ON THE DEGENERATE (h,q)-CHANGHEE NUMBERS AND POLYNOMIALS

YUNJAE KIM[1] AND JIN-WOO PARK[2]

ABSTRACT. In this paper, we investigate a new q-analogue of the higher order degenerate Changhee polynomials and numbers, which are called the Witt-type formula for the q-analogue of degenerate Changhee polynomials of order r. We can derive some new interesting identities related to the degenerate (h,q)-Changhee polynomials and numbers.

1. INTRODUCTION

Let p be chosen as a fixed odd prime number. Throughout this paper, we make use of the following notations. $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completions of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is defined $|p|_p = p^{-1}$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. The q-analogue of number x is defined as

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the *fermionic p-adic q-integral on $\mathbb{Z}_p$* is defined by Kim as follows :

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.1}$$

(see [1–6]). If we put f_1 to the translation of f with $f_1(x) = f(x+1)$, then, by (1.1), we get

$$qI_{-q}(f_1) + I_q(f) = [2]_q f(0). \tag{1.2}$$

As it is well-known fact, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)\, x^l, \tag{1.3}$$

2010 *Mathematics Subject Classification.* 33E20, 05A30, 11B65, 11S05.

Key words and phrases. (h,q)-Euler polynomials, degenerate (h,q)-Changhee polynomials, fermionic p-adic q-integral on $\mathbb{Z}_p$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 October 2018 doi:10.20944/preprints201810.0214.v1

and the *Stirling number of the second kind* is given by the generating function to be

$$\left(e^t-1\right)^m = m!\sum_{l=m}^{\infty} S_2\left(l,m\right)\frac{t^l}{l!},\ \ (\text{see } [7-9]). \tag{1.4}$$

By (1.3), we have

$$(\log(1+x))^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!},\ \ (n\geq 0). \tag{1.5}$$

Unsigned Stirling numbers of the first kind is given by

$$x^{(n)} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^{n}|S_1(n,l)|x^l. \tag{1.6}$$

Note that if we replace x to $-x$ in (1.3), then

$$\begin{aligned}(-x)_n =&(-1)^n x^{(n)} = \sum_{l=0}^{n} S_1(n,l)(-1)^l x^l\\ =&(-1)^n\sum_{l=0}^{n}|S_1(n,l)|x^l.\end{aligned} \tag{1.7}$$

Hence $S_1(n,l) = |S_1(n,l)|(-1)^{n-l}$.

D. S. Kim et. al. introduced the *Changhee polynomials of the first kind of order* r are defined by the generating function to be

$$\left(\frac{2}{2+t}\right)^r(1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x)\frac{t^n}{n!},\ \ (\text{see } [10,\ 11]), \tag{1.8}$$

and E. J. Moon et. al. defined the q-*Changhee polynomials of order* r as follows.

$$\left(\frac{1+q}{q(1+t)+1}\right)^r(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}^{(r)}(x)\frac{t^n}{n!},\ \ (\text{see } [12,\ 13]).$$

In [13], authors defined the generalization of the q-Changhee polynomials which are called by (h,q)-*Changhee polynomials of the first kind* and (h,q)-*Changhee polynomials of the second kind* respectively, defined by the generating function to be

$$\sum_{n=0}^{\infty} Ch_{n,q}^{(h)}(x)\frac{t^n}{n!} = \left(\frac{[2]_q}{q^{h+1}(1+t)+1}\right)(1+t)^x, \tag{1.9}$$

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,q}^{(h)}(x)\frac{t^n}{n!} = \left(\frac{[2]_q}{q^{h+1}++1+t}\right)(1+t)^{1-x}. \tag{1.10}$$

As is well known, the *Euler polynomials* are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},\ \ (\text{see } [1,\ 2,\ 5,\ 14\text{–}19]).$$

In [8], L. Carlitz first introduced the concept of degenerate numbers and polynomials which are related to Euler polynomials as follows

$$\sum_{n=0}^{\infty} E_n(x|\lambda)\frac{t^n}{n!} = \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}, \tag{1.11}$$

where $\lambda \in \mathbb{R}$. Note that, by (1.11), we know that

$$\lim_{\lambda\to 0}\sum_{n=0}^{\infty} E_n(x|\lambda)\frac{t^n}{n!} = \lim_{\lambda\to 0}\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r (1+\lambda t)^{\frac{x}{\lambda}}$$
$$= \left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

and thus, we get

$$\lim_{\lambda\to 0} E_n(x|\lambda) = E_n(x).$$

In the recent year, the degenerate of some special functions are investigated by many authors (see [8, 9, 20–25]). In particular, the *degenerate Changhee polynomials* which are defined by the generating function to be

$$\sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!} = \frac{2\lambda}{2\lambda+\log(1+\lambda t)}\left(1+\log(1+\lambda)^{\frac{1}{\lambda}}\right)^x, \text{ (see [25])}, \tag{1.12}$$

and T. Kim et. al. defined the *degenerate q-Changhee polynomials* as follows.

$$\sum_{n=0}^{\infty} Ch_{n,\lambda,q}(x)\frac{t^n}{n!} = \frac{q\lambda+\lambda}{q\log(1+\lambda t)+q\lambda+\lambda}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x, \text{ (see [22])}.$$

In the past decade, a number of researchers have studied the various generalization of Changhee polynomials (see [10–13, 24–28]), and in [13, 26], authors give new q-analogue of Changhee numbers and polynomials.

In this paper, we introduce a new q-analogue of the Changhee numbers and polynomials of the first kind and the second kind of order r, which are called the Witt-type formula for the q-analogue of Changhee polynomials of order r. We can derive some new interesting identities related to the q-Changhee polynomials of order r.

2. Degenerate q-Changhee polynomials with weight

Let assume that $\lambda \in \mathbb{R}$ with $|\lambda| < p^{-\frac{1}{p-1}}$. By (1.2), we get

$$\int_{\mathbb{Z}_p} q^{hy}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x+y} d\mu_{-q}(y) = \frac{1+q}{q^{h+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x. \tag{2.1}$$

where $h \in \mathbb{Z}$. By (2.1), we define the *degenerate q-Changhee polynomials with weight* by the generating function to be

$$\sum_{n=0}^{\infty} Ch_{n,q}^{(h)}(x|\lambda)\frac{t^n}{n!} = \frac{1+q}{q^{h+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x. \tag{2.2}$$

Note that

$$\lim_{\lambda\to 0}\sum_{n=0}^{\infty} Ch_{n,q}^{(h)}(x|\lambda)\frac{t^n}{n!} = \lim_{\lambda\to 0}\frac{1+q}{q^{h+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x$$
$$= \frac{q+1}{q^{h+1}(1+t)+1}(1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}^{(h)}(x)\frac{t^n}{n!},$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 10 October 2018 doi:10.20944/preprints201810.0214.v1

and so we know that

$$\lim_{\lambda\to 0} Ch_{n,q}^{(h)}(x|\lambda) = Ch_{n,q}^{(h)}(x), \tag{2.3}$$

and, if we put $h=-1$, then

$$\lim_{\lambda\to 0} Ch_{n,q}^{(-1)}(x|\lambda) = Ch_{n,q}(x). \tag{2.4}$$

In addition, by (1.12) and (2.1), we know that for each nonnegative integer n,

$$Ch_{n,\lambda}(x) = \frac{q^{h+1}}{2(1+q)} Ch_{n,q}^{(h)}(x|\lambda). \tag{2.5}$$

By (2.3), (2.4) and (2.5), we know that degenerate q-Changhee polynomials with weight are closely related to the q-Changhee polynomials or degenerate q-Changhee polynomials.

By (2.1), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{hy}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x+y} d\mu_{-q}(y)\\
=&\int_{\mathbb{Z}_p} q^{hy}\sum_{n=0}^{\infty}\binom{x+y}{n}\lambda^{-n}\left(\log(1+\lambda t)\right)^n d\mu_{-q}(y)\\
=&\int_{\mathbb{Z}_p} q^{hy}\sum_{n=0}^{\infty}\binom{x+y}{n}\lambda^{-n} n!\sum_{l=n}^{\infty} S_1(l,n)\frac{(\lambda t)^l}{l!} d\mu_{-q}(y)\\
=&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\lambda^{n-m} m! S_1(n,m)\int_{\mathbb{Z}_p} q^{hy}\binom{x+y}{m} d\mu_{-q}(y)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p} q^{hy}(x+y)_m d\mu_{-q}(y)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.6}$$

By (2.1) and (2.6), we have

$$Ch_{n,q}^{(h)}(x|\lambda) = \sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p} q^{hy}(x+y)_m d\mu_{-q}(y). \tag{2.7}$$

By (1.3), we get

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^{hy}(x+y)_m d\mu_{-q}(y) &= \sum_{l=0}^{m} S_1(m,l)\int_{\mathbb{Z}_p} q^{hy}(x+y)^l d\mu_{-q}(y)\\
&= \sum_{l=0}^{m} S_1(m,l) E_l^{(h)}(x|q),
\end{aligned} \tag{2.8}$$

where $E_n^{(h)}(x|q)$ is the nth *q-Euler polynomials with the weight* which are defined by the generating function to be

$$\begin{aligned}
\sum_{n=0}^{\infty} E_n^{(h)}(x|q)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{hy} e^{t(x+y)} d\mu_{-q}(y)\\
&= \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p} q^{hy}(x+y)^n d\mu_{-q}(y)\right)\frac{t^n}{n!}\\
&= \frac{q+1}{q^{h+1}e^t+1}e^{xt}, \text{ (see [29])}.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\frac{q+1}{q^{h+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\\
=&(q+1)\sum_{m=0}^{\infty}(-1)^{m-1}q^{m(h+1)}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{m}\\
=&(q+1)\sum_{m=0}^{\infty}(1-)^{m-1}q^{m(h+1)}\sum_{l=0}^{m}\lambda^{-l}\left(\log(1+\lambda t)\right)^{l}\\
=&(q+1)\sum_{m=0}^{\infty}q^{m(h+1)}\sum_{l=0}^{m}\lambda^{-l}\sum_{r=l}^{\infty}S_1(r,l)\lambda^{r}\frac{t^r}{r!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}(-1)^{m-1}q^{m(h+1)}\lambda^{n-m}S_1(n-m+l,l)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.9}$$

By (2.7),(2.8) and (2.9), we obtain the following theorem.

Theorem 2.1. *For each nonnegative integer n, we have*

$$\begin{aligned}
Ch_{n,q}^{(h)}(x|\lambda)&=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}q^{hy}(x+y)_m d\mu_{-q}(y)\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\lambda^{n-m}S_1(n,m)S_1(m,l)E_l^{(h)}(x|q),
\end{aligned}$$

and

$$Ch_{n,q}^{(h)}(\lambda)=\sum_{m=0}^{n}\sum_{l=0}^{m}(-1)^{m-1}q^{m(h+1)}\lambda^{n-m}S_1(n-m+l,l).$$

By replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ in (2.2), we have

$$\begin{aligned}
\frac{q+1}{q^{h+1}(1+t)+1}(1+t)^x&=\sum_{n=0}^{\infty}Ch_{n,q}^{(h)}(x|\lambda)\frac{1}{n!}\left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^n\\
&=\sum_{n=0}^{\infty}Ch_{n,q}^{(h)}(x|\lambda)\lambda^{-n}\frac{1}{\lambda}(e^t-1)^n\\
&=\left(\sum_{n=0}^{\infty}Ch_{n,q}^{(h)}(x|\lambda)\lambda^{-n}\frac{1}{n!}\right)\left(n!\sum_{l=n}^{\infty}S_2(l,n)\frac{(\lambda t)^l}{l!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}Ch_{m,q}^{(h)}(x|\lambda)S_2(n,m)\right)\frac{t^n}{n!},
\end{aligned} \tag{2.10}$$

and, thus, by (1.9) and (2.10), we have the following corollary.

Corollary 2.2. *For each nonnegative integer n, we have*

$$Ch_{n,q}^{(h)}(x)=\sum_{m=0}^{n}\lambda^{n-m}S_2(n,m)Ch_{m,q}^{(h)}(x|\lambda).$$

From (2.1) and (3.1), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(q^{h+1}Ch_{n,q}^{(h)}(x+1|\lambda)+Ch_{n,q}^{(h)}(x|\lambda)\right)\frac{t^n}{n!}\\
=&\frac{(q+1)}{q^{h+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\left(q^{h+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x}\\
=&(q+1)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x}\\
=&(q+1)\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(x)_m\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.11}$$

By (2.11), we obtain the following theorem.

Theorem 2.3. *For each nonnegative integer n, we get*

$$q^{h+1}Ch_{n,q}^{(h)}(x+1|\lambda)+Ch_{n,q}^{(h)}(x|\lambda)=(q+1)\sum_{m=0}^{n}(x)_m\lambda^{n-m}S_1(n,m).$$

For positive integer d with $d\equiv 1 \pmod 2$, if we put $f(x)=q^{hx}(1+\log(1+\lambda)^{\frac{x}{\lambda}}$, then, by (1.2), we have

$$\begin{aligned}
&q^d\int_{\mathbb{Z}_p}q^{hx}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{x+d}{\lambda}}d\mu_{-q}(x)+\int_{\mathbb{Z}_p}q^{hx}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{x}{\lambda}}d\mu_{-q}(x)\\
=&(q+1)\sum_{l=0}^{d-1}(-q)^l\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{l}{\lambda}}\\
=&(q+1)\sum_{l=0}^{d-1}(-q)^l\sum_{k=0}^{\infty}\binom{\frac{l}{\lambda}}{k}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^{k}\\
=&(q+1)\sum_{l=0}^{d-1}(-q)^l\sum_{k=0}^{\infty}\left(\frac{l}{\lambda}\right)_k\lambda^l\sum_{r=0}^{\infty}S_1(r+k,k)\frac{t^{r+k}}{(r+k)!}\\
=&\sum_{n=0}^{\infty}\left((q+1)\sum_{l=0}^{d-1}\sum_{k=0}^{n}(-q)^l\left(\frac{l}{\lambda}\right)_k\lambda^lS_1(n,k)\right)\frac{t^n}{n!}\\
=&\sum_{n=0}^{\infty}\left((q+1)\sum_{l=0}^{d-1}\sum_{k=0}^{n}(-q)^l\lambda^{l-k}(l)_{k,\lambda}S_1(n,k)\right)\frac{t^n}{n!},
\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}
&q^d\int_{\mathbb{Z}_p}q^{hx}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{x+d}{\lambda}}d\mu_{-q}(x)+\int_{\mathbb{Z}_p}q^{hx}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{\frac{x}{\lambda}}d\mu_{-q}(x)\\
=&\sum_{n=0}^{\infty}\left(q^dCh_{n,q}^{(h)}(d|\lambda)+Ch_{n,q}^{(h)}(\lambda)\right)\frac{t^n}{n!},
\end{aligned}\tag{2.13}$$

where $(l)_{k,\lambda}=l(l-\lambda)(l-2\lambda)\cdots(l-(k-1)\lambda)$.

By (2.12) and (2.13), we obtain the following theorem.

Theorem 2.4. *For each nonnegative integer n, we have*

$$q^d Ch_{n,q}^{(h)}(d|\lambda) + Ch_{n,q}^{(h)}(\lambda) = (q+1)\sum_{l=0}^{d-1}\sum_{k=0}^{n}(-q)^l \lambda^{l-k}(l)_{k,\lambda} S_1(n,k).$$

3. Higher order degenerate q-Changhee polynomials with weights

In this section, we consider the *higher order degenerate q-Changhee polynomials with weights* which are defined by

$$\begin{aligned} &Ch_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda) \\ &= \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) \underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r-times} q^{\sum_{i=1}^r h_i y_i}(x+y_1+\cdots+y_r)_m d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r), \end{aligned} \tag{3.1}$$

where n is a nonnegative integer, $h_1,\ldots,h_r \in \mathbb{Z}$ and $r \in \mathbb{N}$. By (3.1), we have

$$\begin{aligned} &\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i}(x+y_1+\cdots+y_r)_m d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\right)\frac{t^n}{n!} \\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\lambda^{n-m}m!S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i}\binom{x+y_1+\cdots+y_r}{m}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\frac{t^n}{n!} \\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i}\sum_{n=0}^{\infty}\binom{x+y_1+\cdots+y_r}{n}\lambda^{-n}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{(\lambda t)^l}{l!}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r) \\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i}\sum_{n=0}^{\infty}\binom{x+y_1+\cdots+y_r}{n}\lambda^{-n}\left(\log(1+\lambda t)\right)^n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r) \\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{x+y_1+\cdots+y_r} d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r) \\ &=\prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2), we know that

$$\sum_{n=0}^{\infty} Ch_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{t^n}{n!} = \prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x. \tag{3.3}$$

If we put

$$F_q^{(h_1,\ldots,h_r)}(x,t) = \prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x,$$

then

$$\begin{aligned}F_q^{(-1,\ldots,-1)}(x,t)&=\left(\frac{q+1}{\frac{1}{\lambda}\log(1+\lambda t)+2}\right)^r\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x\\&=\sum_{n=0}^{\infty}Ch_{n,q,\lambda}^{(r)}(x)\frac{t^n}{n!},\end{aligned}$$

and

$$\begin{aligned}\lim_{q\to 1}F_q^{(-1,\ldots,-1)}(x,t)&=\left(\frac{2}{\frac{1}{\lambda}\log(1+\lambda t)+2}\right)^r\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x\\&=\sum_{n=0}^{\infty}Ch_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, $F_q^{(h_1,\ldots,h_r)}(x,t)$ seems to be a new q-extension of the generating function for the Changhee polynomials of order r.

Note that

$$\begin{aligned}&\prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i+1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^x\\&=\left(\sum_{n=0}^{\infty}Ch_{n,q}^{(h_1,\ldots,h_r)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{x}{m}\lambda^{n-m}m!S_1(n,m)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{n-m}\binom{x}{l}\binom{n}{m}\lambda^{n-m-l}l!Ch_{m,q}^{(h_1,\ldots,h_r)}(\lambda)\right)\frac{t^n}{n!}.\end{aligned}\tag{3.4}$$

Since

$$\begin{aligned}&(x+y_1+\cdots+y_r)_n\\&=\sum_{l=0}^{n}S_1(n,l)(x+y_1+\cdots+y_r)^l\\&=\sum_{l=0}^{n}S_1(n,l)\sum_{\substack{l_1+\cdots+l_r=l\\l_1,\ldots,l_r\geq 0}}\binom{n}{l_1,l_2,\ldots,l_r}y_1^{l_1}y_2^{l_2}\cdots(x+y_r)^{l_r},\end{aligned}\tag{3.5}$$

where $\binom{n}{l_1,l+2,\ldots,l_r}=\frac{n!}{l_1!l_2!\cdots l_r!}$,

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}(x+y_1+\cdots+y_r)_m d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\sum_{l=0}^{m}S_1(m,l)(x+y_1+\cdots+y_r)^l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\sum_{l=0}^{m}S_1(m,l)\sum_{\substack{l_1+\cdots+l_r=l\\l_1,\ldots,l_r\geq 0}}\binom{m}{l_1,l_2,\ldots,l_r}y_1^{l_1}y_2^{l_2}\cdots(x+y_r)^{l_r}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\&=\sum_{l=0}^{m}S_1(m,l)\sum_{\substack{l_1+\cdots+l_r=l\\l_1,\ldots,l_r\geq 0}}\binom{m}{l_1,l_2,\ldots,l_r}E_{l_1}^{(h_1)}(q)\cdots E_{l_{r-1}}^{(h_{r-1})}(q)E_{l_r,q}^{(h_r)}(x|q),\end{aligned}\tag{3.6}$$

where $E_n^{(h)}(q) = E_n^{(h)}(0|q)$.

Thus, by (3.4) and (3.6), we obtain the following theorem.

Theorem 3.1. *For $n \geq 0$, we have*

$$\begin{aligned} & Ch_{n,q}^{(h_1,\dots,h_r)}(x|\lambda) \\ &= \sum_{m=0}^{n} \sum_{l=0}^{m} \sum_{\substack{l_1+\cdots+l_r=l \\ l_1,\dots,l_r\geq 0}} \lambda^{n-m} S_1(n,m) S_1(m,l) \binom{m}{l_1,l_2,\dots,l_r} E_{l_1}^{(h_1)}(q) \cdots E_{l_{r-1}}^{(h_{r-1})}(q) E_{l_r,q}^{(h_r)}(x|q). \end{aligned}$$

By replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$ in (3.3),

$$\begin{aligned} & \sum_{n=0}^{\infty} Ch_{n,q}^{(h_1,\dots,h_r)}(x|\lambda) \frac{1}{n!} \left(\frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right)^n \\ &= \sum_{n=0}^{\infty} Ch_{n,q}^{(h_1,\dots,h_r)}(x|\lambda) \frac{1}{n!} \lambda^{-n} n! \sum_{l=n}^{\infty} S_2(l,n) \frac{(\lambda t)^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} Ch_{n,q}^{(h_1,\dots,h_r)}(x|\lambda) \lambda^{n-m} S_2(n,m) \right) \frac{t^n}{n!}, \end{aligned} \tag{3.7}$$

and, by (1.9),

$$\begin{aligned} & \prod_{i=1}^{r} \left(\frac{q+1}{q^{h_i+1}(1+t)+1} \right) (1+t)^x \\ &= \left(\prod_{i=1}^{r-1} \left(\sum_{n=0}^{\infty} Ch_{n,q}^{(h_i)} \frac{t^n}{n!} \right) \right) \left(\sum_{n=0}^{\infty} Ch_{n,q}^{(h_r)}(x) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{\substack{l_1+\cdots+l_r=n \\ l_1,\dots,l_r\geq 0}} \binom{n}{l_1,\dots,l_r} Ch_{l_1,q}^{(h_1)} \cdots Ch_{l_{r-1},q}^{(h_{r-1})} Ch_{l_r,q}^{(h_r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{3.8}$$

Thus, by (3.7) and (3.8), we obtain the following theorem.

Theorem 3.2. *For $n \geq 0$, we have*

$$\sum_{\substack{l_1+\cdots+l_r=n \\ l_1,\dots,l_r\geq 0}} \binom{n}{l_1,\dots,l_r} Ch_{l_1,q}^{(h_1)} \cdots Ch_{l_{r-1},q}^{(h_{r-1})} Ch_{l_r,q}^{(h_r)}(x) = \sum_{m=0}^{n} Ch_{n,q}^{(h_1,\dots,h_r)}(x|\lambda) \lambda^{n-m} S_2(n,m).$$

4. Higher order degenerate q-Changhee polynomials of the second kind with weights

In this section, we consider the *higher order degenerate q-Changhee polynomials of the second kind with weights* is defined as follows:

$$\begin{aligned} & \widehat{Ch}_{n,q}^{(h_1,\dots,h_r)}(x|\lambda) \\ &= \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^{r} h_i y_i} (-x-y_1-\cdots-y_r)_m d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_r) \end{aligned} \tag{4.1}$$

where n is a nonnegative integer. In particular, $\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(0|\lambda)=\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(\lambda)$ are called the *higher order degenerate q-Changhee numbers of the second kind with weight*.

By (1.3) and (4.1), it leads to

$$\begin{aligned}
&\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\\
&=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}(-x-y_1-\cdots-y_r)_m d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}(-1)^m(x+y_1+\cdots+y_r)^{(m)}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&=\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}(-1)^m\sum_{l=0}^{m}|S_1(m,l)|(x+y_1+\cdots+y_r)^l d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}\lambda^{n-m}S_1(n,m)|S_1(m,l)|(-1)^m\\
&\quad\times\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\ldots,l_r\geq 0}}\binom{l}{l_1,\ldots,l_r}\prod_{i=1}^{r-1}\int_{\mathbb{Z}_p}q^{h_iy_i}y_i^{l_i}d\mu_{-q}(y_i)\int_{\mathbb{Z}_p}q^{h_ry_r}(x+y_r)^{l_r}d\mu_{-q}(y_r)\\
&=\sum_{m=0}^{n}\sum_{l=0}^{m}m\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\ldots,l_r\geq 0}}\binom{l}{l_1,\ldots,l_r}\lambda^{n-m}(-1)^lS_1(n,m)S_1(m,l)\left(\prod_{i=1}^{r-1}E_{l_i}^{(h_i)}(q)\right)E_{l_r}^{(h_r)}(x|q).
\end{aligned}\tag{4.2}$$

Thus, we state the following theorem.

Theorem 4.1. *For $n\geq 0$, we have*

$$\begin{aligned}
\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)=&\sum_{m=0}^{n}\sum_{l=0}^{n}\sum_{\substack{l_1+\cdots+l_r=l\\ l_1,\ldots,l_r\geq 0}}\binom{l}{l_1,\ldots,l_r}\lambda^{n-m}(-1)^lS_1(n,m)S_1(m,l)\\
&\times\left(\prod_{i=1}^{r-1}E_{l_i}^{(h_i)}(q)\right)E_{l_r}^{(h_r)}(x|q).
\end{aligned}$$

Now, we consider the generating function of the higher order degenerate q-Changhee polynomials of the second kind with weights as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\lambda^{n-m}S_1(n,m)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_1}(-x-y_1-\cdots-y_r)_n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\frac{t^n}{n!}\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\sum_{n=0}^{\infty}\binom{-x-y_1-\cdots-y_r}{n}\lambda^{-n}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{(\lambda t)^l}{l!}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\sum_{n=0}^{\infty}\binom{-x-y_1-\cdots-y_r}{n}\lambda^{-n}\left(\log(1+\lambda t)\right)^n d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\sum_{i=1}^{r}h_iy_i}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{-x-y_1-\cdots-y_r}d\mu_{-q}(y_1)\cdots d\mu_{-q}(y_r)\\
&=\prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i+1}+1+\frac{1}{\lambda}\log(1+\lambda t)}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{r-x}.
\end{aligned}\tag{4.3}$$

By replacing t by $\frac{1}{\lambda}\left(e^{\lambda t}-1\right)$, we have

$$\begin{aligned}
\left(\prod_{i=1}^{r}\frac{[2]_q}{q^{h_i+1}+(1+t)}\right)(1+t)^{r-x}&=\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^n}{n!}\\
&=\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{1}{n!}\lambda^{-n}n!\sum_{l=n}^{\infty}S_2(l,n)\frac{(\lambda x)^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda)\lambda^{n-m}S_2(n,m)\right)\frac{x^n}{n!},
\end{aligned}\tag{4.4}$$

and, by (1.10), we get

$$\begin{aligned}
&\left(\prod_{i=1}^{r}\frac{[2]_q}{q^{h_i+1}+(1+t)}\right)(1+t)^{r-x}\\
&=\left(\prod_{i=1}^{r-1}\left(\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_i)}\frac{t^n}{n!}\right)\right)\left(\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_r)}(x)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{\substack{i_1+\cdots+i=n\\ i_1,\ldots,i_r\geq 0}}\binom{n}{i_1,\ldots,i_r}\widehat{Ch}_{i_1,q}^{(h_1)}\cdots\widehat{Ch}_{i_{r-1},q}^{(h_{r-1})}\widehat{Ch}_{i_r,q}^{(h_r)}(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{4.5}$$

By (4.4) and (4.5), we obtain the following theorem.

Theorem 4.2. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} \widehat{Ch}_{m,q}^{(h_1,\ldots,h_r)}(x|\lambda)\lambda^{n-m}S_2(n,m) = \sum_{\substack{i_1+\cdots+i=n\\ i_1,\ldots,i_r\geq 0}} \binom{n}{i_1,\ldots,i_r}\widehat{Ch}_{i_1,q}^{(h_1)}\cdots\widehat{Ch}_{i_{r-1},q}^{(h_{r-1})}\widehat{Ch}_{i_r,q}^{(h_r)}(x).$$

Note that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\frac{t^n}{n!}\\
&=\prod_{i=1}^{r}\left(\frac{1+q}{q^{h_i+1}+1+\frac{1}{\lambda}\log(1+\lambda t)}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{r-x}\\
&=\left(\prod_{i=1}^{r}\frac{(1+q)q^{-h_i-1}}{q^{-h_1-1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)+1}\right)\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)^{r-x}\\
&=\left(\prod_{i=1}^{r-1}q^{-h_i-1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)\left(\sum_{n=0}^{\infty}Ch_{n,q}^{(-h_1-2,\ldots,-h_{r-1}-2)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\widehat{Ch}_{n,q}^{(h_r)}(x|\lambda)\frac{t^n}{n!}\right)\\
&=\left(\prod_{i=1}^{r-1}q^{-h_i-1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)\left(\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}Ch_{m,q}^{(-h_1-2,\ldots,-h_{r-1}-2)}(\lambda)\widehat{Ch}_{n-m}^{(h_r)}(x|\lambda)\right)\frac{t^n}{n!}\right),
\end{aligned}$$

and thus we know that

$$\begin{aligned}
&\widehat{Ch}_{n,q}^{(h_1,\ldots,h_r)}(x|\lambda)\\
&=\left(\prod_{i=1}^{r-1}q^{-h_i-1}\left(1+\frac{1}{\lambda}\log(1+\lambda t)\right)\right)\left(\sum_{m=0}^{n}\binom{n}{m}Ch_{m,q}^{(-h_1-2,\ldots,-h_{r-1}-2)}(\lambda)\widehat{Ch}_{n-m}^{(h_r)}(x|\lambda)\right).
\end{aligned}$$

5. Conclusion

The Changhee polynomials were defined by T. Kim, and have been attempted the various generalizations by many researchers (see [10–13, 24–28]). The Changhee numbers(q-Changhee numbers, respectively) are closely relate with the Euler numbers(q-Euler numbers), the Stirling numbers of the first kind and second kind and the harmonic numbers, et. al. which are interesting numbers of the combinatorics, pure and applied mathematics.

In this paper, we defined two types of the degenerate (h, q)-Changhee polynomials and number, and found the relationship between the Stirling numbers of the first kind and second kind, q-Euler numbers, q-Changhee numbers and those polynomials and numbers. It is a further problem to find the relationship between some special polynomials and degenerate (h, q)-Changhee polynomials.

6. Competing interests

The authors declare that they have no competing interests.

7. Author's Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

References

[1] T. Kim, *q-Euler numbers and polynomials associated with p-adic q-integral*, J. Nonlinear Math. Phys., **14** (2007), no. 1, 15-27.

[2] T. Kim, *Some identities on the q-Euler polynomials of higher order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys., **16** (2009), no. 4, 484-491.

[3] T. Kim, *On q-analogye of the p-adic log gamma functions and related integral*, J. Number Theory, **76** (1999), no. 2, 320-329.

[4] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys., **9** (2002), no. 3, 288-299.

[5] T. Kim, *A study on the q-Euler numbers and the fermionic q-integrals of the product of several type q-Bernstein polynomials on $\mathbb{Z}_p$*, Adv. Stud. Contemp. Math., **23** (2013), 5-11.

[6] D. S. Kim and T. Kim, *Some p-adic integrals on $\mathbb{Z}_p$ associated with trigonometric function*, Russ. J. Math. Phys., **25** (2018), No. 3, 300-308.

[7] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[8] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., **15** (1979), 51-88.

[9] D. S. Kim, T. Kim and G. W. Jang, *A note on degenerate Stirling numbers of the first kind*, Proc. Jangjeon Math. Soc., **21** (2018), No. 3, 394-404.

[10] D. S. Kim, T. Kim, J. J. Seo, *A note on Changhee polynomials and numbers*, Adv. Stud. Theor. Phys., **7** (2013), no. 20, 993-1003.

[11] D. S. Kim, T. Kim, J. J. Seo, S. H. Lee, *Higher-order Changhee numbers and polynomials*, Adv. Stud. Theor. Phys., **8** (2014), no. 8, 365-373.

[12] E. -J. Moon, J. W. Park, *A note on the generalized q-Changhee numbers of higher order*, J. Comput. Anal. Appl., **20** (2016), no. 3, 470-479.

[13] J. W. Park and J. Kwon, *A note on the (h,q)-Changhee numbers and polynomials*, submitted.

[14] A. Bayad, T. Kim, *Identities involving values of Bernstein, q- Bernoulli, and q-Euler polynomials*, Russ. J. Math. Phys., **18** (2011), 133-143.

[15] S. Gaboury, R. Tremblay, B.-J. Fugere, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials,* Proc. Jangjeon Math. Soc.**17** (2014), no. 1, 115-123.

[16] D. S. Kim, T. Kim, *Some symmetric identites for the higher-order q-Euler polynomials related to symmetry group S_3 arising from p-adic q-fermionic integral on $\mathbb{Z}_p$,* Filomat, **30** (2016), no. 7, 1717-1721.

[17] A. Sharma, *q-Bernoulli and Euler numbers of higher order,* Duke Math.J. **25** (1958), 343-353.

[18] H. Srivastava, *Some generalizations and basic (or q-)extensions of the Bernoulli, Euler and Genocchi polynomials*, Appl. Math. Inf. Sci., **5** (2011), 390-444.

[19] Z. Zhang, H. Yang, *Some closed formulas for generalized Bernoulli-Euler numbers and polynomials,* Proc. Jangjeon Math. Soc. **11** (2008), no. 2, 191-198.

[20] D. S. Kim, T. Kim and D. V. Dolgy, *Degenerate poly-Cauchy polynomials with a q-parameter*, J. Inequal. Appl., **2015**, 2015:364, 15 pp.

[21] T. Kim and D. S. Kim, *Identities for degenerate Bernoulli polynomials and Korobov polynomials of thd first kind*, Sci. China Math., (2017), doi : 10.1007/s11425-018-9338-5.

[22] T. Kim and D. S. Kim, *Degenerate Laplace transform and degenerate gamma function*, Russ. J. Math. Phys., **24** (2017), no. 2, 241-248.

[23] T. Kim, D. S. Kim and D. V. Dolgy, *Degenerate q-Euler polynomials*, Adv. Difference Equ., **2015** (2015), 13662.

[24] T. Kim, H. I. Kwon and J. J. Seo, *Degenerate q-Changhee polynomials* , J. Nonlinear Sci. Appl., **9** (2016), 2389-2393.

[25] H. I. Kwon, T. Kim and J. J. Seo, *A note on degenerate Changhee numbers and polynomials*, Proc. Jangjeon Math. Soc. **18** (2015), no. 3, 295-305.

[26] S. Arici, E. Ağyüz and M. Acikgoz, *On a q-analogue of some numbers and polynomials*, J. Ineqal. Appl., **2015**, 2015:19.

[27] D. V. Dolgy, T. Kim, S. H. Rim, J. J. Seo, *A note on Changhee polynomials and numbers with q-parameter*, Int. J. Math. Anal., **8** (2014), no. 26, 1255-1264.

[28] T. Kim, T. Mansour, S. H. Rim, J. J. Seo, *A note on q-Changhee polynomials and numbers*, Adv. Stud. Theor. Phys., **8** (2014), no. 1, 35-41.

[29] C. S. Ryoo, *The structure of the roots of (h,q)-Euler polynomials*, Far East J. Math. Sci., **48** (2011), 131-138.

 doi:10.20944/preprints201810.0214.v1

[1] Department of Mathematics, Dong-A University, Busan, 49315, Republic of Korea.
E-mail address: kimholzi@gmail.com

[2] Department of Mathematics Education, Daegu University, Gyeongsan-si, Gyeongsangbuk-do, 712-714, Republic of Korea.
E-mail address: a0417001@knu.ac.kr