CONNECTION PROBLEM FOR SUMS OF FINITE PRODUCTS OF CHEBYSHEV POLYNOMIALS OF THE THIRD AND FOURTH KINDS

DMITRY VICTOROVICH DOLGY[1], DAE SAN KIM[2], TAEKYUN KIM[3], AND JONGKYUM KWON[4]

ABSTRACT. This paper treats the connection problem of expressing sums of finite products of Chebyshev polynomials of the third and fourth kinds in terms of five classical orthogonal polynomials. In fact, by carrying out explicit computations each of them are expressed as linear combinations of Hermite, generalized Laguerre, Legendre, Gegenbauer, and Jacobi polynomials which involve some terminating hypergeometric functions ${}_2F_0, {}_2F_1$, and ${}_3F_2$.

1. INTRODUCTION AND PRELIMINARIES

In this section, we will recall some basic facts about relevant orthogonal polynomials that will be needed throughout this paper. For this, we will first fix some notations. For any nonnegative integer n, the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $< x >_n$ are respectively given by

$$(x)_n = x(x-1)\cdots(x-n+1),\ (n \geq 1),\ (x)_0 = 1, \tag{1.1}$$

$$< x >_n = x(x+1)\cdots(x+n-1),\ (n \geq 1),\ < x >_0 = 1. \tag{1.2}$$

The two factorial polynomials are evidently related by

$$(-1)^n(x)_n =< -x >_n,\ (-1)^n < x >_n = (-x)_n. \tag{1.3}$$

$$\frac{(2n-2s)!}{(n-s)!} = \frac{2^{2n-2s}(-1)^s < \frac{1}{2} >_n}{< \frac{1}{2} - n >_s},\ (n \geq s \geq 0). \tag{1.4}$$

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ (Re\ x, Re\ y > 0). \tag{1.5}$$

$$\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!},\ (n \geq 0). \tag{1.6}$$

2010 *Mathematics Subject Classification.* 11B83, 33C05, 33C20, 33C45.
Key words and phrases. Sums of finite products of Chebyshev polynomials of the third and fourth kinds, Hermite, generalized Laguerre, Legendre, Gegenbauer, Jacobi.

$$\frac{\Gamma(x+1)}{\Gamma(x+1-n)} = (x)_n, \frac{\Gamma(x+n)}{\Gamma(x)} = < x >_n, \ (n \geq 0), \tag{1.7}$$

where $\Gamma(x)$ and $B(x, y)$ are the gamma and beta functions respectively.
The hypergeometric function is defined by

$$\begin{aligned} & {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) \\ & = \sum_{n=0}^{\infty} \frac{< a_1 >_n \cdots < a_p >_n}{< b_1 >_n \cdots < b_q >_n} \frac{x^n}{n!}. \end{aligned} \tag{1.8}$$

We are now ready to state some basic facts about Chebyshev polynomials of the third kind $V_n(x)$, those of the fourth kind $W_n(x)$, Hermite polynomials $H_n(x)$, generalized (extended) Laguerre polynomials $L_n^{\alpha}(x)$, Legendre polynomials $P_n(x)$, Gegenbauer polynomials $C_n^{(\lambda)}(x)$, and Jacobi polynomials $P_n^{(\alpha,\beta)}$. All the necessary facts on those special polynomials can be found in [5-9,11,12]. For the full accounts of this fascinating area of orthogonal polynomials, the reader may refer to [2,3,21].

The above special polynomials are given in terms of generating functions by

$$F(t, x) = \frac{1-t}{1-2xt+t^2} = \sum_{n=0}^{\infty} V_n(x)t^n, \tag{1.9}$$

$$G(t, x) = \frac{1+t}{1-2xt+t^2} = \sum_{n=0}^{\infty} W_n(x)t^n, \tag{1.10}$$

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \tag{1.11}$$

$$(1-t)^{-\alpha-1} \exp(-\tfrac{xt}{1-t}) = \sum_{n=0}^{\infty} L_n^{\alpha}(x)t^n, \ (\alpha > -1), \tag{1.12}$$

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1.13}$$

$$\frac{1}{(1-2xt+t^2)^{\lambda}} = \sum_{n=0}^{\infty} C_n^{(\lambda)}(x)t^n, \ (\lambda > -\frac{1}{2}, \lambda \neq 0, |t| < 1, |x| \leq 1), \tag{1.14}$$

$$\frac{2^{\alpha+\beta}}{R(1-t+R)^{\alpha}(1+t+R)^{\beta}} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n, \tag{1.15}$$

$$(R = \sqrt{1-2xt+t^2}, \ \alpha, \beta > -1).$$

Explicit expressions for the above special polynomials are as in the following.

$$\begin{aligned} V_n(x) &= {}_2F_1(-n, n+1; \tfrac{1}{2}; \tfrac{1-x}{2}) \\ &= \sum_{l=0}^{n} \binom{2n-l}{l} 2^{n-l}(x-1)^{n-l}, \qquad (1.16) \\ W_n(x) &= (2n+1)\,{}_2F_1(-n, n+1; \tfrac{3}{2}; \tfrac{1-x}{2}) \\ &= (2n+1)\sum_{l=0}^{n} \frac{2^{n-l}}{2n-2l+1}\binom{2n-l}{l}(x-1)^{n-l}, \qquad (1.17) \\ H_n(x) &= n! \sum_{l=0}^{[\frac{n}{2}]} \frac{(-1)^l}{l!(n-2l)!}(2x)^{n-2l}, \qquad (1.18) \\ L_n^{\alpha}(x) &= \frac{<\alpha+1>_n}{n!}\,{}_1F_1(-n, \alpha+1; x) \\ &= \sum_{l=0}^{n} \frac{(-1)^l \binom{n+\alpha}{n-l}}{l!} x^l, \qquad (1.19) \\ P_n(x) &= {}_2F_1(-n, n+1; 1; \tfrac{1-x}{2}) \\ &= \frac{1}{2^n}\sum_{l=0}^{[\frac{n}{2}]} (-1)^l \binom{n}{l}\binom{2n-2l}{n} x^{n-2l}, \qquad (1.20) \\ C_n^{(\lambda)}(x) &= \binom{n+2\lambda-1}{n}\,{}_2F_1(-n, n+2\lambda; \lambda+\tfrac{1}{2}; \tfrac{1-x}{2}) \\ &= \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)k!(n-2k)!}(2x)^{n-2k}, \qquad (1.21) \\ P_n^{(\alpha,\beta)}(x) &= \frac{<\alpha+1>_n}{n!}\,{}_2F_1(-n, 1+\alpha+\beta+n; \alpha+1; \tfrac{1-x}{2}) \\ &= \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}(\tfrac{x-1}{2})^k(\tfrac{x+1}{2})^{n-k}. \qquad (1.22) \end{aligned}$$

Next, we state Rodrigues-type formulas for Hermite and generalized Laguerre polynomials and Rodrigues' formulas for Legendre, Gegenbauer and Jacobi polynomials.

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad (1.23)$$

$$L_n^{\alpha}(x) = \frac{1}{n!} x^{-\alpha} e^{x} \frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}), \qquad (1.24)$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \qquad (1.25)$$

$$(1-x^2)^{\lambda-\frac{1}{2}} C_n^{(\lambda)}(x) = \frac{(-2)^n}{n!}\frac{<\lambda>_n}{<n+2\lambda>_n}\frac{d^n}{dx^n}(1-x^2)^{n+\lambda-\frac{1}{2}}, \qquad (1.26)$$

$$(1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}(1-x)^{n+\alpha}(1+x)^{n+\beta}. \qquad (1.27)$$

The last thing we want to mention is the orthogonalities with respect to various weight functions enjoyed by Hermite, generalized Laguerre, Legendre, Gegenbauer and Jacobi polynomials.

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n! \sqrt{\pi} \delta_{n,m}, \tag{1.28}$$

$$\int_{0}^{\infty} x^{\alpha} e^{-x} L_n^{\alpha}(x) L_m^{\alpha}(x)\, dx = \frac{1}{n!}\Gamma(\alpha+n+1)\delta_{n,m}, \tag{1.29}$$

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1}\delta_{n,m}, \tag{1.30}$$

$$\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x)\, dx = \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2}\delta_{n,m}, \tag{1.31}$$

$$\begin{aligned} &\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x)\, dx \\ &= \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)}\delta_{n,m}. \end{aligned} \tag{1.32}$$

For convenience, let us put

$$\gamma_{n,r}(x) = \sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} \binom{r-1+n-l}{r-1} V_{i_1}(x) V_{i_2}(x) \cdots V_{i_{r+1}}(x), \ (n \geq 0, r \geq 1), \tag{1.33}$$

$$\mathcal{E}_{n,r}(x) = \sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{i_1}(x) W_{i_2}(x) \cdots W_{i_{r+1}}(x), \quad (n \geq 0, r \geq 1). \tag{1.34}$$

We observe here that both $\gamma_{n,r}(x)$ and $\mathcal{E}_{n,r}(x)$ have degree n.

In this paper, we will consider the connection problem of expressing the sums of finite products in (1.33) and (1.34) as linear combinations of $H_n(x), L_n^{\alpha}(x), P_n(x), C_n^{(\lambda)}(x)$, and $P_n^{(\alpha,\beta)}(x)$. These will be done by performing explicit computations based on Proposition 2.1. We observe here that the formulas in Proposition 2.1 follow from their orthogonalities, Rodrigues' and Rodrigues-type formulas and integration by parts.

Our main results are the following **Theorem 1.1** and **Theorem 1.2**.

Peer-reviewed version available at Symmetry 2018, 10, 617; doi:10.3390/sym10110617

Theorem 1.1. *Let n, r be any integers with $n \geq 0, r \geq 1$. Then we have the following.*

$$\begin{aligned}
&\sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} \binom{r-1+n-l}{r-1} V_{i_1}(x)V_{i_2}(x)\cdots V_{i_{r+1}}(x) \\
&= \frac{(2n+2r)!}{r!4^{n+r}(n+r-\frac{1}{2})_{n+r}} \sum_{k=0}^{n} \frac{(-2)^k}{(n-k)!} \\
&\times \sum_{j=0}^{[\frac{k}{2}]} \frac{{}_2F_1(2j-k, \frac{1}{2}-n-r; -2n-2r; 2)}{j!4^j(k-2j)!} H_{n-k}(x) \qquad (1.35) \\
&= \frac{1}{r!} \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{(-2)^{n-l}(2n+2r-l)!(n+r-l)!}{l!(2n+2r-2l)!(k-l)!} \\
&\times {}_2F_0(l-k, n-k+\alpha+1; -; 1) L_{n-k}^{\alpha}(x) \qquad (1.36) \\
&= \frac{(-1)^n n!(2n+2r)!}{r!4^r(n+r-\frac{1}{2})_{n+r}} \sum_{k=0}^{n} \frac{(-1)^k(2k+1)}{(n-k)!(n+k+1)!} \\
&\times {}_3F_2(k-n, \frac{1}{2}-n-r, -n-k-1; -2n-2r, -n; 1) P_k(x) \qquad (1.37) \\
&= \frac{(-1)^n(2n+2r)!4^{\lambda-r}\Gamma(\lambda)\Gamma(n+\lambda+\frac{1}{2})}{\sqrt{\pi}r!(n+r-\frac{1}{2})_{n+r}} \sum_{k=0}^{n} \frac{(-1)^k(k+\lambda)}{\Gamma(n+k+2\lambda+1)(n-k)!} \\
&\times {}_3F_2(k-n, \frac{1}{2}-n-r, -n-k-2\lambda; -2n-2r, -n-\lambda+\frac{1}{2}; 1) C_k^{(\lambda)}(x) \qquad (1.38) \\
&= \frac{(-1)^n(2n+2r)!\Gamma(n+\alpha+1)}{r!4^r(n+r-\frac{1}{2})_{n+r}} \sum_{k=0}^{n} \frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{(n-k)!\Gamma(\alpha+k+1)\Gamma(n+k+\alpha+\beta+2)} \\
&\times {}_3F_2(k-n, \frac{1}{2}-n-r, -n-k-\alpha-\beta-1; -2n-2r, -n-\alpha; 1) P_k^{(\alpha,\beta)}(x). \qquad (1.39)
\end{aligned}$$

Theorem 1.2. *Let n, r be any integers with $n \geq 0$, $r \geq 1$. Then we have the following.*

$$\begin{aligned}
&\sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{i_1}(x)W_{i_2}(x)\cdots W_{i_{r+1}}(x) \\
&= \frac{(2n+1)(2n+2r)!}{r!2^{2n+2r+1}(n+r+\frac{1}{2})_{n+r+1}} \sum_{k=0}^{n} \frac{(-2)^k}{(n-k)!} \\
&\times \sum_{j=0}^{[\frac{k}{2}]} \frac{{}_2F_1(2j-k, -n-r-\frac{1}{2}; -2n-2r; 2)}{j!4^j(k-2j)!} H_{n-k}(x) \qquad (1.40) \\
&= \frac{(2n+1)}{r!} \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{(-2)^{n-l}(2n+2r-l)!(n+r-l)!}{l!(2n+2r-2l+1)!(k-l)!} \\
&\times {}_2F_0(l-k, n-k+\alpha+1; -; 1) L_{n-k}^{\alpha}(x) \qquad (1.41)
\end{aligned}$$

$$= \frac{(-1)^n n!(2n+1)(2n+2r)!}{r!2^{2r+1}(n+r+\frac{1}{2})_{n+r+1}} \sum_{k=0}^{n} \frac{(-1)^k(2k+1)}{(n-k)!(n+k+1)!}$$
$$\times\, {}_3F_2(k-n,-n-r-\frac{1}{2},-n-k-1;-2n-2r,-n;1)P_k(x) \quad (1.42)$$
$$= \frac{(-1)^n(2n+2r)!2^{2\lambda-2r-1}(2n+1)\Gamma(\lambda)\Gamma(n+\lambda+\frac{1}{2})}{\sqrt{\pi}r!(n+r+\frac{1}{2})_{n+r+1}}$$
$$\times \sum_{k=0}^{n} \frac{(-1)^k(k+\lambda)}{\Gamma(n+k+2\lambda+1)(n-k)!}$$
$$\times\, {}_3F_2(k-n,-n-r-\frac{1}{2},-n-k-2\lambda;-2n-2r,-n-\lambda+\frac{1}{2};1)C_k^{(\lambda)}(x) \quad (1.43)$$
$$= \frac{(-1)^n(2n+2r)!(2n+1)\Gamma(n+\alpha+1)}{r!2^{2r+1}(n+r+\frac{1}{2})_{n+r+1}}$$
$$\times \sum_{k=0}^{n} \frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{(n-k)!\Gamma(\alpha+k+1)\Gamma(n+k+\alpha+\beta+2)}$$
$$\times\, {}_3F_2(k-n,-n-r-\frac{1}{2},-n-k-\alpha-\beta-1;-2n-2r,-n-\alpha;1)P_k^{(\alpha,\beta)}(x). \quad (1.44)$$

Before closing the section, we are going to mention some of previous results on the related connection problems. The paper [1,17,18] treat the connection problem of expressing sums of finite products of Bernoulli, Euler and Genocchi polynomials in terms of Bernoulli polynomials. In fact, they were carried out by deriving Fourier series expansions for the functions closely related to those sums of finite products. Moreover, the same were done for the sums of finite products of Chebyshev polynomials of the second and of Fibonacci polynomials in [14].

Along the same line as the present paper, sums of finite products of Chebyshev polynomials of the second and Fibonacci polynomials were expressed in [19] as linear combinations of the orthogonal polynomials $H_n(x), L_n^{\alpha}(x), P_n(x), C_n^{(\lambda)}(x)$, and $P_n^{(\alpha,\beta)}(x)$. Also, the connection problem of expressing in terms of all kinds of Chebyshev polynomials were done for sums of finite products of Chebyshev polynomials of the second, third and fourth kinds and of Fibonacci, Legendre and Laguerre polynomials in [10,15,16].

Finally, we let the reader refer to [4,20] for some applications of Chebyshev polynomials.

2. Proof of Theorem 1.1

First, we will state **Proposition 2.1** and **Proposition 2.2** that will be needed in showing Theorem 1.1 and 1.2.

The results in (a), (b), (c), (d) and (e) in Proposition 2.1 follow respectively from (3.7) of [8], (2.3) of [11] (see also (2.4) of [6]), (2.3) of [9], (2.3) of [7] and (2.7) of [12]. They can be derived from their orthogonalities in (28) -(32), Rodrigues-type and Rodrigues' formulas in (23) -(27) and integration by parts.

Peer-reviewed version available at Symmetry 2018, 10, 617; doi:10.3390/sym10110617

Proposition 2.1. Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then we have the following.

$$\begin{aligned}
(a)\quad & q(x) = \sum_{k=0}^{n} C_{k,1} H_k(x), \text{ where}\\
& C_{k,1} = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} q(x) \frac{d^k}{dx^k} e^{-x^2} dx,\\
(b)\quad & q(x) = \sum_{k=0}^{n} C_{k,2} L_k^{\alpha}(x), \text{ where}\\
& C_{k,2} = \frac{1}{\Gamma(\alpha+k+1)} \int_{0}^{\infty} q(x) \frac{d^k}{dx^k} (e^{-x} x^{k+\alpha})\, dx,\\
(c)\quad & q(x) = \sum_{k=0}^{n} C_{k,3} P_k(x), \text{ where}\\
& C_{k,3} = \frac{2k+1}{2^{k+1} k!} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (x^2-1)^k dx,\\
(d)\quad & q(x) = \sum_{k=0}^{n} C_{k,4} C_k^{(\lambda)}(x), \text{ where}\\
& C_{k,4} = \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k \sqrt{\pi}\,\Gamma(k+\lambda+\frac{1}{2})} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k+\lambda-\frac{1}{2}} dx,\\
(e)\quad & q(x) = \sum_{k=0}^{n} C_{k,5} P_n^{(\alpha,\beta)}(x), \text{ where}\\
& C_{k,5} = \frac{(-1)^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+1} \Gamma(\alpha+k+1)\Gamma(\beta+k+1)}\\
& \qquad \times \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\alpha} (1+x)^{k+\beta} dx.
\end{aligned}$$

Proposition 2.2. The following holds true.

(a) For any nonnegative integer m,

$$\int_{-\infty}^{\infty} x^m e^{-x^2} dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!\sqrt{\pi}}{(\frac{m}{2})! 2^m}, & \text{if } m \equiv 0 \pmod 2, \end{cases}$$

(b) For any real numbers $r, s > -1$, we have

$$\int_{-1}^{1} (1-x)^r (1+x)^s dx = 2^{r+s+1} \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)},$$

(c) For any real numbers r, s with $r+s > -1, s > -1$, we have

$$\int_{-1}^{1} (1-x)^r (1-x^2)^s dx = 2^{r+2s+1} \frac{\Gamma(r+s+1)\Gamma(s+1)}{\Gamma(r+2s+2)}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 October 2018 doi:10.20944/preprints201810.0140.v1

Peer-reviewed version available at Symmetry 2018, 10, 617; doi:10.3390/sym10110617

Proof. (a) This is an easy exercise.
(c) This follows from (b) with r replaced by $r+s$.
(b) This follows from the change of variable $1+x=2y$ and (5).

The following lemma can be obtained by differentiating (1.9), as was shown in [13].

Lemma 2.3. Let n, r be integers with $n \geq 0, r \geq 1$. Then we have the following identity.

$$\sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} \binom{r-1+n-l}{r-1} V_{i_1}(x)V_{i_2}(x)\cdots V_{i_{r+1}}(x) = \frac{1}{2^r r!} V_{n+r}^{(r)}(x), \quad (2.1)$$

where the inner sum runs over all nonnegative integers $i_1, i_2, \cdots i_{r+1}$, with $i_1+i_2+\cdots+i_{r+1}=l$.

From (1.16), we see that the rth derivative of $V_n(x)$ is given by

$$V_n^{(r)}(x) = \sum_{l=0}^{n-r} \binom{2n-l}{l} 2^{n-l}(n-l)_r(x-1)^{n-l-r}. \quad (2.2)$$

Especially, we have

$$V_{n+r}^{(r+k)}(x) = \sum_{l=0}^{n-k} \binom{2n+2r-l}{l} 2^{n+r-l}(n+r-l)_{r+k}(x-1)^{n-k-l}. \quad (2.3)$$

Now, we are ready to prove **Theorem 1.1**. As (1.38) and (1.39) can be shown similarly to (1.43) and (1.44) in the next section, we will show only (1.35), (1.36) and (1.37). With $\gamma_{n,r}(x)$ as in (1.33), we let

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,1} H_k(x). \quad (2.4)$$

Then, from (a) of **Proposition 2.1**, (2.1), (2.3), and integration by parts k times, we obtain

$$\begin{aligned} C_{k,1} &= \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \gamma_{n,r}(x) \frac{d^k}{dx^k} e^{-x^2} dx \\ &= \frac{(-1)^k}{2^{k+r} k! r! \sqrt{\pi}} \int_{-\infty}^{\infty} V_{n+r}^{(r)}(x) \frac{d^k}{dx^k} e^{-x^2} dx \\ &= \frac{1}{2^{k+r} k! r! \sqrt{\pi}} \int_{-\infty}^{\infty} V_{n+r}^{(r+k)}(x) e^{-x^2} dx \\ &= \frac{1}{2^{k+r} k! r! \sqrt{\pi}} \sum_{l=0}^{n-k} \binom{2n+2r-l}{l} 2^{n+r-l}(n+r-l)_{r+k} \\ &\quad \times \int_{-\infty}^{\infty} (x-1)^{n-k-l} e^{-x^2} dx. \end{aligned} \quad (2.5)$$

Peer-reviewed version available at Symmetry 2018, 10, 617; doi:10.3390/sym10110617

Before proceeding further, by making use of (a) in **Proposition 2.2**, we note that

$$\begin{aligned}
&\int_{-\infty}^{\infty}(x-1)^m e^{-x^2}dx\\
=&\sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}\int_{-\infty}^{\infty}x^s e^{-x^2}dx\\
=&\sum_{\substack{0\le s\le m\\ s\equiv 0 \ (\mathrm{mod}\ 2)}}\binom{m}{s}(-1)^{m-s}\frac{s!\sqrt{\pi}}{(\frac{s}{2})!2^s}\\
=&(-1)^m\sqrt{\pi}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{2j}\frac{(2j)!}{j!2^{2j}},\ (m\ge 0).
\end{aligned} \tag{2.6}$$

From (2.4) - (2.6), and after simplifications, we have

$$\begin{aligned}
\gamma_{n,r}(x)=&\frac{1}{r!}\sum_{k=0}^{n}\frac{(-2)^k}{(n-k)!}\sum_{l=0}^{k}\sum_{j=0}^{[\frac{k-l}{2}]}\frac{(-\frac{1}{2})^l(2n+2r-l)!(n+r-l)!}{l!(2n+2r-2l)!(k-l-2j)!j!4^j}H_{n-k}(x)\\
=&\frac{1}{r!}\sum_{k=0}^{n}\frac{(-2)^k}{(n-k)!}\sum_{j=0}^{[\frac{k}{2}]}\frac{1}{j!4^j}\sum_{l=0}^{k-2j}\frac{(-\frac{1}{2})^l(2n+2r-l)!(n+r-l)!}{l!(2n+2r-2l)!(k-l-2j)!}H_{n-k}(x)\\
=&\frac{(2n+2r)!}{r!4^{n+r}<\frac{1}{2}>_{n+r}}\sum_{k=0}^{n}\frac{(-2)^k}{(n-k)!}\sum_{j=0}^{[\frac{k}{2}]}\frac{1}{j!4^j(k-2j)!}\\
&\times\sum_{l=0}^{k-2j}\frac{2^l<2j-k>_l<\frac{1}{2}-n-r>_l}{l!<-2n-2r>_l}H_{n-k}(x)\\
=&\frac{(2n+2r)!}{r!4^{n+r}(n+r-\frac{1}{2})_{n+r}}\sum_{k=0}^{n}\frac{(-2)^k}{(n-k)!}\\
&\times\sum_{j=0}^{[\frac{k}{2}]}\frac{{}_2F_1(2j-k,\frac{1}{2}-n-r;-2n-2r;2)}{j!4^j(k-2j)!}H_{n-k}(x).
\end{aligned} \tag{2.7}$$

This shows (1.35) of **Theorem 1.1**.
Next, we let

$$\gamma_{n,r}(x)=\sum_{k=0}^{n}C_{k,2}L_k^{\alpha}(x). \tag{2.8}$$

Then, from (b) of **Proposition 2.1**, (2.1), (2.3) and integration by parts k times , we get

$$
\begin{aligned}
C_{k,2} &= \frac{1}{2^r r!\Gamma(\alpha+k+1)} \int_0^\infty V_{n+r}^{(r)}(x) \frac{d^k}{dx^k}(e^{-x}x^{k+\alpha})dx \\
&= \frac{(-1)^k}{2^r r!\Gamma(\alpha+k+1)} \int_0^\infty V_{n+r}^{(r+k)}(x) e^{-x}x^{k+\alpha}dx \\
&= \frac{(-1)^k}{2^r r!\Gamma(\alpha+k+1)} \sum_{l=0}^{n-k} \binom{2n+2r-l}{l} 2^{n+r-l}(n+r-l)_{r+k} \\
&\quad \times \int_0^\infty (x-1)^{n-k-l} e^{-x}x^{k+\alpha}dx \\
&= \frac{(-1)^k}{2^r r!\Gamma(\alpha+k+1)} \sum_{l=0}^{n-k} \binom{2n+2r-l}{l} 2^{n+r-l}(n+r-l)_{r+k} \\
&\quad \times \sum_{s=0}^{n-k-l} \binom{n-k-l}{s} (-1)^{n-k-l-s}\Gamma(s+k+\alpha+1) \\
&= \frac{(-1)^k}{2^r r!} \sum_{l=0}^{n-k} \binom{2n+2r-l}{l} 2^{n+r-l}(n+r-l)_{r+k} \\
&\quad \times \sum_{s=0}^{n-k-l} \binom{n-k-l}{s} (-1)^{n-k-l-s} < k+\alpha+1 >_s \\
&= \frac{1}{r!} \sum_{l=0}^{n-k} \frac{(2n+2r-l)!(-2)^{n-l}(n+r-l)!}{l!(2n+2r-2l)!(n-k-l)!} \\
&\quad \times \sum_{s=0}^{n-k-l} \frac{1}{s!} < k+l-n >_s < k+\alpha+1 >_s \\
&= \frac{1}{r!} \sum_{l=0}^{n-k} \frac{(2n+2r-l)!(-2)^{n-l}(n+r-l)!}{l!(2n+2r-2l)!(n-k-l)!} \\
&\quad \times {}_2F_0(k+l-n, k+\alpha+1; -; 1).
\end{aligned} \tag{2.9}
$$

Combining (2.8) - (2.9), we finally have

$$
\begin{aligned}
\gamma_{n,r}(x) = \frac{1}{r!} \sum_{k=0}^{n} \sum_{l=0}^{k} &\frac{(2n+2r-l)!(-2)^{n-l}(n+r-l)!}{l!(2n+2r-2l)!(k-l)!} \\
&\times {}_2F_0(l-k, n-k+\alpha+1; -; 1) L_{n-k}^{\alpha}(x).
\end{aligned} \tag{2.10}
$$

This completes the proof for (1.36) of **Theorem 1.1**.

Finally, let us put

$$
\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,3} P_k(x). \tag{2.11}
$$

Then, from (c) of **Proposition 2.1**, (2.1), (2.3) and integration by parts k times, we have

$$\begin{aligned} C_{k,3} &= \frac{2k+1(-1)^k}{2^{k+r+1}k!r!}\int_{-1}^{1} V_{n+r}^{(r+k)}(x)(x^2-1)^k dx \\ &= \frac{(2k+1)(-1)^k}{2^{k+r+1}k!r!}\sum_{l=0}^{n-k}\binom{2n+2r-l}{l}2^{n+r-l}(n+r-l)_{r+k} \\ &\times \int_{-1}^{1}(x-1)^{n-k-l}(x^2-1)^k dx. \end{aligned} \tag{2.12}$$

By making use of (c) in **Proposition 2.2** and after simplifications, from (2.12) we obtain

$$\begin{aligned} C_{k,3} &= \frac{(-1)^{n+k}(2k+1)}{r!}\sum_{l=0}^{n-k} \\ &\times \frac{(-1)^l 4^{n-l}(2n+2r-l)!(n+r-l)!(n-l)!}{l!(2n+2r-2l)!(n-k-l)!(n+k-l+1)!} \\ &= \frac{(-1)^n(2n+2r)!n!}{r!4^r(n+r-\frac{1}{2})_{n+r}}\frac{(-1)^k(2k+1)}{(n-k)!(n+k+1)!} \\ &\times \sum_{l=0}^{n-k}\frac{<k-n>_l<\frac{1}{2}-n-r>_l<-n-k-1>_l}{l!<-2n-2r>_l<-n>_l} \\ &= \frac{(-1)^n(2n+2r)!n!}{r!4^r(n+r-\frac{1}{2})_{n+r}}\frac{(-1)^k(2k+1)}{(n-k)!(n+k+1)!} \\ &\times {}_3F_2(k-n,\frac{1}{2}-n-r,-n-k-1;-2n-2r,-n;1). \end{aligned} \tag{2.13}$$

From (2.11) and (2.13), we get

$$\begin{aligned} \gamma_{n,r}(x) &= \frac{(-1)^n(2n+2r)!n!}{r!4^r(n+r-\frac{1}{2})_{n+r}}\sum_{k=0}^{n}\frac{(-1)^k(2k+1)}{(n-k)!(n+k+1)!} \\ &\times {}_3F_2(k-n,\frac{1}{2}-n-r,-n-k-1;-2n-2r,-n;1)P_k(x). \end{aligned} \tag{2.14}$$

This proves (1.37) of **Theorem 1.1**.

3. Proof of Theorem 1.2

Here we will show only (1.43) and (1.44) in **Theorem 1.2**, as (1.40), (1.41), (1.42) can be shown analogously to the proofs for (1.35), (1.36), (1.37), respectively. The following can be derived by differentiating the equation (1.10) and is stated in [13].

Peer-reviewed version available at Symmetry 2018, 10, 617; doi:10.3390/sym10110617

Lemma 3.1. Let n, r be integers with $n \geq 0$, $r \geq 1$. Then we have the following identity.

$$\sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{i_1}(x)W_{i_2}(x)\cdots W_{i_{r+1}}(x) = \frac{1}{2^r r!} W_{n+r}^{(r)}(x), \tag{3.1}$$

where the inner sum runs over all nonnegative integers $i_1, i_2, \cdots, i_{r+1}$, with $i_1 + i_2 + \cdots + i_{r+1} = l$.

From (1.17), the rth derivative of $W_n(x)$ is given by

$$W_n^{(r)}(x) = (2n+1) \sum_{l=0}^{n-r} \frac{2^{n-l}}{2n+1-2l} \binom{2n-l}{l} (n-l)_r (x-1)^{n-l-r}. \tag{3.2}$$

In particular, we have

$$W_{n+r}^{(r+k)}(x) = (2n+1) \sum_{l=0}^{n-k} \frac{2^{n+r-l}}{2n+2r+1-2l} \binom{2n+2r-l}{l} (n+r-l)_{r+k} (x-1)^{n-k-l}. \tag{3.3}$$

With $\mathcal{E}_{n,r}(x)$ as in (1.34), we let

$$\mathcal{E}_{n,r}(x) = \sum_{k=0}^{n} C_{k,4} C_k^{(\alpha)}(x). \tag{3.4}$$

Then, from (d) of **Proposition 2.1**, (3.1),(3.3) and integration by parts k times, we get

$$\begin{aligned} C_{k,4} &= \frac{(k+\lambda)\Gamma(\lambda)}{2^{k+r} r! \sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \\ &\times \int_{-1}^{1} W_{n+r}^{(r+k)}(x)(1-x^2)^{k+\lambda-\frac{1}{2}} dx \\ &= \frac{(k+\lambda)\Gamma(\lambda)(2n+1)}{2^{k+r} r! \sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \\ &\times \sum_{l=0}^{n-k} \frac{2^{n+r-l}}{2n+2r+1-2l} \binom{2n+2r-l}{l} (n+r-l)_{r+k} \\ &\times \int_{-1}^{1} (x-1)^{n-k-l}(1-x^2)^{k+\lambda-\frac{1}{2}} dx \\ &= \frac{(k+\lambda)\Gamma(\lambda)(2n+1)(-2)^{n-k}}{r! \sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})} \\ &\times \sum_{l=0}^{n-k} \frac{(-\frac{1}{2})^l (2n+2r-l)!(n+r-l)!}{(2n+2r-2l+1)l!(2n+2r-2l)!(n-k-l)!} \\ &\times \int_{-1}^{1} (1-x)^{n-k-l}(1-x^2)^{k+\lambda-\frac{1}{2}} dx. \end{aligned} \tag{3.5}$$

Invoking (c) of **Proposition 2.2** and after simplifications, from (3.5) we obtain

Peer-reviewed version available at Symmetry 2018, 10, 617; doi:10.3390/sym10110617

$$\begin{aligned}
C_{k,4} &= \frac{(-1)^{n-k}(k+\lambda)\Gamma(\lambda)(2n+1)2^{2n+2\lambda+1}\Gamma(n+\lambda+\frac{1}{2})(2n+2r)!}{\Gamma(n+k+2\lambda+1)(n-k)!r!\sqrt{\pi}} \\
&\times \sum_{l=0}^{n-k} \frac{(-\frac{1}{4})^l(2n+2r-l)!(n+r-l+1)!(n-k)!(n+k+2\lambda)_l}{l!(2n+2r)!(2n+2r-2l+2)!(n-k-l)!(n+\lambda-\frac{1}{2})_l} \\
&= \frac{(-1)^k(k+\lambda)\Gamma(\lambda)(2n+1)2^{2\lambda-2r-1}(-1)^n\Gamma(n+\lambda+\frac{1}{2})(2n+2r)!}{\Gamma(n+k+2\lambda+1)(n-k)!r!\sqrt{\pi}(n+r+\frac{1}{2})_{n+r+1}} \\
&\times \sum_{l=0}^{n-k} \frac{<k-n>_l<-n-r-\frac{1}{2}>_l<-n-k-2\lambda>_l}{l!<-2n-2r>_l<-n-\lambda+\frac{1}{2}>_l} \\
&= \frac{(-1)^k(k+\lambda)\Gamma(\lambda)(2n+1)2^{2\lambda-2r-1}(-1)^n\Gamma(n+\lambda+\frac{1}{2})(2n+2r)!}{\Gamma(n+k+2\lambda+1)(n-k)!r!\sqrt{\pi}(n+r+\frac{1}{2})_{n+r+1}} \\
&\times {}_3F_2(k-n,-n-r-\frac{1}{2},-n-k-2\lambda;-2n-2r,-n-\lambda+\frac{1}{2};1).
\end{aligned} \tag{3.6}$$

From (3.4) and (3.6), we have

$$\begin{aligned}
\mathcal{E}_{n,r}(x) &= \frac{\Gamma(\lambda)(2n+1)2^{2\lambda-2r-1}(-1)^n\Gamma(n+\lambda+\frac{1}{2})(2n+2r)!}{r!\sqrt{\pi}(n+r+\frac{1}{2})_{n+r+1}} \\
&\times \sum_{k=0}^{n} \frac{(-1)^k(k+\lambda)}{\Gamma(n+k+2\lambda+1)(n-k)!} \\
&\times {}_3F_2(k-n,-n-r-\frac{1}{2},-n-k-2\lambda;-2n-2r,-n-\lambda+\frac{1}{2};1)C_k^{(\alpha)}(x).
\end{aligned} \tag{3.7}$$

This shows (1.43) of **Theorem 1.2**.

Next, we let

$$\mathcal{E}_{n,r}(x) = \sum_{k=0}^{n} C_{k,5}P_n^{(\alpha,\beta)}(x). \tag{3.8}$$

Then, from (e) of **Proposition 2.1**, and (3.1), (3.3), and integrating by parts k times, we obtain

$$\begin{aligned}
C_{k,5} &= \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{\alpha+\beta+k+r+1}r!\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\
&\times \int_{-1}^{1} W_{n+r}^{(r+k)}(x)(1-x)^{k+\alpha}(1+x)^{k+\beta}dx \\
&= \frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)(2n+1)}{2^{\alpha+\beta+k+r+1}r!\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \\
&\times \sum_{l=0}^{n-k} \frac{2^{n+r-l}}{2n+2r-2l+1}\binom{2n+2r-l}{l}(n+r-l)_{r+k}(-1)^{n-k-l} \\
&\times \int_{-1}^{1}(1-x)^{n+\alpha-l}(1+x)^{k+\beta}dx.
\end{aligned} \tag{3.9}$$

By exploiting (b) in **Proposition 2.2** and after simplifications, from (3.9) we get

$$\begin{aligned} C_{k,5} &= \frac{(-1)^{n-k}(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)2^{2n+1}(2n+1)\Gamma(n+\alpha+1)}{\Gamma(\alpha+k+1)\Gamma(n+k+\alpha+\beta+2)r!} \\ &\times \sum_{l=0}^{n-k} \frac{(-\frac{1}{4})^l(2n+2r-l)!(n+r-l+1)!(n+k+\alpha+\beta+1)_l}{l!(2n+2r-2l+2)!(n-k-l)!(n+\alpha)_l} \\ &= \frac{(-1)^{n-k}(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)(2n+1)\Gamma(n+\alpha+1)(2n+2r)!}{\Gamma(\alpha+k+1)\Gamma(n+k+\alpha+\beta+2)(n-k)!r!2^{2r+1}(n+r+\frac{1}{2})_{n+r+1}} \\ &\times \sum_{l=0}^{n-k} \frac{<k-n>_l<-n-r-\frac{1}{2}>_l<-n-k-\alpha-\beta-1>_l}{l!<-2n-2r>_l<-n-\alpha>_l} \\ &= \frac{(-1)^{n-k}(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)(2n+1)\Gamma(n+\alpha+1)(2n+2r)!}{\Gamma(\alpha+k+1)\Gamma(n+k+\alpha+\beta+2)(n-k)!r!2^{2r+1}(n+r+\frac{1}{2})_{n+r+1}} \\ &\times {}_3F_2(k-n,-n-r-\frac{1}{2},-n-k-\alpha-\beta-1;-2n-2r,-n-\alpha;1). \end{aligned} \tag{3.10}$$

Thus, from (3.8) and (3.10), we have

$$\begin{aligned} \mathcal{E}_{n,r}(x) &= \frac{(-1)^n(2n+1)\Gamma(n+\alpha+1)(2n+2r)!}{r!2^{2r+1}(n+r+\frac{1}{2})_{n+r+1}} \\ &\times \sum_{k=0}^{n} \frac{(-1)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(\alpha+k+1)\Gamma(n+k+\alpha+\beta+2)(n-k)!} \\ &\times {}_3F_2(k-n,-n-r-\frac{1}{2},-n-k-\alpha-\beta-1;-2n-2r,-n-\alpha;1)P_n^{(\alpha,\beta)}(x). \end{aligned}$$

4. Conclusion

In this paper, we considered sums of finite products of Chebyshev polynomials of the third and fourth kinds and expressed each of them in terms of five orthogonal polynomials, namely Hermite, generalized Laguerre, Legendre, Gegenbauer and Jacobi polynomials. This can be viewed as a generalization of the classical connection problem. Those sums of finite products were also represented by all kinds of Chebyshev polynomials in [15]. In addition, the same had been done for sums of finite products of Chebyshev polynomials of the second, Fibonacci polynomials, Legendre polynomials and Laguerre polynomials.

Competing interests
The authors declare that they have no competing interests.

Authors contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Funding
This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2017R1E1A1A03070882).

References

[1] R. P. Agarwal, D. S. Kim, T. Kim, J. Kwon, *Sums of finite products of Bernoulli functions,* Adv. Difference Equ. **2017**, 2017:237, 15pp.

[2] G. E. Andrews, R. Askey, R. Roy, *Special functions,* Encyclopedia of Mathematics and its Applications 71, Cambridge University Press, Cambridge, **1999**.

[3] R. Beals, R. Wong, *Special functions and orthogonal polynomials,* Cambridge Studies in Advanced Mathematics 153, Cambridge University Press, Cambridge, **2016**.

[4] E. H. Doha, W. M. Abd-Elhameed, M. M. Alsuyuti, *On using third and fourth kinds Chebyshev polynomials for solving the integrated forms of high odd-order linear boundary value problems,* J. Egyptian Math. Soc. **23** (2015), no. 2, 397–405.

[5] D. S. Kim, D. V. Dolgy, T. Kim, S. -H. Rim, *Identities involving Bernoulli and Euler polynomials arising from Chebyshev polynomials,* Proc. Jangjeon Math. Soc. **15** (2012), no. 4, 361–370.

[6] D. S. Kim, T. Kim, D. V. Dolgy, *Some identities on Laguerre polynomials in connection with Bernoulli and Euler numbers,* Discrete Dyn. Nat. Soc. **2012**, Art. ID 619197, 10 pp.

[7] D. S. Kim, T. Kim, S.-H. Rim, *Some identities involving Gegenbauer polynomials,* Adv. Difference Equ. **2012**, 2012:219, 11pp.

[8] D. S. Kim, T. Kim, S.-H. Rim, S. H. Lee, *Hermite polynomials and their applications associated with Bernoulli and Euler numbers,* Discrete Dyn. Nat. Soc. **2012**, Art. ID 974632, 13 pp.

[9] D. S. Kim, S.-H. Rim, T. Kim, *Some identities on Bernoulli and Euler polynomials arising from orthogonality of Legendre polynomials,* J. Inequal. Appl. **2012**, 2012:227. 8 pp.

[10] T. Kim, D. V. Dolgy, D. S. Kim, *Representing sums of finite products of Chebyshev polynomials of the second kind and Fibonacci polynomials in terms of Chebyshev polynomials,* Adv. Stud. Contemp. Math.(Kyungshang), **28** (2018), no. 3, 321–335.

[11] T. Kim, D. S. Kim, *Extended Laguerre polynomials associated with Hermite, Bernoulli, and Euler numbers and polynomials,* Abstr. Appl. Anal. **2012**, Art. ID. 957350, 15 pp.

[12] T. Kim, D. S. Kim, D. V. Dolgy, *Some identities on Bernoulli and Hermite polynomials associated with Jacobi polynomials,* Discrete Dyn. Nat. Soc. **2012**, Art. ID 584643, 11 pp.

[13] T. Kim, D. S. Kim, D. V. Dolgy, J. Kwon, *Sums of finite products of Chebyshev polynomials of the third and fourth kinds,* Adv. Difference Equ. **2018**, 2018:283, 17pp.

[14] T. Kim, D. S. Kim, D. V. Dolgy, J.-W. Park, *Sums of finite products of Chebyshev polynomials of the second kind and Fibonacci polynomials,* J. Inequal. Appl. **2018**, 2018:148, 14pp.

[15] T. Kim, D. S. Kim, D. V. Dolgy, C. -S. Ryoo, *Representing sums of finite products of Chebyshev polynomials of the third and fourth kinds by Chebyshev polynomials,* Symmetry **10** (2018), no. 7, Art. 258, 10pp.

[16] T. Kim, D. S. Kim, G. -W. Jang, J. Kwon, *Representing sums of finite products of Legendre and Laguerre polynomials by Chebyshev polynomials,* Adv. Stud. Contemp. Math.(Kyungshang), **28** (2018), no. 4, in press.

[17] T. Kim, D. S. Kim, G. -W. Jang, J. Kwon, *Sums of finite products of Euler functions,* Advances in Real and Complex Analysis with Applications, 243-260, Trends in Mathematics, Springer, **2017**.

[18] T. Kim, D. S. Kim, L. C. Jang, G.-W. Jang, *Sums of finite products of Genocchi functions,* Adv. Difference Equ. **2017**, 2017:268, 17pp.

[19] T. Kim, D. S. Kim, J. Kwon, D. V. Dolgy, *Expressing sums of finite products of Chebyshev polynomials of the second kind and Fibonacci polynomials by several orthogonal polynomials,* Mathematics(communicated)

[20] J. C. Mason, *Chebyshev polynomials of the second, third and fourth kinds in approximation, indefinite integration, and integral transforms,* J. Comput. Appl. Math. **49** (1993), no. 1-3, 169–178.

[21] Z. X. Wang, D. R. Guo, *Special functions,* Translated from the Chinese by Guo and X. J. Xia, World Scientific Publishing Co., Inc., Teaneck, NJ, **1989**.

Peer-reviewed version available at *Symmetry* **2018**, *10*, 617; doi:10.3390/sym10110617

[1] Institute of National Sciences, Far Eastern Federal University, 690950 Vladivostok, Russia
E-mail address: d_dol@mail.ru

[2] Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr

[3] Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin 300160, China, Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr

[4] Department of Mathematics Education and ERI, Gyeongsang National University, Jinju, Gyeongsangnamdo, 52828, Republic of Korea
E-mail address: mathkjk26@gnu.ac.kr