target is justified by the fact that is shows good statistical properties: it is always well-conditioned and hence invertible (Ledoit and Wolf, 2003). Besides the identity, we also consider a multiple of the identity, named the Identity Variance. This is given by:

$$\mathrm{T}_{id} \equiv I_p diag(S) I_p, \tag{8}$$

here $diag(S)$ is the main diagonal of the sample covariance matrix (hence the assets variances) and $\mathrm{I_p}$ the identity matrix of dimension p.

The Single Index Model (Sharpe, 1963) assumes that the returns $\boldsymbol{r}_t$ can be described by a one-factor model, resembling the impact of the whole market:

$$\boldsymbol{r}_t = \boldsymbol{\alpha} + \boldsymbol{\beta} r_{mkt} + \varepsilon_t, \\ with\ t = 1, \dots, n$$

Where r_{mkt} is the overall market returns; $\boldsymbol{\beta}$ is the vector of factor estimates for each asset; $\boldsymbol{\alpha}$ is the market mispricing and ε_t the model error. The Single-Index market model represents a practical way of reducing the dimension of the problem, measuring how much each asset is affected by the market factor. The model implies the covariance structure among asset returns is given by:

$$T_{si} \equiv s_{mkt}^2 \boldsymbol{\beta}\boldsymbol{\beta}' + \Omega \tag{9}$$

where s_m^2 is the sample variance of asset returns; $\boldsymbol{\beta}$ is the vector of beta estimates and Ω contains the residual variance estimates.

The Common Covariance model is aimed at minimizing the heterogeneity of assets variances and covariances by averaging both of them (Pantaleo et al., 2011). Let $\mathrm{var_{ij,i=j}}$ and $\mathrm{covar_{ij,i\neq j}}$ being respectively the variances and covariances of the sample covariance matrix, their averages are given by:

$$\overline{var} = \frac{1}{p}\sum_{k=1}^{p} \mathrm{var_{k,i=j}}\,;$$

$$\overline{covar} = \frac{1}{\mathrm{p(p-1)/2}} \sum_{k=1}^{\mathrm{p(p-1)/2}} \mathrm{covar}_{k,\mathrm{i\neq j}},$$

where p is the number of securities. The resulting target matrix T_{cv} has its diagonal elements all equal to the average of the sample covariance, while non-diagonal elements are all equal to the average of sample covariances.

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In the Constant Correlation model the main diagonal is filled with sample variances, and elsewhere a constant covariance parameter which is equal for all assets. The matrix can be written according to the following decomposition:

$$T_{cc} \equiv \mathrm{P}\,\mathrm{diag}(S)\,\mathrm{P}, \tag{10}$$

where P is the lower triangular matrix filled with the constant correlation parameter $\bar{\rho} = \frac{1}{p(p-1)/2}\sum_{i=1}^{p} \rho_{ij}$ for $i < j$ and ones in the main diagonal. $\mathrm{diag}(S)$ represents the main diagonal of the sample covariance matrix.

The Exponential Weighted Moving Average (EWMA) model (J. P. Morgan and Reuters Ltd 1996) which was introduced by the JP Morgan's research team to provide an easy but consistent way to assess portfolio covariance. RiskMetrics EWMA considers the variances and covariance driven by an IGARCH process:

$$T_{EWMA,t} \equiv (1-\lambda)X^{'}X + \lambda T_{EWMA,t-1} \tag{11}$$

with $T_{EWMA,0} = I_p$. $T_{EWMA,t-1}$ is the target matrix at time $t-1$ and λ is the smoothing parameter: the higher λ, the higher the persistence in the variance.

2.3. The impact of misspecification in the target matrix

We are now going to show to which extent risk-based portfolios can be affected by misspecification in the target matrix. To do so, we provide a numerical illustration, merely inspired by the one in (Ardia et al., 2017). Assume an investment universe made by 3 securities: a sovereign bond (Asset-1), a corporate bond (Asset-2) and equity (Asset-3), we are able to impose an arbitrary structure to the related 3×3 *true* covariance matrix[4]. We preliminary recall that Σ can be written according to the following decomposition:

$$\Sigma \equiv (\mathrm{diag}(\Sigma))^{1/2} \mathrm{P}_{\Sigma} (\mathrm{diag}(\Sigma))^{1/2},$$

where $(\mathrm{diag}(\Sigma))^{1/2}$ is a diagonal matrix with volatilities on the diagonal and zeros elsewhere and P_{Σ} is the related correlation matrix, with ones on the diagonal and correlations symmetrically displaced elsewhere. We impose

$$(\Sigma_{1,1}^{1/2},\ \Sigma_{2,2}^{1/2},\ \Sigma_{3,3}^{1/2},\ ) = (0.1, 0.1, 0.2),$$

and

$$(\mathrm{P}_{\Sigma;1,2}, \mathrm{P}_{\Sigma;1,3},\ \mathrm{P}_{\Sigma;2,3}) = \ (-0.1, -0.2, 0.7),$$

hence, the *true* covariance matrix is:

$$\Sigma \ \equiv \begin{bmatrix} 0.010 & -0.001 & -0.004 \\ -0.001 & 0.010 & 0.014 \\ -0.004 & 0.014 & 0.040 \end{bmatrix}.$$

Now assume that the *true* covariance matrix Σ is equal to its shrunk counterpart when $\delta = \frac{1}{2}$:

$$\Sigma \equiv \Sigma_s = \frac{1}{2}S + \frac{1}{2}\mathrm{T},$$

[4] (Ardia et al. 2017) imposes Asset-1 and Asset-2 to have 10% annual volatility; Asset-3 to have 20% annual volatility; correlations between Asset-1/Asset-2 and Asset-1/Asset-3 are set negative and correlation between corporate bonds and equities (Asset-2/Asset-3) is set positive. They give as motivation for the selection of these values the fact that they precisely resemble the real-world scenario of the past recent years.

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That is both the sample covariance matrix S and the target matrix T must be equal to $\frac{1}{2}\Sigma$ and the *true* target matrix is:

$$\mathrm{S} \equiv \mathrm{T} \equiv \begin{bmatrix} 0.005 & -0.0005 & -0.002 \\ -0.0005 & 0.005 & 0.007 \\ -0.002 & 0.007 & 0.020 \end{bmatrix},$$

with few algebraic computations, we can obtain the volatilities and correlations simply by applying the covariance decomposition, ending up with

$$(\mathrm{T}_{1,1}^{1/2}, \mathrm{T}_{2,2}^{1/2}, \mathrm{T}_{3,3}^{1/2}) = (0.0707, 0.0707, 0.1414);$$
$$(\mathrm{P}_{\mathrm{T};1,2}, \mathrm{P}_{\mathrm{T};1,3}, \mathrm{P}_{\mathrm{T};2,3}) = (-0.1, -0.2, 0.7)$$

In this case, we can conclude that the target matrix T is undervaluing all the covariance and correlation values.

At this point, some remarks are needed. First, as summarised in Table 2, we work out the *true* risk-based portfolio weights, which are equal to the ones in (Ardia et al. 2017) as expected. Weights are differently spread out: the MV equally allocates wealth to the first two assets, excluding equities. This because it mainly relies upon the asset variance, limiting the diversification of the resulting portfolio. The remaining portfolios allocate wealth without excluding any asset; however, the MD overvalues Asset-1 assigning to it more than 56% of total wealth. The IV and ERC seem to maximise diversification under a risk-parity concept, similarly allocating wealth among the investment universe.

Table 2. *True* weights of the four risk-based portfolios and maximum and minimum of the Frobenius norm for the misspecification in the variance and covariance, respectively.

	MV	**IV**	**ERC**	**MD**
Asset-1	0.500	0.400	0.428	0.566
Asset-2	0.500	0.400	0.335	0.226
Asset-3	0.000	0.200	0.181	0.207
Max FN	0.171	0.137	0.125	0.156
Min FN	0.127	8.0e-17	0.039	0.136

Second, assuming Σ as the *true* covariance matrix allows us to simulate misspecification both in the variance and in the covariance components of the target matrix T simply increasing or decreasing the imposed *true* values. Since we are interested in investigating misspecification impact on the *true* risk-based portfolio weights, we measure its effects after each shift with the Frobenius norm between the *true* weights and the misspecified ones:

$$\|\widetilde{\boldsymbol{\omega}}\|_F^2 = \sum_{i=1}^{p} \widetilde{\omega}_i^2,$$

where $\widetilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \widehat{\boldsymbol{\omega}}$.

Third, turning the discussion on the working aspects of this toy example, we will separately shift the volatility and the correlation of Asset-3, as in (Ardia et al. 2017). The difference with them is that we modify the values in the *true* target matrix T. Moreover, in order to understand also how shrinkage intensity affects the portfolio weights, we perform this analysis for 11 values of δ, spanning from 0 to 1 (with step 0.1). This allows us to understand both extreme cases, i.e. when the *true* covariance matrix is only estimated with the sample estimator ($\delta = 0$) and only with the target matrix ($\delta = 1$). Remember that the *true* shrinkage intensity is set at $\delta = \frac{1}{2}$.

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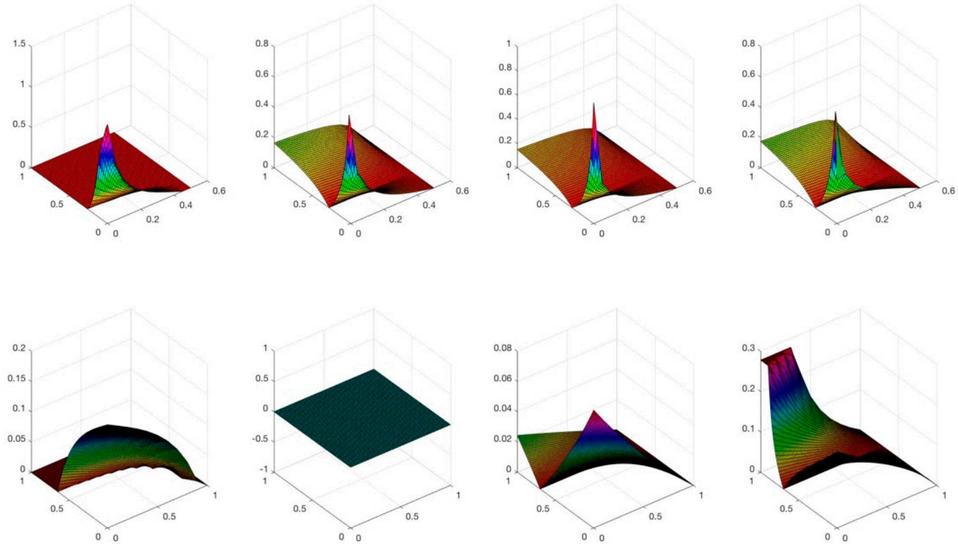

Figure 1. Frobenius norm between *true* and estimated weights; first row reports misspecification in variance, while second row in covariance. The surfaces' three dimensions are: the shrinkage intensity in y axis (from 0 to 1); the misspecification in the variance (from 0 to 0.5) or in the covariance (from 0 to 1) in x axis and the Frobenius norm in z axis. Each column refers to a specific risk-based portfolio. From the left to the right: MV, IV, ERC, MD, respectively.

Moving to the core of this numerical illustration, we proceed as follows. First, for what is concerning the volatility, we let $\mathrm{T}_{3,3}^{1/2}$ to vary between 0 and 0.5, ceteris paribus. Results are summarised in Figure 1, row 1. As expected, there is no misspecification in all the risk-based portfolio at the initial state $\mathrm{T}_{3,3}^{1/2} = 0.1414$, i.e. the *true* value. All the portfolio weights are misspecified in the range $[0; 0.1414)$, with MV showing the greatest departure from the *true* portfolio weights when the Asset-3 volatility is undervalued below 0.12. The absence of misspecification effects in the MV weights is due to the initial high-risk attributed to Asset-3: in fact, it is already excluded from the optimal allocation at the initial non-perturbated state. The IV, ERC and MD portfolio weights show nearly the same distance from their not misspecified counterpart. The same applies in the range $(0.1414; 0.5]$, with MD (ERC) showing more (less) misspecification as 0.5 is reached, compared to the others. MV is again not misspecified, since Asset-3 is always excluded from the allocation. This allows the MV portfolio not to be affected by shifts in the shrinkage intensity when there is over-misspecification. On the other hand, the remaining portfolios react in the same way to shrinkage intensity misspecification, showing an increase in the Frobenius norm especially for low values of Asset-3 variance. All the portfolios share the same effect when the weights are estimated with the sample covariance only: in this case the distance from *true* portfolios is at maximum.

Second, we assess the misspecification impact when it arises in the correlation. We let the correlation between Asset-3 and 2 ($\mathrm{P}_{\mathrm{T};2,3}$) to vary from 0 to 1, ceteris paribus. In this case, we have signs of perturbation in the MV and the MD portfolios, while the ERC shows far less distortion, as presented in Figure 1, row 2. Surprisingly, the IV is not to impacted at all by misspecification in the correlation structure of the target matrix T. Moreover, IV is also the only one not be impacted by the shrinkage intensity misspecification. Both effects are due to the specific characteristics of Asset-3 and the way in which IV selects to allocate weights under a risk-parity scheme. Lastly, MV and ERC show the greatest distortion and hence higher distance from the *true* weights for small values of shrinkage intensity, while for the MD the Frobenius norm attains its maximum when the target matrix is the estimator ($\delta = 1$).

In conclusion, we started this numerical illustration to assess the effects of target matrix misspecification in risk-based portfolios: as in (Ardia et al. 2017), the four risk-based portfolios reacts

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similarly to perturbation in volatility and correlation (even if for us they originate in the target matrix), with the MV being the most affected when the variance is misspecified and the IV being the less affected from covariance shifts. In particular, MV performs very poorly when Asset-3 volatility tends to zero. This portfolio is less sensitive to overvalued variance misspecification in very risky assets, but very sensitive in the opposite sense, and it is one of the most affected to perturbations in the correlation. The remaining three portfolios react similarly to variance misspecification, while MD shows a similar sensitivity as the MV to perturbation in the correlation. The IV does not show any sign of distortion when covariance is shifted. Moreover, we improve previous findings showing how weights are affected by shifts in the shrinkage intensity: when sample covariance is the estimator ($\delta = 0$), the distance from the *true* weights stands at maximum level.

4. Case Study – Monte Carlo Analysis

This section offers a comprehensive comparison of the six target matrix estimators by mean of an extensive Monte Carlo (MC) study. The aim of this analysis is twofold: (i) assessing estimators' statistical properties and similarity with the *true* target matrix; (ii) addressing the problem of how selecting a specific target estimator impacts on the portfolio weights. This investigation is aimed at giving a very broad overview about *(i)* and *(ii)* since we monitor both the p/n ratio and the whole spectrum of shrinkage intensity. We run simulations for 15 combinations of p and n, and for 11 different shrinkage intensities spanning in the interval $[0\,;1]$, for an overall number of 165 scenarios.

The MC study is designed as follows. Returns are simulated assuming a factor model is the data generating process, as in (MacKinlay and Pastor 2000). In details, we impose a one-factor structure for the returns generating process:

$$r_t = \xi . f_t + \varepsilon_t; \\ with\, t = 1, \dots, n$$

where f_t is the $k \times 1$ vector of returns on the factor, ξ is the $p \times 1$ vector of factor loadings and ε_t the vector of residuals of p length. Under this framework returns are simulated implying multivariate normality and absence of serial correlation. The asset factor loadings are drawn from a uniform distribution and equally spread, while returns on the single factor are generated from a Normal distribution. The bounds for the uniform distribution and the mean and the variance for the Normal one are calibrated on real market data, specifically on the empirical dataset "49-Industry portfolios" with monthly frequency, available at Kennet French website[5]. Residuals are drawn from a uniform distribution in the range $[0.10; 0.30]$ so that the related covariance matrix is diagonal with an average annual volatility of 20%.

For each of the 165 scenarios, we apply the same strategy. First, we simulate the $n \times p$ matrix of asset log-returns, then we estimate the six target matrices and their corresponding shrunk matrices $\hat{\Sigma}_s$. At last, we estimate the weights of the four risk-based portfolios. Some remarks are needed. First, we consider the number of assets as $p = \{10{,}50{,}100\}$ and number of observations as $n = \{60{,}120{,}180{,}3000{,}6000\}$ months, which correspond to 5, 10, 15, 250 and 500 years. Moreover, the shrinkage intensity is let to vary between their lower and upper bounds as $\delta = \{0{,}0.1{,}0.2{,}0.3{,}0.4{,}0.5{,}0.6{,}0.7{,}0.8{,}0.9{,}1\}$. For each of the 165 scenarios we run 100 Monte Carlo trials[6], giving robustness to the results.

We stress again the importance of Monte Carlo simulations, which allow us to impose the *true* covariance Σ and hence the *true* portfolio weights $\boldsymbol{\omega}$. This is crucial because we can compare the *true* quantities with their estimated counterparts.

[5] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

[6] Simulations were done in MATLAB setting the random seed generator at its default value, thus ensuring the full reproducibility of the analisys.

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With respect to the point *(i)*, we use two criteria to assess and compare the statistical properties of target matrices: the reciprocal 1-norm condition number (RCN) and the Frobenius Norm. Being the 1-norm condition number (CN) defined as:

$$CN(A) = \kappa(A) = \|A^{-1}\|,$$

for a given A. It measures the matrix sensitivity to changes in the data: when is large, it indicates that a small shift causes important changes, offering a measure of the ill-conditioning of A. Since CN takes value in the interval $[0\,;+\infty)$, it is more convenient to use its scaled version, the RCN:

$$RCN = 1/\,\kappa(A). \tag{12}$$

It is defined in the range [0 ; 1]: the matrix is well-conditioned if the reciprocal condition number is close to 1 and ill-conditioned vice-versa. Under the Monte Carlo framework, we will study its MC estimator:

$$E[CN] = \frac{1}{M}\sum_{m=1}^{M} CN_m, \tag{13}$$

where M is the number of MC simulations. On the other hand, the Frobenius norm is employed to gauge the similarity between the estimated target matrix and the *true* one. We define it for the $p \times p$ symmetric matrix Z as:

$$FN(Z) = \|Z\|_F^2 = \sum_{i=1}^{p}\sum_{j=1}^{p} z_{ij}^2.$$

In our case, $Z = \hat{\Sigma}_s - \Sigma$. Its Monte Carlo estimator is given by the following

$$E[FN] = \frac{1}{M}\sum_{m=1}^{M} FN_m. \tag{14}$$

Regarding (ii), we assess the discrepancy between *true* and estimated weights again with the Frobenius norm. In addition, we report the values at which the Frobenius norm attains its best results, i.e. when the shrinkage intensity is optimal.

4.1. Main Results

Figure 2 summarises the statistical properties of the various target matrices.

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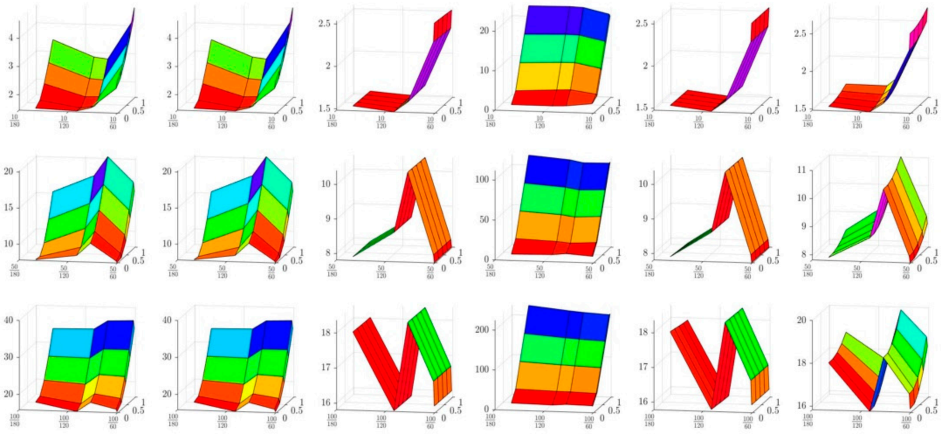

Figure 2. The condition number (y-axis) as the p/n ratio moves from $\frac{p}{60}$ to $\frac{p}{6000}$. Each column corresponds to a specific target matrix: from left to right, the Identity, the Variance Identity, the Single-Index, the Common Covariance, the Constant Correlation and the EWMA, respectively. Each row corresponds to a different p: in ascendant order from 10 (first row) to 100 (third row).

The Figure 2 shows from left to right the condition numbers for the Identity, the Variance Identity, the Market model, the Common Variance, the Constant Correlation and the EWMA, respectively. Each column corresponds to a specific target, while each rows refer to a different number of assets p: the first column to 10, the secondo to 50 and the third to 100. For each sub-figure, on the x-axis we show the p/n ratio in ascendant order and on the y-axis the condition number: the matrix is well-conditioned when its value is closer to 1, vice-versa is ill-conditioned the more it tends zero.

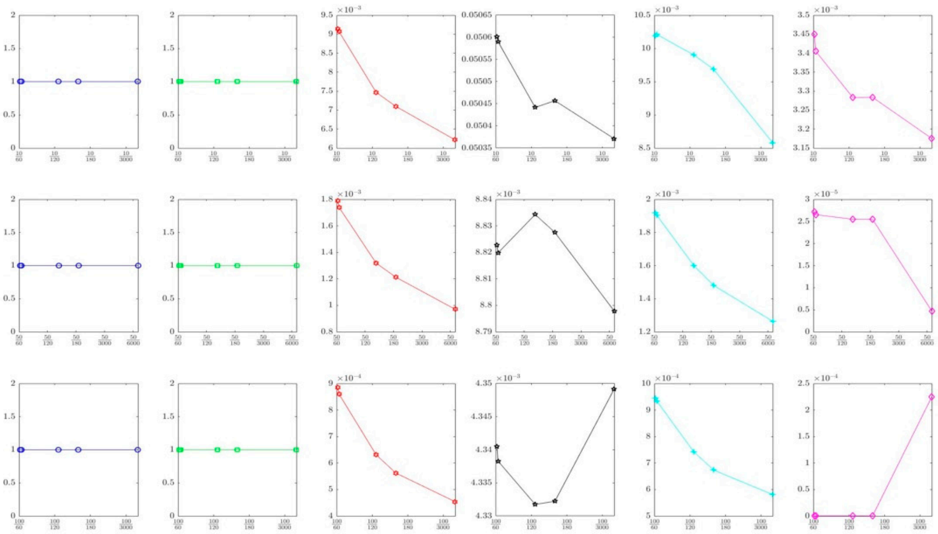

Figure 3. Surfaces representing the Frobenius norm (z-axis) between the *true* and the estimated target matrices, considering the shrinkage intensity (y-axis) and the p/n ratio (x-axis). Each column corresponds to a specific target matrix: from left to right, the Identity, the Variance Identity, the Single-Index, the Common Covariance, the Constant Correlation and the EWMA, respectively. Each row corresponds to a different p: in ascendant order from $p = 10$ (first row) to $p = 100$ (third row).

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Then, we turn to the study of similarity among *true* and estimated target matrices. Figure 3 represents the Monte Carlo Frobenius norm between the *true* and the estimated target matrices. The surfaces give a clear overview about the relation among the Frobenius norm itself, the p/n ratio and the shrinkage intensity. Overall, the Frobenius norm is minimised by the Single-Index and the CC: in these cases the target matrices are not particularly affected by the shrinkage intensity, while their reaction to increases in the p/n ratio are controversial. In fact, quite surprisingly the distance between *true* and estimated weights diminishes as both p and n increases. For $p = 50$ and $p = 100$ there is a hump for small p/n values; however, the Frobenius norm increases when $\frac{p}{n} \geq 1$. Despite of the low condition number, the EWMA shows a similar behaviour to the Single-Index and the Constant Correlation target matrices, especially w.r.t. p/n values. On the other hand, it is more affected by shifts in the shrinkage parameters; the distance from the *true* weights increases moving towards the target matrix. Lastly, the Common Covariance and the Variance Identity are very far away from the *true* target matrix: they are very sensitive to high p/n and δ values.

To conclude, the identity is the most well-conditioned matrix, and it is stable across all the examined p/n combinations. Nevertheless, the Single-Index and the CC target matrices show the greater similarity with the *true* target matrix minimizing Frobenius norm, while the identity seems less similar to the *true* target.

4.1.1. Results on Portfolio Weights

Table 3 and Table 4 present main results of the Monte Carlo study: for each combination of p and n, we report the Monte Carlo estimator of the Frobenius norm between *true* and estimated weights. In particular, Table 3 reports averaged Frobenius norm along the shrinkage intensity (excluding the case $\delta = 0$, which corresponds to the sample covariance matrix), while Table 4 lists the minimum values for the optimal shrinkage intensity.

Table 3. Frobenius norm for the portfolio weights. Values are averaged along the shrinkage intensity (excluding the case $\delta = 0$). For each n, the first line reports the Frobenius norm for the sample covariance matrix. Abbreviations in use are: S for sample covariance; Id for identity matrix; Vid for Variance Identity; SI for Single-Index; CV for Common Covariance; CC for Constant Correlation and EWMA for Exponentially Weighted Moving Average.

	P=10				P=50				P=100			
	MV	**IV**	**ERC**	**MD**	**MV**	**IV**	**ERC**	**MD**	**MV**	**IV**	**ERC**	**MD**
Panel A: n=60												
S	0.834	0.1585	0.1736	0.5842	0.7721	0.0573	0.0637	0.4933	0.7555	0.0409	0.0447	0.4565
Id	0.6863	0.1425	0.1528	0.5045	0.6215	0.0559	0.0631	0.3873	0.4967	0.0404	0.0451	0.3652
VId	0.6935	0.1583	0.1732	0.5176	0.5999	0.0567	0.0634	0.4092	0.5901	0.0404	0.0445	0.3686
SI	0.838	0.1585	0.1736	0.5678	0.7685	0.0573	0.0637	0.4709	0.75	0.0409	0.0447	0.4288
CV	1.2438	0.1583	0.1731	1.011	1.1484	0.0567	0.0628	0.9381	1.1386	0.0404	0.0438	0.9185
CC	0.8353	0.1585	0.1733	0.5361	0.7808	0.0573	0.0635	0.4328	0.7663	0.0409	0.0445	0.3922
EWMA	0.8473	0.1593	0.1745	0.595	0.7811	0.0575	0.064	0.5142	0.7325	0.0411	0.045	0.4431
Panel B: n=120												
S	0.9064	0.0877	0.0989	0.4649	0.7814	0.059	0.0656	0.5065	0.6519	0.0424	0.0472	0.4332
Id	0.8157	0.087	0.0983	0.4256	0.6259	0.0613	0.0688	0.4354	0.6307	0.0389	0.0431	0.328
VId	0.8235	0.0871	0.0985	0.4284	0.6259	0.0613	0.0688	0.4354	0.489	0.0421	0.0471	0.3712
SI	0.9097	0.0877	0.0989	0.4563	0.7777	0.059	0.0656	0.4925	0.6458	0.0424	0.0472	0.419
CV	1.3269	0.0871	0.0982	0.9667	1.1806	0.0587	0.0651	1.0138	1.0974	0.0421	0.0467	0.8951
CC	0.905	0.0877	0.0988	0.4357	0.7822	0.059	0.0655	0.4636	0.6566	0.0424	0.0471	0.3856
EWMA	0.9281	0.0883	0.0996	0.4859	0.7994	0.0592	0.0658	0.5246	0.6788	0.0427	0.0475	0.4601
Panel C: n=180												
S	0.7989	0.1311	0.1423	0.5007	0.7932	0.0564	0.0627	0.4631	0.6905	0.0404	0.044	0.4065
Id	0.7206	0.1308	0.142	0.4736	0.6705	0.0562	0.0625	0.405	0.5477	0.0375	0.0399	0.3748
VId	0.7273	0.1308	0.1421	0.4757	0.6838	0.0562	0.0626	0.4127	0.5754	0.0402	0.044	0.3556
SI	0.8001	0.1311	0.1423	0.4954	0.7904	0.0564	0.0627	0.4545	0.6873	0.0404	0.044	0.3982
CV	1.2715	0.1308	0.1419	0.9961	1.2073	0.0562	0.0624	0.9988	1.1422	0.0402	0.0437	0.8705
CC	0.7957	0.1311	0.1423	0.4803	0.792	0.0564	0.0626	0.4259	0.692	0.0404	0.044	0.3672
EWMA	0.8415	0.1322	0.1435	0.526	0.8284	0.0567	0.0631	0.5005	0.7206	0.0408	0.0445	0.4429
Panel D: n=3000												
S	0.7504	0.1476	0.1596	0.3957	0.734	0.049	0.0539	0.3988	0.513	0.0384	0.0428	0.3259
Id	0.7441	0.1477	0.1597	0.3946	0.7009	0.049	0.0539	0.3872	0.4615	0.0384	0.0428	0.3096
VId	0.7437	0.1477	0.1596	0.3945	0.7043	0.049	0.0539	0.3886	0.4673	0.0384	0.0428	0.312
SI	0.7516	0.1476	0.1596	0.3955	0.7339	0.049	0.0539	0.3984	0.5123	0.0384	0.0428	0.3252
CV	1.2864	0.1477	0.1597	0.963	1.2281	0.049	0.0538	0.9954	1.1041	0.0384	0.0428	0.6822
CC	0.7488	0.1476	0.1596	0.3949	0.7316	0.049	0.0539	0.3904	0.5096	0.0384	0.0428	0.3143
EWMA	0.8563	0.1489	0.1611	0.4452	0.8161	0.0497	0.0547	0.4652	0.6244	0.0389	0.0435	0.4076
Panel E: n=6000												
S	0.9672	0.1302	0.1409	0.4821	0.5737	0.0539	0.0589	0.3481	0.5772	0.0402	0.0437	0.3436
Id	0.9496	0.1301	0.1408	0.4813	0.6095	0.0575	0.0639	0.4076	0.5449	0.0402	0.0437	0.3342
VId	0.951	0.1301	0.1409	0.4815	0.5419	0.054	0.0589	0.3401	0.5483	0.0402	0.0437	0.3354
SI	0.9688	0.1302	0.1409	0.482	0.574	0.0539	0.0589	0.3479	0.5772	0.0402	0.0437	0.3434
CV	1.4142	0.1301	0.1408	1.0034	1.1436	0.054	0.0589	0.9706	1.1422	0.0402	0.0437	0.7031
CC	0.9656	0.1302	0.1409	0.4814	0.5709	0.0539	0.0589	0.3415	0.575	0.0402	0.0437	0.3368
EWMA	1.0432	0.1312	0.1422	0.5232	0.6946	0.0547	0.0599	0.4319	0.681	0.0407	0.0444	0.4229

In both tables, we compare the six target matrices by examining one risk-based portfolio at time and the effect of increasing p for fixed n. Special attention is devoted to the cases when $p > n$: the high-dimensional sample. We have this scenario only when $p = 100$ and $n = 60$. Here, the sample

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covariance matrix becomes ill-conditioned (Marčenko and Pastur 1967), thus it is interesting to evaluate gains obtained with shrinkage. The averaged Frobenius norm values in Table 3 give us a general overview about how target matrices perform across the whole shrinkage intensity spectrum in one goal. We aim to understand if, in average terms, shrinking the covariance matrix benefits risk-portfolio weights. On the other hand, the minimum Frobenius norm values help us understanding to what extent the various target matrices can help reproducing the *true* portfolio weights: the more intensity we need, the better is the target. In both tables, sample values are listed in the first row of each Panel.

Starting from Table 3, Panel A, the MV allocation seems better described by the Identity and the Variance Identity regardless the number of assets p. In particular, we look at the difference between the weights calculated entirely on the sample covariance matrix and the those of the targets: the Identity and the Variance Identity are the only estimator to perform better. In fact, shrinking towards the sample is not as bad as shrinking towards the Common Covariance. Increasing n and moving to Panel B, similar results are obtained. This trend is confirmed in Panel C, while in the cases of $n = 3000$ and $n = 6000$ all the estimators perform similarly. Hence, for the MV portfolio the Identity matrix works at best in reproducing portfolio weights very similar to the *true* ones. The same conclusions applies for the MD portfolio: when p and n are small, the Identity and the Variance Identity overperform other alternatives. On the other hand, we get very different results for the IV and ERC. Both portfolios seem not gaining benefits from the shrinkage procedure, as the Frobenius norm is very similar to that of the sample covariance matrix for all the target matrices under consideration. This is *true* for all pairs of p and n. In the high-dimensional case ($p = 100$; $n = 60$) the Identity matrix works best in reducing the distance between *true* and estimated portfolio weights, both for the MV and MD portfolios. In average, shrinkage does not help too much when alternative target matrices are used; only in the case of Common Covariance shrinking is worse than using the sample covariance matrix. All these effects vanish when we look at the IV and ERC portfolios: here, shrinkage does not help too much, whatever the target is.

Overall, the results are in line with the conclusions of the numerical illustrations in Section 3. Indeed, the MV portfolio shows the highest distance between *true* and estimated weights, similarly to the MD. Both portfolios are affected by the dimensionality of the sample: shrinkage always help in reducing weights misspecification; it improves in high-dimensional cases. On contrary, estimated weights for the IV and the ERC portfolios are close to the *true* ones by construction, hence, shrinkage does not help too much.

Switching to Table 4, results illustrate again the Identity and the Variance Identity attaining the best reduction of the Frobenius norm for the MV and MD portfolios. If results are similar to those of Table 3 for the MV, results for the MD show an improvement in using the shrinkage estimators. The Identity, Variance Identity, Common Covariance and Constant Correlation target matrices overperform all the alternatives, including the sample estimator, minimising the Frobenius norm in a similar fashion. This is true also for the high-dimensional case. On the contrary, the IV and the ERC do not benefit from shrinking the sample covariance matrix, even in high-dimensional samples, confirming Table 3 insights. Lastly, we look at the shrinkage intensity at which target matrices attain the highest Frobenius norm reduction. The intensity is comprised in the interval $[0; 1]$: the more it is close to 1, the more the target matrix helps in reducing the estimation error of the sample covariance matrix. interestingly, the Identity and the Variance Identity show shrinkage intensities always close to 1, meaning that shrinking towards them is highly beneficial, as they are fairly better than the sample covariance matrix. This is verified either for the high-dimensional case and for those risk portfolios (IV and ERC) who do not show great improvements from shrinkage.

Table 4. Frobenius norm for the portfolio weights. Values corresponds to the optimal shrinkage intensity, listed after the Frobenius norm for each portfolio. We report values for the sample covariance matrix ($\delta = 0$) separately in the first row of each panel. For each n, the first line reports the Frobenius norm for the sample covariance matrix. Abbreviations stand for: S for sample covariance; Id for identity matrix; VId for Variance Identity; SI for Single-Index; CV for Common Covariance; CC for Constant Correlation and EWMA for Exponentially Weighted Moving Average.

	P=10								P=50								P=100							
	MV		IV		ERC		MD		MV		IV		ERC		MD		MV		IV		ERC		MD	
Panel A: n=60																								
S	0.834	0	0.1585	0	0.1736	0	0.5842	0	0.7721	0	0.0573	0	0.0637	0	0.4933	0	0.7555	0	0.0409	0	0.0447	0	0.4565	0
Id	0.6778	0.7	0.1424	0.4	0.1525	0.8	0.501	0.8	0.5997	1	0.0558	1	0.0624	1	0.3704	1	0.471	1	0.0403	0.8	0.0446	1	0.3462	1
VId	0.6689	0.9	0.1581	0.5	0.173	0.9	0.5084	0.9	0.5539	1	0.0565	0.9	0.0627	1	0.3795	1	0.5428	1	0.0402	1	0.0437	1	0.3331	1
SI	0.8345	0.1	0.1585	0.1	0.1735	1	0.558	1	0.7666	1	0.0573	0.1	0.0637	0.1	0.4633	0.9	0.7479	1	0.0409	0.1	0.0447	0.1	0.4195	0.9
CV	1.2392	0.1	0.1581	0.5	0.1729	0.2	0.509	1	1.117	0.1	0.0565	0.9	0.0627	0.5	0.3795	1	1.1068	0.1	0.0402	1	0.0437	1	0.3331	1
CC	0.8335	0.3	0.1585	0.1	0.1731	1	0.5081	1	0.7733	0.1	0.0573	0.1	0.0634	1	0.3795	1	0.757	0.1	0.0409	0.1	0.0444	1	0.3332	1
EWMA	0.8331	0.1	0.1586	0.1	0.1737	0.1	0.5852	0.1	0.7706	0.1	0.0573	0.1	0.0637	0.1	0.4953	0.1	0.7213	1	0.0409	0.1	0.0447	0.1	0.4395	0.6
Panel B: n=120																								
S	0.9064	0	0.0877	0	0.0989	0	0.4649	0	0.7814	0	0.059	0	0.0656	0	0.5065	0	0.6519	0	0.0424	0	0.0472	0	0.4332	0
Id	0.8121	0.6	0.087	0.6	0.0981	0.8	0.4241	0.7	0.6119	1	0.0613	0.9	0.0685	1	0.4255	1	0.613	1	0.0388	0.9	0.0428	1	0.3111	1
VId	0.8121	0.8	0.087	0.8	0.0982	0.9	0.4242	0.9	0.6119	1	0.0613	0.9	0.0685	1	0.4255	1	0.4425	1	0.042	1	0.0467	1	0.3445	1
SI	0.907	0.1	0.0877	0.1	0.0989	1	0.4526	1	0.776	1	0.059	0.1	0.0656	1	0.4872	1	0.6431	1	0.0424	0.1	0.0472	0.1	0.414	0.9
CV	1.3269	0.1	0.087	0.8	0.0981	0.3	0.4245	1	1.1756	0.1	0.0586	0.9	0.0651	0.5	0.4302	1	1.0916	0.1	0.042	1	0.0467	1	0.3445	1
CC	0.9043	0.8	0.0877	0.1	0.0987	1	0.4241	0.9	0.781	0.2	0.059	0.2	0.0654	1	0.4302	1	0.6527	0.1	0.0424	0.1	0.0471	1	0.3446	1
EWMA	0.9052	0.1	0.0876	0.2	0.0988	0.2	0.4651	0.1	0.7797	0.2	0.0589	0.2	0.0655	0.2	0.5056	0.1	0.6554	0.1	0.0424	0.1	0.0472	0.1	0.4331	0.1
Panel C: n=180																								
S	0.7989	0	0.1311	0	0.1423	0	0.5007	0	0.7932	0	0.0564	0	0.0627	0	0.4631	0	0.6905	0	0.0404	0	0.044	0	0.4065	0
Id	0.7177	0.5	0.1307	0.7	0.1419	0.9	0.4724	0.8	0.6613	0.8	0.0562	0.6	0.0624	1	0.3977	0.9	0.534	1	0.0375	0.6	0.0398	1	0.3645	1
VId	0.718	0.7	0.1307	0.9	0.1419	1	0.4724	0.9	0.6614	0.9	0.0562	0.8	0.0624	1	0.3979	1	0.5428	1	0.0402	1	0.0437	1	0.3331	1
SI	0.799	0.2	0.1311	0.1	0.1423	1	0.4929	1	0.7897	0.7	0.0564	0.1	0.0627	1	0.4515	0.9	0.6863	0.8	0.0404	0.1	0.044	1	0.3955	0.8
CV	1.2715	0.1	0.1307	0.9	0.1418	0.4	0.4724	1	1.2073	0.1	0.0562	0.8	0.0624	0.4	0.3979	1	1.1422	0.1	0.0402	1	0.0437	0.9	0.3331	1
CC	0.7942	1	0.1311	0.1	0.1422	1	0.4725	1	0.7912	0.4	0.0564	0.1	0.0626	1	0.3977	1	0.6904	0.1	0.0404	0.1	0.0439	1	0.3331	1
EWMA	0.8035	0.1	0.1312	0.1	0.1424	0.1	0.5008	0.1	0.7951	0.1	0.0564	0.1	0.0626	0.1	0.4653	0.1	0.6938	0.1	0.0404	0.1	0.044	0.1	0.4074	0.1
Panel D: n=3000																								
S	0.7504	0	0.1476	0	0.1596	0	0.3957	0	0.734	0	0.049	0	0.0539	0	0.3988	0	0.513	0	0.0384	0	0.0428	0	0.3259	0
Id	0.7425	0.1	0.1477	0.1	0.1596	0.1	0.3941	0.1	0.6988	1	0.049	1	0.0538	1	0.3859	1	0.4573	1	0.0384	1	0.0428	1	0.3072	1
VId	0.7426	0.3	0.1476	0.1	0.1596	0.1	0.3941	0.3	0.6988	1	0.049	1	0.0538	1	0.3859	1	0.4573	1	0.0384	1	0.0428	1	0.3072	1
SI	0.7506	0.1	0.1476	0.1	0.1596	0.1	0.3953	1	0.7339	0.5	0.049	0.1	0.0539	1	0.3983	0.8	0.512	1	0.0384	0.2	0.0428	1	0.325	0.9
CV	1.2864	0.1	0.1476	0.1	0.1596	0.1	0.3951	1	1.2281	0.1	0.049	1	0.0538	1	0.3859	1	1.1041	0.1	0.0384	1	0.0428	1	0.3072	1
CC	0.7477	1	0.1476	0.1	0.1596	0.1	0.3946	0.7	0.7299	1	0.049	0.1	0.0539	1	0.386	1	0.5073	1	0.0384	0.2	0.0428	1	0.3072	1
EWMA	0.7615	0.1	0.1477	0.1	0.1597	0.1	0.3981	0.1	0.7439	0.1	0.0491	0.1	0.0539	0.1	0.4043	0.1	0.5263	0.1	0.0384	0.1	0.0429	0.1	0.3346	0.1
Panel E: n=6000																								
S	0.9672	0	0.1302	0	0.1409	0	0.4821	0	0.5737	0	0.0539	0	0.0589	0	0.3481	0	0.5772	0	0.0402	0	0.0437	0	0.3436	0
Id	0.9486	1	0.13	1	0.1408	1	0.4811	1	0.6085	0.7	0.0575	0.1	0.0639	0.1	0.4072	0.8	0.5428	1	0.0402	1	0.0437	1	0.3331	1
VId	0.9486	1	0.13	1	0.1408	1	0.4811	1	0.5365	1	0.054	0.1	0.0589	0.7	0.3381	1	0.5428	1	0.0402	1	0.0437	1	0.3331	1
SI	0.9675	0.1	0.1302	0.1	0.1409	1	0.482	1	0.5738	0.1	0.0539	0.1	0.0589	1	0.3478	1	0.5772	0.4	0.0402	0.1	0.0437	1	0.3433	0.8
CV	1.4142	0.1	0.13	1	0.1408	1	0.4811	1	1.1436	0.1	0.054	0.1	0.0589	0.1	0.3381	1	1.1422	0.1	0.0402	1	0.0437	1	0.3331	1
CC	0.9644	1	0.1302	0.1	0.1409	1	0.4812	1	0.5687	1	0.0539	0.1	0.0589	1	0.3381	1	0.5733	1	0.0402	0.9	0.0437	1	0.3331	1
EWMA	0.9765	0.1	0.1302	0.1	0.1409	0.1	0.4832	0.1	0.5901	0.1	0.054	0.1	0.059	0.1	0.3561	0.1	0.59	0.1	0.0402	0.1	0.0438	0.1	0.3524	0.1

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4.2.3 Sensitivity to shrinkage intensity

To have a view on the whole shrinkage intensity spectrum (i.e. the interval $[0;1]$) we refer to Figure 3, where we report the Frobenius Norms for the weights (y-axis) w.r.t. the shrinkage intensity (x-axis). Each column corresponds to a specific risk-based portfolio: from left to right, the Minimum Variance, the Inverse Volatility, the Equal-Risk-Contribution and the Maximum Diversification, respectively. Each row corresponds to the p/n ratio in n ascending order. For each subfigure, the Identity is blue o-shaped, the Variance Identity is green square-shaped, the Single-Index is red hexagram-shaped, the Common Covariance is black star-shaped, the Constant Correlation is cyan +-shaped and the EWMA is magenta diamond-shaped.

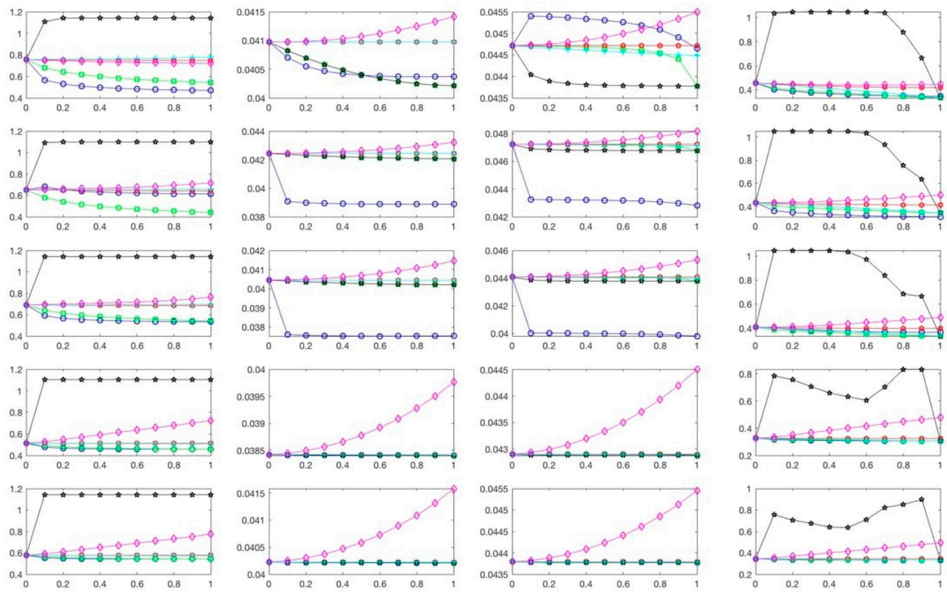

Figure 4. Frobenius norm for portfolio weights with respect to the shrinkage parameter, when $p = 100$.

Figure 4 illustrates the case $p = 100$, so to include the high-dimensional scenario. Starting from the latter (first row, $n = 60$), the Variance Identity is the only target matrix to always reduce weight misspecification for all the considered portfolios, for all shrinkage levels. The Identity do the same, excluding the ERC case where it performs worse than the sample covariance matrix. the remaining targets behave very differently across the four risk-based portfolios: the Common Covariance is the worst in both the MV and MD and the EWMA is the worst in both ERC and IV. The Market Model and the Constant Correlation do not improve much from the sample estimator across all portfolios.

Looking at the second row ($n = 120$), the Identity is the most efficient target, reducing the distance between estimated and *true* portfolio weights in all the considered portfolios. The Variance Identity is also very efficient in MV and MD portfolios, while the remaining targets show similar results as in the previous case. The same conclusions apply for the case $n = 180$.

When the number of observations is equal or higher than $n = 3000$, results do not change much. The Identity, the Variance Identity, the Market model and the Constant Correlation are the most efficient target matrices towards to shrink, while the EWMA is the worst for both IV and ERC portfolios and the Common Covariance is the worst for the MV and MD ones.

In conclusion, for the MV portfolio the Common Covariance should not be used, since it always produces weights very distant from the *true* ones being very unstable. At the same time, the EWMA should not be used to shrink the covariance matrix in the IV and ERC portfolios. The most convenient matrices towards which to shrink are the Identity and the Variance Identity. Overall, the MV and the MD portfolios gain more from shrinkage than the IV and ERC.

5. Conclusions

In this article, we provide a comprehensive overview about shrinkage in risk-based portfolios. Portfolios solely based on the asset returns covariance matrix are usually perceived as "robust" since they avoid to estimate the asset returns mean. However, they still suffer from estimation error when the sample estimator is used, affecting with misspecification the portfolio weights. Shrinkage estimators have been proved to reduce the estimation error by pulling the sample covariance towards a more structured target.

By the mean of an extensive Monte Carlo study, we compare six different target matrices: the Identity, the Variance Identity, the Single-index model, the Common Covariance, the Constant Correlation and the Exponential Weighted Moving Average, respectively. We do so considering their effects on weights for the Minimum Variance, Inverse Volatility, Equal-risk-contribution and Maximum diversification portfolios. Moreover, we control for the whole shrinkage intensity spectrum and for dataset size, changing observation length and number of assets. Therefore, we are able to *(i)* assess estimators' statistical properties and similarity with the *true* target matrix; *(ii)* address the problem of how selecting a specific target estimator impacts on the portfolio weights.

Regarding *(i)*, findings suggest the identity matrix held the best statistical properties, being well-conditioned across all the combinations of observations/assets, especially for high-dimensional dataset. Nevertheless, this target is not very similar to the *true* target matrix. The Single-Index and the Constant Correlation target matrices show the greater similarity with the *true* target matrix, minimizing the Frobenius norm, albeit they are poor-conditioned when observations and assets share similar sizes. Turning to *(ii)*, the identity attains the best results in terms of distance reduction between the *true* and estimated portfolio weights for both the Minimum Variance and Maximum Diversification portfolio construction techniques. The identity matrix is also stable against shifts in the shrinkage intensity.

Overall, selecting the target matrix is very important, since we verified there are large shifts in the distance between *true* and estimated portfolio weights when shrinking towards different targets. In risk-based portfolio allocations the Identity and the Variance Identity matrices represent the best target among the six considered in this study, especially in the case of Minimum Variance and Maximum Diversification portfolios. In fact, they are always well-conditioned and overperform their competitor in deriving the most similar weights to the *true* ones.

Lastly, findings confirm that the Minimum Variance and Maximum Diversification portfolios are more sensitive to misspecification in the covariance matrix, therefore they benefit the most when the sample covariance matrix is shrunk. Findings are in line to what previously found in (Ardia et al. 2017): the Inverse Volatility and the Equal-Risk-Contribution are more robust to covariance misspecification; hence, allocations do not improve significantly when shrinkage is used.

Conflicts of Interest: "The authors declare no conflict of interest."

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