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Analog of Hayman Conjecture for linear difference polynomials

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Abstract

We investigate the value distribution of difference polynomials of entire and meromorphic functions, which can be viewed as the Hayman's conjecture. And we also study the uniqueness of difference polynomials sharing a common value.

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INTRODUCTION AND DEFINITIONS:

A meromorphic (respectively entire) function always means a non-constant function meromorphic (respectively analytic) in the complex plane. Nevanlinna theory of value distribution is concerned with the density of points where a meromorphic function takes a certain value in the complex plane. It is also assumed that the reader is familiar with the basic concepts of Nevanlinna Theory, see e.g. ([4],[9]), such as the characteristic function T(r, f), proximity function m(r, f), counting function N(r, f) and so on. In addition, S(r, f) denotes any quantity that satisfies the condition that S(r, f)= o(T(r, f)) as r tends to infinity outside of a possible exceptional set of finite logarithimic measure. In the sequel, a meromorphic function a(z)is called a small function with respect to f if and only if T[r, a (z)] = o(T(r, f)) as r tends to infinity outside of a possible exceptional set of finite logarithimic measure. We denote by S(f), the family of all such small meromorphic functions.

We say that two meromorphic functions f and g share the value a (belonging to extended complex plane) CM (IM)

provided that

$$f(z) \equiv a$$

if and only if

$$g(z) \equiv a,$$

counting multiplicity (ignoring multiplicity).

DEFINITION 1 :
Let c be a non-zero complex costant then for a meromorphic function f(z) , we define its shift by f(z+c) and its difference operator by

$$\Delta_c f(z) = f(z+c) - f(z),$$

$$\Delta_{mc} f(z) = f(z+mc) - f(z),$$

where m is a positive integer

$$\Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)),$$

n$\in$ $\mathbf{N}, n \geq 2,$

$$= \sum_{k=0}^{n} \frac{(-1)^k.n!}{k!.(n-k)!} f(z + \overline{n-k}.c).$$

In particular,

$$\Delta_c^n f(z) = \Delta^n f(z)$$

for c=1.
We define **Differential - difference Monomial** as

$$M[f] = \prod_{i=0}^{k} \prod_{j=0}^{m} [f^{(j)}(z+c_{ij})]^n ij$$

where c_{ij} are complex constants , and n_{ij} are natural numbers , i= 0, 1, ... ,k and j=0, 1, ... ,m.

Then the degree of M[f] will be the sum of all the powers in the product on the right hand side.

DEFINITION 2 : Let

$$M_1[f], M_2[f], ...$$

denote the distinct monomials in f, and

$$a_1(z), a_2(z), ...$$

be the small meromorphic functions including complex numbers then

$$P[f] = P[z, f] = \sum_{j \in \Delta} a_j(z).M_j[f]$$

where Δ is a finite set of multi- indices, $a_j(z)$ are small functions of f, $M_j[f]$ are differential- difference monomials,
will be called a differential- difference polynomial in f, which is a finite sum of products of f , derivatives of f, their shifts, and derivatives of its shifts. **W**e define the total degree d of P[z, f] in f as

$$d = \underbrace{Max.}_{j \in \Delta} d_{M_j}.$$

If all the terms in the summation of P[f] have same degrees, then P[f] is known as homogeneous differential- difference polynomial. Usually, we take P[f] such that T(r, P) $\neq S(r, f)$.

Linear Difference Polynomial is defined as the Difference polynomial of degree one e.g.

$$\Delta_c^n f(z).$$

Uniqueness Theory of Meromorphic functions is an important part of Nevanlinna Theory. Recently number of papers have focussed on the Nevanlinna Theory with respect to difference operators. Then many authors started to investigate the uniqueness of meromorphic functions sharing values with their shifts or difference operators.
The classical result due to Nevanlinna theory of meromorphic functions is the five point theorem i.e. if two non-constant meromorphic functions f and g share five distinct values ignoring multiplicities(IM) then

$$f(z) \equiv g(z).$$

The number 5 is best possible. If the number of shared values is decreased, then an additional assumptions on value distribution needs to be introduced in order to obtain uniqueness.

DEFINITION 3 :

Let k be a positive integer and a be a complex number. We denote by $N_{k)}(r, 1/(f-a))$, the counting function of a- points of f with multiplicity $\leq k$, by $N_{(k}(r, 1/(f-a))$, the counting function of a- points of f with multiplicity $\geq k$ then

Set $N_k(r, 1/(f-a)) = \bar{N}(r, 1/(f-a)) + \bar{N}_{(2}(r, 1/(f-a)) + ... + \bar{N}_{(k}(r, 1/(f-a))$

A finite value a is called the Picard exceptional value of f, if f - a has no zeros. The Picard theorem shows that a transcendental entire function has at most one Picard exceptional value, a transcendental meromorphic functions has at most two picard exceptional values. The Hayman conjecture [4] is that if f is

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a transcendental meromorphic function and n $\in$ N, *then* f$^n f'$ takes every finite non-zero value infinitely often which means that the Picard exceptional value of f$^n f'$ may only be zero. Laine and Yang[5] has proved this conjecture for shifts and difference operators as following:

THEOREM A[5]: Let f be a transcendental entire function with finite order and c be a non-zero complex constant. Then for n $\geq$ 2, f(z)$^n f(z+c)$ assumes every non-zero complex value a infinitely often.
Liu K. et. al.[6] proved the above result for the meromorphic functions and obtained the follwing result:

THEOREM B[6]: Let f be a transcendental meromorphic function function with finite order and c be a non-zero complex constant. Then for n $\geq$ 6, f(z)$^n f(z+c) - a(z)$ has infinitely many zeros.

THEOREM C[6]: Let f be a transcendental meromorphic function function with finite order and c be a non-zero complex constant. Then for n $\geq$ 7, then the difference polynomial f(z)$^n[f(z+c) - f(z)]a(z)$ has infinitely many zeros.

We will consider the general linear difference polynomials in place of shift or difference operator and prove the following results:

MAIN RESULTS:

THEOREM 1.1: Let f be a transcendental entire function with finite order and as in definition 2, P[f] be a linear difference polynomial defined as $P[f] = c_0 f(z) + c_1 f(z+c) + c_2 f(z+2c) + ... + c_n f(z+nc); T(r, P[f]) \neq$ S(r, f), where c$\neq$ 0 and c$_j, j = 0, 1, ..., n$, are complex constants then f$^l P[f] - a(z), a(z) \neq$ 0, ∞ has infinitely many zeros provided l > 2n + 1.

THEOREM 1.2: Let f be a transcendental meromorphic function with finite order and as in definition 2, P[f] be a linear difference polynomial defined as $P[f] = c_0 f(z) + c_1 f(z+c) + c_2 f(z+2c) + ... + c_n f(z+nc); T(r, P[f]) \neq$ S(r, f), where c$\neq$ 0 and c$_j, j = 0, 1, ..., n$, are complex constants then f$^l P[f] - a(z), a(z) \neq$ 0, ∞ has infinitely many zeros provided l > 4n + 3.

SOME COROLLARIES:

1. If n = 0 in above theorems and P[f] = f(z+c), then the following results are improvement and generalizations of Theorem A and Theorem B:
i. for transcendental entire function f, f$^l f(z+c)$ assumes every non-zero complex value 'a' infinitely often provided l > 1.
ii. In case of transcendental meromorphic function f, f$^l f(z+c)$ assumes every non-zero and finite complex value 'a' infinitely often provided l > 3.

2. If n = 1 in above theorems and P[f] = f(z+c) - f(z) , then

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i. for transcendental entire function f, $f^l[f(z+c)-f(z)]$ $(\neq S(r,f))$ assumes every non-zero complex value 'a' infinitely often provided l > 3.
ii. In case of transcendental meromorphic function f, $f^l[f(z+c)-f(z)]$ $(\neq S(r,f))$ assumes every non-zero and finite complex value 'a' infinitely often provided l > 7.

3. Similar results can be obtained for $f^l[\Delta_c^n f(z)]$ for all n.

EXAMPLES:

1. Let f(z) = $e^{zi}+1, c = \Pi$ then f(z).f(z + c) $\neq$ 1 identically. Therefore, Cor. 1 (i) does not hold for l= 1.

2. Let f(z) = tan z, c = $\frac{\Pi}{2}$, $f^3.f(z+c) = -tan^2 z \neq 1$ identically. So cor. 1(ii) does not hold for l = 3.

3. Let f(z) = $\frac{e^z+1}{e^z-1}$, c= Πi then $f(z)^2.f(z+c) \neq$ -1 identically. So cor. 1(ii) does not hold for l = 2.

4. Let f(z) = e$\uparrow -e^z$, then $f^2.f(z+c)-2=-1$ and order of f is infinite, where c is non-zero constant satisfying $e^c=-2$. So $f^2.f(z+c)-2$ has no zeros. This shows that the main results do not hold for infinite ordered f.

For the proof of the results we need the following lemmas:

LEMMA 1 ([2],[3]): Let f be a non- constant meromorphic function of finite order and c be a non- zero complex constant, then

$m(r, \frac{f(z+c)}{f(z)})$ = S(r, f),

for all r outside a possible exceptional set of finite logarithmic measure.

LEMMA 2 [1]: Let c be a non-zero complex constant, and let f be a meromorphic function of finite order then

T(r, f(z+c)) = T(r, f) + S(r, f)

N(r, f(z+c)) = N(r, f) + S(r, f)

N(r, 0, f(z+c)) = N(r, 0, f) + S(r, f)

LEMMA 3 ([8]): LetF and G be two non-constant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:

i. max.(T(r, F), T(r, G)) $\leq N_2(r,0,F)+N_2(r,0,G)+N_2(r,F)+N_2(r,G)+S(r,F)+S(r,G)$

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$ii. F \equiv G$

$iii. F.G \equiv 1$.

PROOF OF THEOREM 1.1:

Let G[z] = f$^l P[f]$ where f is an entire function and suppose G[z]- a(z), a(z)$\neq$ 0, ∞ has finitely many zeros. Then we get by using Lemma 1 and Lemma 2

T(r, G[z]) = T(r,f$^l[c_0 f(z) + c_1 f(z+c) + c_2 f(z+2c) + ... + c_n f(z+nc)]$)

=T(r,f$^{l+1}[c_0 + c_1\frac{f(z+c)}{f(z)} + c_2\frac{f(z+2c)}{f(z)} + ... + c_n\frac{f(z+nc)}{f(z)}]$

$\geq$ (l + 1) T(r, f) - T(r, [c$_0 + c_1\frac{f(z+c)}{f(z)} + c_2\frac{f(z+2c)}{f(z)} + ... + c_n\frac{f(z+nc)}{f(z)}$])

But T(r, [c$_0 + c_1\frac{f(z+c)}{f(z)} + c_2\frac{f(z+2c)}{f(z)} + ... + c_n\frac{f(z+nc)}{f(z)}$])

$\leq$ T(r, $\frac{f(z+c)}{f(z)}$) + T(r, $\frac{f(z+2c)}{f(z)}$) + ... + T(r, $\frac{f(z+nc)}{f(z)}$) + S(r, f)

= N(r, $\frac{f(z+c)}{f(z)}$) + N(r, $\frac{f(z+2c)}{f(z)}$) + ... + N(r, $\frac{f(z+nc)}{f(z)}$) + S(r, f)

= N(r, $\frac{1}{f(z)}$) + N(r, $\frac{1}{f(z)}$) + ... + N(r, $\frac{1}{f(z)}$) + S(r, f)

$\leq$ n T(r, f)+ S(r, f)

Therefore, we have

T(r, G[z])
$\geq$ (l + 1) T(r, f) - n T(r, f)+ S(r, f) ...(1)

Since f is entire, therefore, by using Nevanlinna's second main theorem and lemma , we get

[l + 1 - n] T(r, f) $\leq$ T(r, G[z]) $\leq \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, \frac{1}{G(z)-a(z)})$ + S(r, G)

= $\bar{N}(r, \frac{1}{G(z)})$ + S(r, f)

$\leq$ (n + 2) N(r, 0, f)+$S(r, f)$

$\leq$ (n + 2) T(r, f) +$S(r, f)$

So we get

l T(r, f) $\leq$ (2n + 1)T(r, f) +$S(r, f)$

which is a contradiction as l > 2n + 1. Thus our supposition is wrong and hence, $f^l P[f]-a(z), a(z) \neq 0, \infty$ has infinitely many zeros.

PROOF OF THEOREM 1.2:

Let G[z] = $f^l P[f]$ where f is a meromorphic function and suppose G[z]- a(z), a(z)$\neq$ 0, ∞ has finitely many zeros. Then we get by using Lemma 1 and Lemma 2

$$T(r, G[z]) = T(r, f^l[c_0 f(z)+c_1 f(z+c)+c_2 f(z+2c)+...+c_n f(z+nc)]\ )$$

$$= T(r, f^{l+1}[c_0+c_1\frac{f(z+c)}{f(z)}+c_2\frac{f(z+2c)}{f(z)}+...+c_n\frac{f(z+nc)}{f(z)}]$$

$$\geq (l+1)\ T(r, f) - T(r, [c_0+c_1\frac{f(z+c)}{f(z)}+c_2\frac{f(z+2c)}{f(z)}+...+c_n\frac{f(z+nc)}{f(z)}])$$

But $T(r, [c_0+c_1\frac{f(z+c)}{f(z)}+c_2\frac{f(z+2c)}{f(z)}+...+c_n\frac{f(z+nc)}{f(z)}])$

$$\leq T(r, \frac{f(z+c)}{f(z)}) + T(r, \frac{f(z+2c)}{f(z)}) + ... + T(r, \frac{f(z+nc)}{f(z)}) + S(r, f)$$

$$= N(r, \frac{f(z+c)}{f(z)}) + N(r, \frac{f(z+2c)}{f(z)}) + ... + N(r, \frac{f(z+nc)}{f(z)}) + S(r, f)$$

$$\leq 2n\ T(r, f) + S(r, f)$$

Therefore, we have

$$T(r, G[z]) \geq (l+1)\ T(r, f) - 2n\ T(r, f) + S(r, f)$$

Since f is meromorphic, therefore, by using Nevanlinna's second main theorem and lemma , we get

$$[\,l+1-2n]\ T(r, f) \leq T(r, G[z]) \leq \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, G(z)) + \bar{N}(r, \frac{1}{G(z)-a(z)}) + S(r, G)$$

$$= \bar{N}(r, \frac{1}{G(z)}) + \bar{N}(r, G(z)) + S(r, f)$$

$$\leq (2n+4)\ T(r, f) + S(r, f)$$

So we get

$$l\ T(r, f) \leq (4n+3)T(r, f) + S(r, f)$$

which is a contradiction as l > 4n + 3. Thus our supposition is wrong and hence, $f^l P[f]-a(z), a(z) \neq 0, \infty$ has infinitely many zeros.

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APPLICATIONS:

As applications of the above main results, we present the following result:

THEOREM 2.1: Let f and g be transcendental entire functions with finite order and as in definition 2, P[f] and P[g] be two linear difference polynomials defined as
$P[f] = c_0 f(z) + c_1 f(z+c) + c_2 f(z+2c) + ... + c_n f(z+nc); T(r, P[f]) \neq$ S(r, f), where c$\neq$ 0 and c$_j, j = 0, 1, ..., n,$ are complex constants, and $f^l P[f]$ and $g^l P[g]$ share a(z), a(z)$\neq$ 0, ∞ CM, then $f^l P[f] = g^l P[g]$ or $f^l P[f].g^l P[g] = (a(z))^2$. provided l > 3n + 5.

PROOF OF THEOREM 2.1:

Let F(z) = $\frac{f^l P[f]}{a(z)}$ and G(z) = $\frac{g^l P[g]}{a(z)}$, then F(z) and G(z) share 1 CM except the zeros or poles of a(z). We have by using Lemma 2,

N$_2(r, 0, F) = N_2(r, 0, f^l) + N_2(r, 0, P[f]) + S(r, f)$

$\leq N_{1)}(r, 0, f) + N(r, 0, f) + N(r, 0, P[f]) + S(r, f)$ by definition 3

$\leq (n+3)T(r, f) + S(r, f)$

Similarly, we have N$_2(r, 0, G) \leq (n+3)T(r, g) + S(r, g)$

By Lemma 3, suppose i, holds, then since f, g are entire functions

max.(T(r, F), T(r, G)) $\leq N_2(r, 0, F) + N_2(r, 0, G) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G)$

$\leq (n+3)[T(r, f) + T(r, g)] + S(r, f) + S(r, g)$

Thus, we have

T(r, F) + T(r, G) $\leq 2(n+3)[T(r, f) + T(r, g)] + S(r, f) + S(r, g)$

Using eq. 1 we have

[l + 1 - n] [T(r, f) + T(r, g)] $\leq$ T(r, F) + T(r, G)

$\leq 2(n+3)[T(r, f) + T(r, g)] + S(r, f) + S(r, g)$

Thus, $l.[T(r, f) + T(r, g)] \leq (3n+5)[T(r, f) + T(r, g)] + S(r, f) + S(r, g)$

which contradicts the given condition that l > 3n + 5.

Hence by Lemma 3, result holds.
REMARK: Similar result can be proved when f, g are meromorphic functions.

SOME COROLLARIES:
1. Let f and g be transcendental entire functions with finite order, c be non-zero complex constant and if F = f$^n f(z+c)$ and G = g$^n g(z+c)$ share 1 CM, then for $n > 5$, F$\equiv G$ or F.G $\equiv$ 1 which corresponds with the result in [7].

2. Let f and g be transcendental entire functions with finite order, c be non-zero complex constant and if F = f$^n[f(z+c) - f(z)]$ and G = g$^n[g(z+c) - g(z)]$ share 1 CM, then for $n > 8$, F$\equiv G$ or F.G $\equiv$ 1.

3. Similar results can be obtained for f$^l[\Delta_c^n f(z)]$ and g$^l[\Delta_c^n g(z)]$ for all n.

EXAMPLES:

1. Let f(z) = sin z and g(z) = cos z, l = 1, c = π, then $f^l P[f] = f^l f(z+c) =$ -sin$^2 z$ and $g^l P[g] = g^l g(z+c) =$ -cos$^2 z$. Here -sin$^2 z$ and -cos$^2 z$ share -1/2,CM which proves that the Theorem 2.1 may not be true when l = 1.

2. Let f(z) = e^z and g(z) = e$^{-z}, l = 1, c =\pi.i$, then $f^l P[f] = f^l f(z+c) =$ -e^{2z} and $g^l P[g] = g^l g(z+c) =$ -e^{-2z} Here -e^{2z} and -e^{-2z} share -1,CM and $f^l P[f].g^l P[g] = 1$. This holds for $l > 1$ too.

3. In case of meromorphic functions, let $f(z) = tanz$ and $g(z) = cotz$, l = 1, c = π, then $f^l P[f] = f^l f(z+c) = \tan^2 z$ and $g^l P[g] = g^l g(z+c) = \cot^2 z$. Here $\tan^2 z$ and $\cot^2 z$ share 1,CM and $f^l P[f].g^l P[g] = 1$.

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