CONNECTION PROBLEM FOR SUMS OF FINITE PRODUCTS
OF CHEBYSHEV POLYNOMIALS OF THE THIRD AND
FOURTH KINDS

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Abstract. This paper treats the connection problem of expressing sums of
finite products of Chebyshev polynomials of the third and fourth kinds in
terms of five classical orthogonal polynomials. In fact, by carrying out explicit
computations each of them are expressed as linear combinations of Hermite,
generalized Laguerre, Legendre, Gegenbauer, and Jacobi polynomials which
involve some terminating hypergeometric functions \(_2F_0, _2F_1, \text{and } _3F_2\).

1. Introduction and preliminaries

In this section, we will recall some basic facts about relevant orthogonal polynomi-
als that will be needed throughout this paper. For this, we will first fix some
notations. For any nonnegative integer \(n\), the falling factorial polynomials \((x)_n\)
and the rising factorial polynomials \(<x>_n\) are respectively given by

\[
(x)_n = x(x-1)\cdots(x-n+1), \ (n \geq 1), \ (x)_0 = 1,
\]

\[
<x>_n = x(x+1)\cdots(x+n-1), \ (n \geq 1), \ <x>_0 = 1.
\]

The two factorial polynomials are evidently related by

\[
(-1)^n(x)_n = <x>_n, \ (-1)^n <x>_n = (-x)_n.
\]

\[
\frac{(2n-2s)!}{(n-s)!} = 2^{2n-2s}(-1)^s \frac{<\frac{1}{2}>_n}{<\frac{1}{2} - n>_s}, \ (n \geq s \geq 0).
\]

\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \ (Re \ x, Re \ y > 0).
\]

\[
\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}, \ (n \geq 0).
\]

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Sums of finite products of Chebyshev polynomials of the third and fourth kinds

\[
\frac{\Gamma(x + 1)}{\Gamma(x + 1 - n)} = (x)_n, \quad \frac{\Gamma(x + n)}{\Gamma(x)} = <x>_n, \quad (n \geq 0), \quad (1.7)
\]

where \(\Gamma(x)\) and \(B(x, y)\) are the gamma and beta functions respectively.

The hypergeometric function is defined by

\[
_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; x) = \sum_{n=0}^{\infty} \frac{<a_1>_n \cdots <a_p>_n}{<b_1>_n \cdots <b_q>_n} \frac{x^n}{n!}. \quad (1.8)
\]

We are now ready to state some basic facts about Chebyshev polynomials of the third kind \(V_n(x)\), those of the fourth kind \(W_n(x)\), Hermite polynomials \(H_n(x)\), generalized (extended) Laguerre polynomials \(L_\alpha^n(x)\), Legendre polynomials \(P_n(x)\), Gegenbauer polynomials \(C_\lambda^n(x)\), and Jacobi polynomials \(P_{\alpha,\beta}^n(x)\). All the necessary facts on those special polynomials can be found in [5-9,11,12]. For the full accounts of this fascinating area of orthogonal polynomials, the reader may refer to [2,3,21].

The above special polynomials are given in terms of generating functions by

\[
F(t, x) = \frac{1 - t}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} V_n(x)t^n, \quad (1.9)
\]

\[
G(t, x) = \frac{1 + t}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} W_n(x)t^n, \quad (1.10)
\]

\[
e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \quad (1.11)
\]

\[
(1 - t)^{-\alpha-1} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_\alpha^n(x)\frac{t^n}{n!}, \quad (\alpha > -1), \quad (1.12)
\]

\[
(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}, \quad (1.13)
\]

\[
\frac{1}{(1 - 2xt + t^2)^\lambda} = \sum_{n=0}^{\infty} C_\lambda^n(x)\frac{t^n}{n!}, \quad (\lambda > -\frac{1}{2}, \lambda \neq 0, |t| < 1, |x| \leq 1), \quad (1.14)
\]

\[
\frac{2^{\alpha+\beta}}{R(1-t+R)^\alpha(1+t+R)^\beta} = \sum_{n=0}^{\infty} P_{\alpha,\beta}^n(x)\frac{t^n}{n!}, \quad (1.15)
\]

\[
(R = \sqrt{1 - 2xt + t^2}; \quad \alpha, \beta > -1).
\]

Explicit expressions for the above special polynomials are as in the following.
\[ V_n(x) = 2F_1(-n, n+1; \frac{1}{2}; \frac{1-x}{2}) \]
\[ = \sum_{l=0}^{n} \binom{n-1}{l} 2^{n-1} (x-1)^{n-l}, \quad (1.16) \]
\[ W_n(x) = (2n+1)2F_1(-n, n+1; \frac{3}{2}; \frac{1-x}{2}) \]
\[ = (2n+1) \sum_{l=0}^{n} \frac{2^{n-l}}{2n-2l+1} \binom{n-1}{l} (x-1)^{n-l}, \quad (1.17) \]
\[ H_n(x) = n! \sum_{l=0}^{\left[ \frac{n}{2} \right]} \binom{n}{l} (-1)^l (2x)^{n-2l}, \quad (1.18) \]
\[ L_n^\alpha(x) = \frac{\alpha+1}{n!} \binom{n}{l} 2F_1(-n, \alpha+1; x) \]
\[ = \sum_{l=0}^{n} \frac{(-1)^l (n+\alpha)}{n-l} x^l, \quad (1.19) \]
\[ P_n(x) = 2F_1(-n, n+1; \frac{1-x}{2}) \]
\[ = \frac{1}{2^n} \sum_{l=0}^{\left[ \frac{n}{2} \right]} (-1)^l \binom{n}{l} (2n-2l) x^{n-2l}, \quad (1.20) \]
\[ C_n^{(\lambda)}(x) = \binom{n+2\lambda-1}{n} 2F_1(-n, n+2\lambda; \lambda+\frac{1}{2}; \frac{1-x}{2}) \]
\[ = \sum_{k=0}^{\left[ \frac{n}{2} \right]} (-1)^k \frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)k!(n-2k)!} (2x)^{n-2k}, \quad (1.21) \]
\[ P_n^{(\alpha, \beta)}(x) = \frac{\alpha+1}{n!} \binom{n+\alpha+\beta+n}{n} 2F_1(-n, \alpha+\beta+n; \alpha+1; \frac{1-x}{2}) \]
\[ = \sum_{k=0}^{n} \binom{n+\beta}{n-k} \binom{n+\beta}{k} (\frac{1-x}{2})^{-k}(\frac{x+1}{2})^{n-k}. \quad (1.22) \]

Next, we state Rodrigues-type formulas for Hermite and generalized Laguerre polynomials and Rodrigues’ formulas for Legendre, Gegenbauer and Jacobi polynomials.

\[ H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad (1.23) \]
\[ L_n^\alpha(x) = \frac{1}{n!} x^{-\alpha} e^x \frac{d^n}{dx^n} (e^{-x} x^n), \quad (1.24) \]
\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \quad (1.25) \]
\[ (1-x^2)^{\frac{\lambda}{2}} C_n^{(\lambda)}(x) = \frac{(-2)^n}{n!} \sum_{\lambda + n \geq 2} \frac{\lambda - n}{x} \frac{d^n}{dx^n} (1-x)^{2n+\lambda-\frac{1}{2}}, \quad (1.26) \]
\[ (1-x)^\alpha (1+x)^\beta P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1-x)^{n+\alpha}(1+x)^{n+\beta}. \quad (1.27) \]
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The last thing we want to mention is the orthogonalities with respect to various weight functions enjoyed by Hermite, generalized Laguerre, Legendre, Gegenbauer and Jacobi polynomials.

\[
\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \, dx = 2^n n! \sqrt{\pi} \delta_{n,m}, \quad (1.28)
\]

\[
\int_0^{\infty} x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) \, dx = \frac{1}{n!} \Gamma(\alpha + n + 1) \delta_{n,m}, \quad (1.29)
\]

\[
\int_{-1}^{1} P_n(x) P_m(x) \, dx = \frac{2}{2n+1} \delta_{n,m}, \quad (1.30)
\]

\[
\int_{-1}^{1} (1-x^2)^{\lambda-\frac{1}{2}} C_n^{(\lambda)}(x) C_m^{(\lambda)}(x) \, dx = \frac{\pi 2^{1-2\lambda} \Gamma(n+2\lambda)}{n!(n+\lambda)\Gamma(\lambda)^2} \delta_{n,m}, \quad (1.31)
\]

\[
\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) \, dx = \frac{2^{n+\beta+1} \Gamma(n+\alpha + 1) \Gamma(n + \beta + 1)}{(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\Gamma(n+1)} \delta_{n,m}. \quad (1.32)
\]

For convenience, let us put

\[
\gamma_{n,r}(x) = \sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} \binom{r-1+n-l}{r-1} V_{i_1}(x) V_{i_2}(x) \cdots V_{i_{r+1}}(x), \quad (n \geq 0, r \geq 1), \quad (1.33)
\]

\[
\mathcal{E}_{n,r}(x) = \sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{i_1}(x) W_{i_2}(x) \cdots W_{i_{r+1}}(x), \quad (n \geq 0, r \geq 1). \quad (1.34)
\]

We observe here that both \( \gamma_{n,r}(x) \) and \( \mathcal{E}_{n,r}(x) \) have degree \( n \).

In this paper, we will consider the connection problem of expressing the sums of finite products in (1.33) and (1.34) as linear combinations of \( H_n(x), L_n^\alpha(x), P_n(x), C_n^{(\lambda)}(x), \) and \( P_n^{(\alpha,\beta)}(x) \). These will be done by performing explicit computations based on Proposition 2.1. We observe here that the formulas in Proposition 2.1 follow from their orthogonalities, Rodrigues’ and Rodrigues-type formulas and integration by parts.

Our main results are the following Theorem 1.1 and Theorem 1.2.
Theorem 1.1. Let $n, r$ be any integers with $n \geq 0, r \geq 1$. Then we have the following.

\[
\sum_{l=0}^{n} \sum_{i_1+i_2+\ldots+i_{r+1}=l} \binom{r-1+n-l}{r-1} V_{i_1}(x)V_{i_2}(x) \cdots V_{i_{r+1}}(x) = (2n + 2r)! r^{4n+r}(n + r - \frac{1}{2})_{n+r} \sum_{k=0}^{n} (-2)^k \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} F_1(2j - k, \frac{1}{2} - n - r; -2n - 2r; \frac{1}{2}) j!4^l(k - 2j)!
\]

\[
= \frac{1}{r!} \sum_{k=0}^{n} \binom{2n+2r}{k} \binom{n-r}{k} (n + r - l)! (n + r - l)! \times 2F_0(l - k, n - k + \alpha + 1; -; 1)^L_{n-k}(x)
\]

\[
= \frac{(-1)^n(2n + 2r)! r^{4n+2r}(n + r - \frac{1}{2})_{n+r} \sum_{k=0}^{n} (-1)^k(2k + 1) \binom{2n+2r}{k} \binom{n-r}{k} (n + r - l)! (n + r - l)! \times 2F_0(l - k, n - k + \alpha + 1; -; 1)^L_{n-k}(x)
\]

Theorem 1.2. Let $n, r$ be any integers with $n \geq 0, r \geq 1$. Then we have the following.

\[
\sum_{l=0}^{n} \sum_{i_1+i_2+\ldots+i_{r+1}=l} (-1)^{n-l} \binom{r-1+n-l}{r-1} W_{i_1}(x)W_{i_2}(x) \cdots W_{i_{r+1}}(x) = (2n + 1)(2n + 2r)! r^{4n+2r+1}(n + r + \frac{1}{2})_{n+r+1} \sum_{k=0}^{n} (-2)^k \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} F_1(2j - k, -n - r - \frac{1}{2}; -2n - 2r; 1) j!4^l(k - 2j)!
\]

\[
= \frac{1}{r!} \sum_{k=0}^{n} \binom{2n+2r}{k} \binom{n-r}{k} (n + r - l)! (n + r - l)! \times 2F_0(l - k, n - k + \alpha + 1; -; 1)^L_{n-k}(x)
\]
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\[
\begin{align*}
&= \frac{(-1)^n n! (2n+1) (2n+2r)!}{r! 2^{2r+1} (n+r+\frac{1}{2})^{n+r+1}} \sum_{k=0}^{n} \frac{(-1)^k (2k+1)}{(n-k)! (n+k+1)!}
\times \frac{\sqrt{\pi} r! (n+r+\frac{1}{2})^{n+r+1}}{2^k} F_2\left(k-n, -n-r - \frac{1}{2}, -n-k+1; -2n-2r, -n; 1\right) P_k(x) \\
&= \frac{(-1)^n (2n+2r)! 2^{2\lambda-2r-1} (2n+1) \Gamma(\lambda) \Gamma(n+\lambda+\frac{1}{2})}{\sqrt{\pi} r! (n+r+\frac{1}{2})^{n+r+1}} \\
&\times \sum_{k=0}^{n} \frac{(-1)^k (k+\lambda)}{(n+k+2\lambda+1)(n-k)!}
\times F_2\left(k-n, -n-r - \frac{1}{2}, -n-k-2\lambda; -2n-2r, -n-\lambda+\frac{1}{2}; 1\right) C_k^{(\lambda)}(x)
\end{align*}
\]

Before closing the section, we are going to mention some of previous results on the related connection problems. The paper [1,17,18] treat the connection problem of expressing sums of finite products of Bernoulli, Euler and Genocchi polynomials in terms of Bernoulli polynomials. In fact, they were carried out by deriving Fourier series expansions for the functions closely related to those sums of finite products. Moreover, the same were done for the sums of finite products of Chebyshev polynomials of the second and of Fibonacci polynomials in [14].

Along the same line as the present paper, sums of finite products of Chebyshev polynomials of the second and Fibonacci polynomials were expressed in [19] as linear combinations of the orthogonal polynomials \(H_n(x), L_n(x), P_n(x), C_n^{(\lambda)}(x)\), and \(P_n^{(\alpha,\beta)}(x)\). Also, the connection problem of expressing in terms of all kinds of Chebyshev polynomials were done for sums of finite products of Chebyshev polynomials of the second, third and fourth kinds and of Fibonacci, Legendre and Laguerre polynomials in [10,15,16].

Finally, we let the reader refer to [4,20] for some applications of Chebyshev polynomials.

2. Proof of Theorem 1.1

First, we will state Proposition 2.1 and Proposition 2.2 that will be needed in showing Theorem 1.1 and 1.2.

The results in (a), (b), (c), (d) and (e) in Proposition 2.1 follow respectively from (3.7) of [8], (2.3) of [11] (see also (2.4) of [6]), (2.3) of [9], (2.3) of [7] and (2.7) of [12]. They can be derived from their orthogonalities in (28) - (32), Rodrigues-type and Rodrigues’ formulas in (23) - (27) and integration by parts.
Proposition 2.1. Let \( q(x) \in \mathbb{R}[x] \) be a polynomial of degree \( n \). Then we have the following.

(a) \( q(x) = \sum_{k=0}^{n} C_{k,1} H_k(x), \) where
\[
C_{k,1} = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} q(x) \frac{d^k}{dx^k} e^{-x^2} \, dx,
\]

(b) \( q(x) = \sum_{k=0}^{n} C_{k,2} L_k^0(x), \) where
\[
C_{k,2} = \frac{1}{\Gamma(\alpha + k + 1)} \int_{0}^{\infty} q(x) \frac{d^k}{dx^k} (e^{-x} x^{k+\alpha}) \, dx,
\]

(c) \( q(x) = \sum_{k=0}^{n} C_{k,3} P_k(x), \) where
\[
C_{k,3} = \frac{2k+1}{2^{k+1} k!} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (x^2 - 1)^k \, dx,
\]

(d) \( q(x) = \sum_{k=0}^{n} C_{k,4} C_k^{(\lambda)}(x), \) where
\[
C_{k,4} = \frac{(k + \lambda) \Gamma(\lambda)}{(-2)^k \sqrt{\pi} \Gamma(k + \lambda + \frac{1}{2})} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1 - x^2)^{k+\lambda-\frac{1}{2}} \, dx,
\]

(e) \( q(x) = \sum_{k=0}^{n} C_{k,5} P_n^{(\alpha,\beta)}(x), \) where
\[
C_{k,5} = \frac{(-1)^k (2k + \alpha + \beta + 1) \Gamma(k + \alpha + \beta + 1)}{2^{\alpha+\beta+k+1} \Gamma(\alpha + k + 1) \Gamma(\beta + k + 1)} \\
\times \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1 - x)^{k+\alpha}(1 + x)^{k+\beta} \, dx.
\]

Proposition 2.2. The following holds true.

(a) For any nonnegative integer \( m \),
\[
\int_{-\infty}^{\infty} x^m e^{-x^2} \, dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod{2}, \\ \frac{m! \sqrt{\pi}}{2^{m/2}}, & \text{if } m \equiv 0 \pmod{2}, \end{cases}
\]

(b) For any real numbers \( r, s > -1 \), we have
\[
\int_{-1}^{1} (1 - x)^r (1 + x)^s dx = 2^{r+s+1} \frac{\Gamma(r+1) \Gamma(s+1)}{\Gamma(r+s+2)},
\]

(c) For any real numbers \( r, s > -1 \), we have
\[
\int_{-1}^{1} (1 - x)^r (1 - x^2)^s dx = 2^{r+2s+1} \frac{\Gamma(r+s+1) \Gamma(s+1)}{\Gamma(r+2s+2)}.
\]
Proof. (a) This is an easy exercise.
(c) This follows from (b) with \( r \) replaced by \( r + s \).
(b) This follows from the change of variable \( 1 + x = 2y \) and (5).

The following lemma can be obtained by differentiating (1.9), as was shown in [13].

Lemma 2.3. Let \( n, r \) be integers with \( n \geq 0, r \geq 1 \). Then we have the following identity.

\[
\sum_{l=0}^{n} \sum_{i_1 + i_2 + \cdots + i_{r+1} = l} \binom{r - 1 + n - l}{r - 1} V_{i_1}(x) V_{i_2}(x) \cdots V_{i_{r+1}}(x) = \frac{1}{2^r r!} V_{n+r}^{(r)}(x),
\]

where the inner sum runs over all nonnegative integers \( i_1, i_2, \ldots, i_{r+1} \), with \( i_1 + i_2 + \cdots + i_{r+1} = l \).

From (1.16), we see that the \( r \)th derivative of \( V_n(x) \) is given by

\[
V_n^{(r)}(x) = \sum_{l=0}^{n-r} \binom{2n - l}{l} 2^{n-l} (n-l)_r (x-1)^{n-l-r}.
\]

Especially, we have

\[
V_{n+r}^{(r+k)}(x) = \sum_{l=0}^{n-k} \binom{2n + 2r - l}{l} 2^{n-r-l} (n-r-l)_{r+k} (x-1)^{n-r-l}.
\]

Now, we are ready to prove Theorem 1.1. As (1.38) and (1.39) can be shown similarly to (1.43) and (1.44) in the next section, we will show only (1.35), (1.36) and (1.37). With \( \gamma_{n,r}(x) \) as in (1.33), we let

\[
\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,1} H_k(x).
\]

Then, from (a) of Proposition 2.1, (2.1), (2.3), and integration by parts \( k \) times, we obtain

\[
C_{k,1} = \frac{(-1)^k}{2^{k+1} k! \sqrt{\pi}} \int_{-\infty}^{\infty} \gamma_{n,r}(x) \frac{d^k}{dx^k} e^{-x^2} dx
= \frac{(-1)^k}{2^{k+r} k! r! \sqrt{\pi}} \int_{-\infty}^{\infty} V_n^{(r)}(x) \frac{d^k}{dx^k} e^{-x^2} dx
= \frac{1}{2^{k+r} k! r! \sqrt{\pi}} \int_{-\infty}^{\infty} V_{n+r}^{(r+k)}(x) e^{-x^2} dx
= \frac{1}{2^{k+r} k! r! \sqrt{\pi}} \sum_{l=0}^{n-k} \binom{2n + 2r - l}{l} 2^{n+r-l} (n-r-l)_{r+k} \times \int_{-\infty}^{\infty} (x-1)^{n-k-l} e^{-x^2} dx.
\]
Before proceeding further, by making use of (a) in Proposition 2.2, we note that

\[
\int_{-\infty}^{\infty} (x - 1)^m e^{-x^2} dx = \sum_{s=0}^{m} \binom{m}{s} (-1)^{m-s} \int_{-\infty}^{\infty} x^s e^{-x^2} dx = \sum_{0 \leq s \leq m (\mod 2)} \binom{m}{s} (-1)^{m-s} \frac{s! \sqrt{\pi}}{\left(\frac{s}{2}\right)!^{2s}}
\]

\[
= (-1)^m \frac{\left[2\right]}{\left[2j\right]} \binom{m}{2j} \frac{(2j)!}{j!^{2j}} (m \geq 0).
\]

From (2.4) - (2.6), and after simplifications, we have

\[
\gamma_{n,r}(x) = \frac{1}{r!} \sum_{k=0}^{n} \frac{(-2)^k}{(n-k)!} \sum_{l=0}^{k} \frac{\left[\frac{k}{2}\right]}{l!} \frac{(2n + 2r - 2l)!}{(n + r - l)!} \frac{(2n + 2r - 2l)!}{(k - l)!} \frac{1}{j!^{4j}(k - 2j)!} H_{n-k}(x)
\]

\[
= \frac{(2n + 2r)!}{r!^{4n+2r} \frac{1}{2} > n+r} \sum_{k=0}^{n} \frac{(-2)^k}{(n-k)!} \sum_{l=0}^{k} \frac{\left[\frac{k}{2}\right]}{l!} \frac{(2n + 2r - 2l)!}{(n + r - l)!} \frac{1}{j!^{4j}(k - 2j)!} H_{n-k}(x)
\]

\[
\times \sum_{l=0}^{k-2j} \sum_{t=0}^{2j-k-t} \frac{\left[\frac{k}{2}\right]}{l!} \frac{(2n - 2r - 2l)!}{(n - r - 2l)!} H_{n-k}(x)
\]

\[
= \frac{(2n + 2r)!}{r!^{4n+2r} \frac{1}{2} n+r} \sum_{k=0}^{n} \frac{(-2)^k}{(n-k)!} \sum_{l=0}^{k} \frac{\left[\frac{k}{2}\right]}{l!} \frac{(2n + 2r - 2l)!}{(n + r - l)!} \frac{1}{j!^{4j}(k - 2j)!} H_{n-k}(x)
\]

\[
\times \sum_{j=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{2F_1}{j!^{4j}(k - 2j)!} H_{n-k}(x).
\]

This shows (1.35) of Theorem 1.1.

Next, we let

\[
\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,2} L_{n-k}^r(x).
\]

Then, from (b) of Proposition 2.1, (2.1), (2.3) and integration by parts \(k\) times, we get
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Theorem 1.1

This completes the proof for (1.36) of Theorem 1.1.

Finally, let us put

$$\gamma_{n,r}(x) = \sum_{k=0}^{n} C_{k,3} P_k(x).$$  (2.11)
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Then, from (c) of Proposition 2.1, (2.1), (2.3) and integration by parts \( k \) times, we have

\[
C_{k,3} = \frac{2k + 1}{2^{r+1}kr!} \int_{-1}^{1} V_{n+r}^{(r+k)}(x)(x^2 - 1)^k dx
= \frac{(2k + 1)(-1)^{n-k}2n + 2r - l}{2^{r+1}kr!} \sum_{l=0}^{n-k} \binom{2n + 2r - l}{l} 2^{n+r-l}(n + r - l)_{r+k} \tag{2.12}
\times \int_{-1}^{1} (x - 1)^{n-k-l}(x^2 - 1)^k dx.
\]

By making use of (c) in Proposition 2.2 and after simplifications, from (2.12) we obtain

\[
C_{k,3} = \frac{(-1)^{n+k}(2k + 1)}{r!} \sum_{l=0}^{n-k} \frac{(-1)^{l}4^{n-l}(2n + 2r - l)!}{l!(2n + 2r - 2l)!} \frac{1}{(n - k)!} \frac{1}{(n + k + 1)!} \tag{2.13}
\times \sum_{l=0}^{n-k} \frac{(-1)^{l}n!}{(l! < k - n >_l < \frac{1}{2} - n - r >_l < -n - k - 1 >_l)}
\times 3F_2(k - n, \frac{1}{2} - n - r, -n - k - 1; -2n + 2r, -n; 1).
\]

From (2.11) and (2.13), we get

\[
\gamma_{n,r}(x) = \frac{(-1)^{n}(2n + 2r)!n!}{r!4^{r}(n + r - \frac{1}{2})^{n+r} (n - k)! (n + k + 1)!} \frac{(-1)^{k}(2k + 1)}{n^{2k + 1}r^k}
\times 3F_2(k - n, \frac{1}{2} - n - r, -n - k - 1; -2n + 2r, -n; 1)P_k(x).
\tag{2.14}
\]

This proves (1.37) of Theorem 1.1.

3. Proof of Theorem 1.2

Here we will show only (1.43) and (1.44) in Theorem 1.2, as (1.40), (1.41), (1.42) can be shown analogously to the proofs for (1.35), (1.36), (1.37), respectively. The following can be derived by differentiating the equation (1.10) and is stated in [13].
Lemma 3.1. Let \( n, r \) be integers with \( n \geq 0, r \geq 1 \). Then we have the following identity.

\[
\sum_{l=0}^{n} \sum_{i_1+i_2+\cdots+i_{r+1}=l} (-1)^{n-l} \binom{n-l}{l} W_{i_1}(x)W_{i_2}(x)\cdots W_{i_{r+1}}(x) = \frac{1}{2r+1} W_{n+r}(x),
\]

(3.1)

where the inner sum runs over all nonnegative integers \( i_1, i_2, \ldots, i_{r+1} \), with \( i_1 + i_2 + \cdots + i_{r+1} = l \).

From (1.17), the \( r \)-th derivative of \( W_n(x) \) is given by

\[
W^{(r)}_n(x) = (2n+1) \sum_{l=0}^{n-r} \frac{2^{n-l}}{2n+1-2l} \binom{2n-l}{l}(n-l)_r(x-1)^{n-l-r}.
\]

(3.2)

In particular, we have

\[
W^{(r+k)}_{n+r}(x) = (2n+1) \sum_{l=0}^{n-k} \frac{2^{n+r-l}}{2n+2r+1-2l} \binom{2n+2r-l}{l}(n+r-l)_{r+k}(x-1)^{n-k-l}.
\]

(3.3)

With \( \mathcal{E}_{n,r}(x) \) as in (1.34), we let

\[
\mathcal{E}_{n,r}(x) = \sum_{k=0}^{n} C_{k,4} C_k^{(a)}(x).
\]

(3.4)

Then, from (d) of Proposition 2.1, (3.1),(3.3) and integration by parts \( k \) times, we get

\[
C_{k,4} = \frac{(k+\lambda)\Gamma(\lambda)}{2^{k+r+1}r!\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}
\]

\[
\times \int_{-1}^{1} W^{(r+k)}_{n+r}(x)(1-x^2)^{k+\lambda-\frac{1}{2}} dx
\]

\[
= \frac{(k+\lambda)\Gamma(\lambda)(2n+1)}{2^{k+r+1}r!\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}
\]

\[
\times \sum_{l=0}^{n-k} \frac{2^{n+r-l}}{2n+2r+1-2l} \binom{2n+2r-l}{l}(n+r-l)_{r+k}
\]

\[
\times \int_{-1}^{1} (x-1)^{n-k-l}(1-x^2)^{k+\lambda-\frac{1}{2}} dx
\]

(3.5)

\[
= \frac{(k+\lambda)\Gamma(\lambda)(2n+1)(-2)^{n-k}}{r!\sqrt{\pi}\Gamma(k+\lambda+\frac{1}{2})}
\]

\[
\times \sum_{l=0}^{n-k} \frac{(-\frac{1}{2})^l(2n+2r-l)!(n+r-l)!}{(2n+2r-2l+1)!!(2n+2r-2l)!(n-k-l)!}
\]

\[
\times \int_{-1}^{1} (1-x)^{n-k-l}(1-x^2)^{k+\lambda-\frac{1}{2}} dx.
\]

Invoking (c) of Proposition 2.2 and after simplifications, from (3.5) we obtain
\[ C_{k,4} = \frac{(-1)^{n-k}(k + \lambda)(n + \lambda + \frac{1}{2})}{\Gamma(n + k + 2\lambda + 1)(n - k)!\sqrt{n!}} \times \sum_{l=0}^{n-k} \frac{(-1)^{l}(2n + 2r - l)![(n + r - l + 1)!(n - k)!(n + k + 2\lambda)_l]}{l!(2n + 2r)![(n + 2r - 2l + 2)!(n - k + l)!(n + \lambda - \frac{1}{2})_l]} \]

\[ = \frac{(-1)^k(k + \lambda)(n + \lambda + \frac{1}{2})}{\Gamma(n + k + 2\lambda + 1)(n - k)!\sqrt{n!}(n + r + \frac{1}{2})_{n+r+1}} \times \sum_{l=0}^{n-k} \frac{(-1)^{k}(k + \lambda)(2n + 1)2^{2\lambda - 2r - 1}(-1)^n\Gamma(n + \lambda + \frac{1}{2})(2n + 2r)!}{\Gamma(n + k + 2\lambda + 1)(n - k)!\sqrt{n!}(n + r + \frac{1}{2})_{n+r+1}} \times 3F2(k-n, -n - r - \frac{1}{2}, -n - k + 2\lambda; -2n - 2r, -n - \lambda + \frac{1}{2}; 1). \tag{3.6} \]

From (3.4) and (3.6), we have

\[ E_{n,r}(x) = \frac{\Gamma(n + 2n + 1)2^{2\lambda - 2r - 1}(-1)^n\Gamma(n + \lambda + \frac{1}{2})(2n + 2r)!}{r!\sqrt{n!}(n + r + \frac{1}{2})_{n+r+1}} \times \sum_{k=0}^{n} \frac{(-1)^k(k + \lambda)}{\Gamma(n + k + 2\lambda + 1)(n - k)!} \times 3F2(k-n, -n - r - \frac{1}{2}, -n - k + 2\lambda; -2n - 2r, -n - \lambda + \frac{1}{2}; 1)C_k^{(\alpha, \beta)}(x). \tag{3.7} \]

This shows (1.43) of Theorem 1.2.

Next, we let

\[ E_{n,r}(x) = \sum_{k=0}^{n} C_{k,5} P_n^{(\alpha, \beta)}(x). \tag{3.8} \]

Then, from (e) of Proposition 2.1, and (3.1), (3.3), and integrating by parts \( k \) times, we obtain

\[ C_{k,5} = \frac{(2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1)}{2^{2\alpha + \beta + k + r + 1}d\Gamma(\alpha + k + 1)\Gamma(\beta + k + 1)} \times \int_{-1}^{1} W_n^{(r+k)}(x)(1 - x)^{k+\alpha}(1 + x)^{k+\beta}dx \]

\[ = \frac{(2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1)(2n + 1)}{2^{2\alpha + \beta + k + r + 1}d\Gamma(\alpha + k + 1)\Gamma(\beta + k + 1)} \times \sum_{l=0}^{n-k} \frac{2^{n+r-l}}{2n + 2r - 2l + 1} \left(\frac{2n + 2r - l}{l}\right)(n + r - l)_{r+k}(-1)^{n-k-l} \times \int_{-1}^{1} (1 - x)^{n+\alpha-l}(1 + x)^{k+\beta}dx. \tag{3.9} \]
Sums of finite products of Chebyshev polynomials of the third and fourth kinds

By exploiting (b) in Proposition 2.2 and after simplifications, from (3.9) we get

\[
C_{k,5} = \frac{(-1)^{n-k}(2k + \alpha + \beta + 1)\Gamma(k + \alpha + \beta + 1)2^{2n+1}(2n + 1)\Gamma(n + \alpha + 1)}{\Gamma(n + k + \alpha + \beta + 2) r!} \times \sum_{l=0}^{n-k} \frac{(-1)^l(2n + 2r - l)!/(n + r - l + 1)!/(n + k + \alpha + \beta + 1)l}{l!(2n + 2r - 2l + 2)!/(n - k - l)!/(n + \alpha)l} = \frac{(-1)^{n-k}(2k + \alpha + \beta + 1)\Gamma(n + k + \alpha + \beta + 1)(2n + 1)\Gamma(n + \alpha + 1)(2n + 2r)!}{\Gamma(n + k + 1)\Gamma(n + k + \alpha + \beta + 2)(n - k)!r!2^{2r+1}(n + r + \frac{1}{2})_{n+r+1}} \times \sum_{l=0}^{n-k} \frac{(-1)^l(2n + 2r - l)!/(n - r - \frac{1}{2})l}{l!} < -n - r - \frac{1}{2} \times \frac{(-1)^l(2n + 2r - l)!/(n - r - \frac{1}{2})l}{l!} < -n - k - \alpha - \beta - 1 \times \frac{(-1)^l(2n + 2r - l)!/(n - r - \frac{1}{2})l}{l!} < -n - k - \alpha - \beta - 1; -2n - 2r, -n - \alpha, 1}. \quad (3.10)
\]

Thus, from (3.8) and (3.10), we have

\[
E_{n,r}(x) = \frac{(-1)^n(2n + 1)\Gamma(n + \alpha + 1)(2n + 2r)!}{r!2^{2r+1}(n + r + \frac{1}{2})_{n+r+1}} \times \sum_{k=0}^{n} \frac{(-1)^k(2k + \alpha + \beta + 1)\Gamma(n + k + \alpha + \beta + 1)}{\Gamma(n + k + 1)\Gamma(n + k + \alpha + \beta + 2)(n - k)!} \times \frac{(-1)^k(2k + \alpha + \beta + 1)\Gamma(n + k + \alpha + \beta + 1)}{\Gamma(n + k + 1)\Gamma(n + k + \alpha + \beta + 2)(n - k)!} \times \frac{(-2n + 2r, -n - \alpha; 1)P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(x)}.
\]

4. CONCLUSION

In this paper, we considered sums of finite products of Chebyshev polynomials of the third and fourth kinds and expressed each of them in terms of five orthogonal polynomials, namely Hermite, generalized Laguerre, Legendre, Genegbauer and Jacobi polynomials. This can be viewed as a generalization of the classical connection problem. Those sums of finite products were also represented by all kinds of Chebyshev polynomials in [15]. In addition, the same had been done for sums of finite products of Chebyshev polynomials of the second, Fibonacci polynomials, Legendre polynomials and Laguerre polynomials.

Competing interests

The authors declare that they have no competing interests.

Authors contributions

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