ON THE FULLY DEGENERATE DAEHEE NUMBERS AND POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. In this paper, we investigate the new degenerate Daehee polynomials and numbers which are called the degenerate Daehee polynomials of the second kind, and derive some new and interesting identities and properties of those polynomials.

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$.

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the *p-adic invariant integral on $\mathbb{Z}_p$* is defined as

$$I_0(f)=\int_{\mathbb{Z}_p} f(x)d\mu_0(x)=\lim_{N\to\infty}\frac{1}{p^N}\sum_{n=0}^{p^N-1} f(x),\ \text{(see [8, 16, 17])}. \tag{1.1}$$

From (1.1), we have

$$I_0(f_n)-I_0(f)=\sum_{l=0}^{n-1} f^{'}(l) \text{ where } f_n(x)=f(x+n) \text{ and } f^{'}(l)=\left.\frac{df(x)}{dx}\right|_{x=l}. \tag{1.2}$$

In particular,

$$I_0(f_1)-I_0(f)=f^{'}(0). \tag{1.3}$$

The *Stirling numbers of the first kind* are defined by

$$(x)_n=\sum_{l=0}^{n} S_1(n,l)x^l,\ (n\geq 0), \tag{1.4}$$

and the *Stirling numbers of the second kind* are given by

$$x^n=\sum_{l=0}^{n} S_2(n,l)(x)_l, \tag{1.5}$$

where $(x)_0=1$ and $(x)_n=x(x-1)\cdots(x-n+1)$, $(n\geq 1)$ (see [3, 28, 23]).

From (1.4) and (1.5), we can derive the following equations

$$(e^t-1)^n=n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \tag{1.6}$$

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and

$$(\log(x+1))^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!},\ (n\geq 0),\ (\text{see [3, 11, 28]}). \tag{1.7}$$

In addition,

$$\log(1+t) = \sum_{n=1}^{\infty}(-1)^{n+1}(n-1)!\frac{t^n}{n!},\ [3, 11, 28]. \tag{1.8}$$

The *Bernoulli polynomials of order* r are defined by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!},\ (\text{see [1, 4, 12, 25, 29]}). \tag{1.9}$$

The *Carlitz's degenerate Bernoulli polynomials of order* r is defined by the generating function to be

$$\sum_{n=0}^{\infty}\beta_n^{(r)}(x|\lambda)\frac{t^n}{n!} = \left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}, \tag{1.10}$$

where $\lambda\in\mathbb{R}$ (see [2, 10, 15, 19, 22]). By (1.10), we know that

$$\begin{aligned}\lim_{\lambda\to 0}\sum_{n=0}^{\infty}\beta_n^{(r)}(x|\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}\\ &= \left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{t^n}{n!},\end{aligned}$$

and thus, we get

$$\lim_{\lambda\to 0}\beta_n^{(r)}(x|\lambda) = B_n^{(r)}(x).$$

In [21], T. Kim et. al. defined the degenerate Bernoulli numbers and polynomials to be

$$\left(\frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}b_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \tag{1.11}$$

which are different from Carlitz's degenerate numbers and polynomials. By (1.11), we know that

$$\lim_{\lambda\to 0}b_{n,\lambda}(x) = B_n^{(r)}(x).$$

Note that, by (1.3),

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1+\cdots+x_r}{\lambda}}\,d\mu_0(x_1)\cdots d\mu_0(x_r)\\ &=\left(\frac{\frac{1}{\lambda}\log(1+\lambda t)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}b_{n,\lambda}^{(r)}(x)\frac{t^n}{n!},\end{aligned} \tag{1.12}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1+\cdots+x_r}{\lambda}}d\mu_0\cdots d\mu_0(x_r)\\ &=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\binom{\frac{x+x_1+\cdots+x_r}{\lambda}}{n}\lambda^n t^n d\mu_0(x_1)\cdots d\mu_0(x_r)\\ &=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_{n,\lambda}d\mu_0(x_1)\cdots d\mu_0(x_r)\frac{t^n}{n!}.\end{aligned} \tag{1.13}$$

By (1.12) and (1.13), we know that

$$b_{n,\lambda}^{(r)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_{n,\lambda}d\mu_0(x_1)\cdots d\mu_0(x_r). \tag{1.14}$$

The *higher-order Daehee polynomials* are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)^r(1+t)^x=\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!},\ \text{(see [7, 12, 13, 30])}. \tag{1.15}$$

In particular, if $r=1$, then $D_n^{(1)}(x)=D_n(x)$ are called the *Daehee polynomials*.

By replacing t as $\log(1+t)$ in (1.15), we have

$$\begin{aligned}\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{1}{n!}\left(\log(1+t)\right)^n\\&=\left(\sum_{n=0}^{\infty}B_n^{(r)}(x)\frac{1}{n!}\right)\left(n!\sum_{l=0}^{\infty}S_1(l,n)\frac{t^l}{l!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_m^{(r)}(x)S_1(n,m)\right)\frac{t^n}{n!},\end{aligned}$$

and so, we get

$$D_n^{(r)}(x)=\sum_{m=0}^{n}B_m^{(r)}(x)S_1(n,m).$$

Note that by (1.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!}&=\left(\frac{\log(1+t)}{t}\right)^r(1+t)^x\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x+x_1+\cdots+x_r}d\mu_0(x_1)\cdots d\mu_0(x_r)\\&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{x+x_1+\cdots+x_r}{n}t^n d\mu_0(x_1)\cdots d\mu_0(x_r)\\&=\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_n d\mu_0(x_1)\cdots d\mu_0(x_r)\right)\frac{t^n}{n!},\end{aligned}$$

and thus, we know that

$$D_n^{(r)}(x)=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_n d\mu_0(x_1)\cdots d\mu_0(x_r). \tag{1.16}$$

L. Carlitz introduced the degenerate Bernoulli polynomials in [2], and the degenerate of some special functions have studied by many researchers (see [2, 5, 6, 9, 10, 11, 14, 15, 19, 20, 22, 24, 26, 27]).

In particular, authors defined the *degenerate Stirling numbers of the second kind* which are defined by the generating function

$$\frac{1}{m!}\left((1+\lambda t)^{\frac{1}{\lambda}}-1\right)^m=\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!}, \tag{1.17}$$

where m is a given nonnegative integer in [11, 14, 18, 20, 23].

Daehee numbers and polynomials are introduced by Kim et. al. in [7], and those polynomials have been generalized and obtained interesting properties by many researchers (see [4, 7, 8, 9, 12, 13, 23, 25, 26]).

In this paper, we consider the new degenerate Daehee polynomials and numbers which are called the degenerate Daehee polynomials of the second kind, and derive some new and interesting identities and properties of those polynomials.

2. Fully degenerate Daehee polynomials of the second kind

Let us assume that $\lambda \in \mathbb{R}$. By (1.3), we have

$$\int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{x+y}{\lambda}}\, d\mu_0(y) = \frac{\frac{1}{\lambda}\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}. \tag{2.1}$$

By (2.1), we define the *degenerate Daehee polynomials of the second kind* by the generating function to be

$$\frac{\frac{1}{\lambda}\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}(1+\lambda\log(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} D_n(x|\lambda)\frac{t^n}{n!}. \tag{2.2}$$

In the special case, $x = 0$, $D_n(\lambda) = D_n(0|\lambda)$ are called the *degenerate Daehee numbers of the second kind.*

Note that

$$\begin{aligned}\lim_{\lambda\to 0}\sum_{n=0}^{\infty} D_n(x|\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{\frac{1}{\lambda}\log(1+\lambda\log(1+t))}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\ &= \frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!},\end{aligned}$$

and thus, we know that

$$\lim_{\lambda\to 0} D_n(x|\lambda) = D_n(x), \ (n \geq 0).$$

From (1.7) and (2.1), we have

$$\begin{aligned}&\int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{x+y}{\lambda}}\, d\mu_0(y)\\ =&\sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}\lambda^l(\log(1+t))^l\, d\mu_0(y)\\ =&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} S_1(n,l)\lambda^l l!\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}d\mu_0(y)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we have

$$D_{n,\lambda} = \sum_{k=0}^{n} S_1(n,l)\lambda^l l!\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}d\mu_0(y), \ (n \geq 0). \tag{2.4}$$

Since

$$\begin{aligned}\binom{\frac{x+y}{\lambda}}{l} &= \frac{\left(\frac{x+y}{\lambda}\right)\left(\frac{x+y}{\lambda}-1\right)\left(\frac{x+y}{\lambda}-2\right)\cdots\left(\frac{x+y}{\lambda}-l+1\right)}{l!}\\ &= \frac{(x+y)(x+y-\lambda)(x+y-2\lambda)\cdots(x+y-(l-1)\lambda)}{\lambda^l l!},\end{aligned} \tag{2.5}$$

by (2.4) and (2.5), we have

$$D_n(\lambda)=\sum_{k=0}^{n}S_1(n,l)\int_{\mathbb{Z}_p}(x+y)_{l,\lambda}d\mu_0(y),\ (n\geq 0), \tag{2.6}$$

where $(x)_{n,\lambda}=x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda)$.

Thus, by (2.4) and (2.5), we have the following theorem.

Theorem 2.1. *For each $n\geq 0$, we have*

$$\begin{aligned}D_n(\lambda)&=\sum_{k=0}^{n}S_1(n,l)\lambda^l l!\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}d\mu_0(y)\\&=\sum_{k=0}^{n}S_1(n,l)\int_{\mathbb{Z}_p}(x+y)_{l,\lambda}d\mu_0(y).\end{aligned}$$

By replacing t as e^t-1 in (2.2), we obtain the following.

$$\begin{aligned}\sum_{n=0}^{\infty}D_n(x|\lambda)\frac{(e^t-1)^n}{n!}&=\sum_{n=0}^{\infty}D_n(x|\lambda)\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}D_m(x|\lambda)S_2(n,m)\right)\frac{t^n}{n!}.\end{aligned}\tag{2.7}$$

On the other hand,

$$\begin{aligned}\sum_{n=0}^{\infty}D_n(x|\lambda)\frac{(e^t-1)^n}{n!}&=\frac{\frac{1}{\lambda}\log(1+\lambda t)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}\\&=\sum_{n=0}^{\infty}B_n(x|\lambda)\frac{t^n}{n!},\end{aligned}\tag{2.8}$$

where $B_n(x|\lambda)$ is the *degenerate Bernoulli polynomials of the second kind of order* $r\in\mathbb{Z}$ which are defined by the generating function to be

$$\left(\frac{\frac{1}{\lambda}\log(1+\lambda t)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}B_n^{(r)}(x|\lambda)\frac{t^n}{n!}\ \text{(see [10, 15, 22])}. \tag{2.9}$$

In the special case $r=1$, $B_n^{(1)}(x|\lambda)=B_n(x|\lambda)$ is called the *degenerate Bernoulli polynomials of the second kind.*

For positive integer d with $d\equiv 1 \pmod 2$, if we put $f(x)=(1+\lambda\log(1+t))^{\frac{x}{\lambda}}$, then, by (1.2) we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{x+d}{\lambda}}d\mu_0(y)-\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}d\mu_0(y)\\&=\sum_{l=0}^{d-1}\frac{1}{\lambda}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\log\left(1+\lambda\log(1+t)\right).\end{aligned}\tag{2.10}$$

By (2.10), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (1+\lambda\log(1+t))^{\frac{x}{\lambda}}\, d\mu_0(x)\\
=&\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{d}{\lambda}}-1}\sum_{l=0}^{d-1}(1+\lambda\log(1+t))^{\frac{l}{\lambda}}\\
=&d\sum_{l=0}^{d-1}\frac{\frac{1}{\lambda}\log\left(1+\frac{\lambda}{d}d\log(1+t)\right)}{\left(1+\frac{\lambda}{d}d\log(1+t)\right)^{\frac{d}{\lambda}}-1}\left(1+\frac{\lambda}{d}d\log(1+t)\right)^{\frac{d}{\lambda}\frac{l}{d}}.
\end{aligned} \tag{2.11}$$

Note that, by (1.6),

$$\begin{aligned}
&\frac{\frac{1}{\lambda}\log\left(1+\frac{\lambda}{d}d\log(1+t)\right)}{\left(1+\frac{\lambda}{d}d\log(1+t)\right)^{\frac{d}{\lambda}}-1}\left(1+\frac{\lambda}{d}d\log(1+t)\right)^{\frac{d}{\lambda}\frac{l}{d}}\\
=&\sum_{n=0}^{\infty}B_n\left(\frac{l}{d}\middle|\frac{\lambda}{d}\right)\frac{(d\log(1+t))^n}{n!}\\
=&\sum_{n=0}^{\infty}B_n\left(\frac{l}{d}\middle|\frac{\lambda}{d}\right)\frac{d^n}{n!}n!\sum_{m=n}^{\infty}S_1(m,n)\frac{x^m}{m!}\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_m\left(\frac{l}{d}\middle|\frac{\lambda}{d}\right)d^mS_1(n,m)\right)\frac{t^n}{n!}
\end{aligned} \tag{2.12}$$

By (2.1), (2.11) and (2.12), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}D_n(\lambda)\frac{t^n}{n!}=&\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\,d\mu_0(x)\\
=&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{d-1}B_m\left(\frac{l}{d}\middle|\frac{\lambda}{d}\right)d^{m+1}S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.13}$$

Hence, by (2.7), (2.8) and (2.13), we obtain the following theorem.

Theorem 2.2. *For non-negative integer n and $d\in\mathbb{N}$ with $d\equiv 1$ (mod 2), we have*

$$B_n(x|\lambda)(x)=\sum_{m=0}^{n}D_{m,\lambda}(x)S_2(n,m),$$

and

$$D_{n,\lambda}=\sum_{m=0}^{n}\sum_{l=0}^{d-1}B_m\left(\frac{l}{d}\middle|\frac{\lambda}{d}\right)d^{m+1}S_1(n,m).$$

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By (2.2), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} D_n(x|\lambda)\frac{t^n}{n!} &= \left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}\right)(1+\lambda\log(1+t))^{\frac{x}{\lambda}} \\
&= \left(\sum_{n=0}^{\infty} D_n(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{\frac{x}{\lambda}}{n}\lambda^n\left(\log(1+t)\right)^n\right) \\
&= \left(\sum_{n=0}^{\infty} D_n(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{\frac{x}{\lambda}}{n}\lambda^n n!\sum_{l=n}^{\infty} S_1(l,n)\frac{x^l}{l!}\right) \qquad (2.14) \\
&= \left(\sum_{n=0}^{\infty} D_n(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}(x)_{l,\lambda}S_1(n,l)\frac{x^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(x)_{l,\lambda}S_1(k,l)D_{n-k}(\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides of (2.14), we obtain the following theorem.

Theorem 2.3. *For non-negative integer n, we have*

$$D_n(x|\lambda) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}(x)_{l,\lambda}S_1(k,l)D_{n-k}(\lambda).$$

Note that, if we put $f(x) = (1+\lambda\log(1+t))^{\frac{x}{\lambda}}$, then

$$\begin{aligned}
f^{'}(0) &= \frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right) \\
&= \frac{1}{\lambda}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\lambda^n}{n}\left(\log(1+t)\right)^n \\
&= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\lambda^{n-1}}{n}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{t^l}{l!} \qquad (2.15) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{n}(-1)^{m+1}\lambda^{m-1}(m-1)!S_1(n,m)\right)\frac{t^n}{n!},
\end{aligned}$$

and thus, by (1.3), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} f(x_1)d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)d\mu_0(x) \\
&= \frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}(1+\lambda\log(1+t))^{\frac{1}{\lambda}} - \frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1} \qquad (2.16) \\
&= \sum_{n=0}^{\infty}\left(D_n(1|\lambda)-D_n(\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left(\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)\right)\sum_{l=0}^{n-1}\left(1+\lambda\log(1+t)\right)^{\frac{l}{\lambda}} \\
=&\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}}\left(1+\lambda\log(1+t)\right)^{n}-\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}} \qquad (2.17)\\
=&\sum_{r=0}^{\infty}\left\{D_r(n|\lambda)-D_r(\lambda)\right\}\frac{t^r}{r!},
\end{aligned}$$

and, by (2.15), we get

$$\begin{aligned}
&\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)\sum_{l=0}^{n-1}\left(1+\lambda\log(1+t)\right)^{\frac{l}{\lambda}} \\
=&\left(\sum_{p=0}^{\infty}\sum_{m=1}^{p}(-\lambda)^{m-1}(m-1)!S-1(p,m)\frac{t^p}{p!}\right)\left(\sum_{l=0}^{n-1}\sum_{s=0}^{\infty}\sum_{q=0}^{s}(l)_{q,\lambda}S_1(s,q)\frac{t^s}{s!}\right) \\
=&\sum_{r=0}^{\infty}\left(\sum_{l=0}^{n-1}\sum_{p=0}^{r}\sum_{m=1}^{p}\sum_{q=0}^{r-p}\binom{r}{p}(l)_{q,\lambda}(-\lambda)^{m-1}(m-1)!S_1(p,m)S_1(r-p,q)\right)\frac{t^r}{r!}
\end{aligned} \qquad (2.18)$$

From (2.15), (2.16) and (2.18), we obtain the following corollary.

Theorem 2.4. *For each nonnegative integer r, we have*

$$D_r(1|\lambda)-D_r(\lambda)=\sum_{m=1}^{r}(-\lambda)^{m-1}(m-1)!S_1(r,m).$$

Moreover, for each positive integer $n\geq 2$,

$$D_r(n|\lambda)-D_r(\lambda)=\sum_{l=0}^{n-1}\sum_{p=0}^{r}\sum_{m=1}^{p}\sum_{q=0}^{r-p}\binom{r}{p}(l)_{q,\lambda}(-\lambda)^{m-1}(m-1)!S_1(p,m)S_1(r-p,q).$$

3. Higher-order Degenerate Daehee polynomials of the second kind

In this section, we consider the *higher-order degenerate Daehee polynomials of the second kind* given by the generating function as follows: For given positive real number r,

$$\left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}-1}\right)^{r}\left(1+\lambda\log(1+t)\right)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}D_n^{(r)}(x|\lambda)\frac{t^n}{n!}. \qquad (3.1)$$

In the special case $x=0$, $D_n^{(r)}(0|\lambda)=D_n^{(r)}(\lambda)$ are called the *higher-order degenerate Daehee numbers of the second kind.*

Note that

$$\begin{aligned}
\lim_{\lambda\to 0}\sum_{n=0}^{\infty}D_n^{(r)}(x|\lambda)\frac{t^n}{n!}&=\lim_{\lambda\to 0}\left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{\left(1+\lambda\log(1+t)\right)^{\frac{1}{\lambda}}-1}\right)^{r}\left(1+\lambda\log(1+t)\right)^{\frac{x}{\lambda}} \\
&=\left(\frac{\log(1+t)}{t}\right)^{r}(1+t)^{x}=\sum_{n=0}^{\infty}D_n^{(r)}(x)\frac{t^n}{n!}.
\end{aligned}$$

From (2.9), we note that

$$\begin{aligned}&\left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\&=\sum_{n=0}^{\infty}B_n^{(r)}(x|\lambda)\frac{(\log(1+t))^n}{n!}=\sum_{n=0}^{\infty}\frac{B_n^{(r)}(x|\lambda)}{n!}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{t^l}{l!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_m^{(r)}(x|\lambda)S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned}\tag{3.2}$$

In addition, by replacing t by e^t-1 in (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}D_n^{(r)}(x|\lambda)\frac{(e^t-1)^n}{n!}&=\sum_{n=0}^{\infty}\frac{D_n^{(r)}(x|\lambda)}{n!}n!\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}D_m^{(r)}(x|\lambda)S_2(n,m)\right)\frac{t^n}{n!},\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}D_n^{(r)}(x|\lambda)\frac{(e^t-1)^n}{n!}&=\left(\frac{\frac{1}{\lambda}\log(1+\lambda t)}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda t)^{\frac{x}{\lambda}}\\&=\sum_{n=0}^{\infty}B_n^{(r)}(x|\lambda)\frac{t^n}{n!}.\end{aligned}\tag{3.4}$$

Hence, by (3.2), (3.3) and (3.4), we obtain the following theorem.

Theorem 3.1. *For $n\geq 0$, we have*

$$D_n^{(r)}(x|\lambda)=\sum_{m=0}^{n}B_m^{(r)}(x|\lambda)S_1(n,m).$$

Moreover,

$$B_n^{(r)}(x|\lambda)=\sum_{m=0}^{n}D_m^{(r)}(x|\lambda)S_2(n,m).$$

Note that for each $k,r\in\mathbb{N}$

$$\begin{aligned}&\left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\&=\left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}\right)^{k}\left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}\right)^{r-k}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\&=\left(\sum_{n=0}^{\infty}D_n^{(k)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}D_n^{(r-k)}(x|\lambda)\frac{t^n}{n!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}D_m^{(k)}(\lambda)D_{n-m}^{(r-k)}(x|\lambda)\right)\frac{t^n}{n!}.\end{aligned}\tag{3.5}$$

It is well known that for each $k \in \mathbb{Z}$,

$$\left(\frac{t}{\log(1+t)}\right)^{k}(1+t)^{x-1}=\sum_{n=0}^{\infty}B_n^{(n-k+1)}(x)\frac{t^n}{n!}, \tag{3.6}$$

By (1.17), (1.7) and (3.6), we have

$$\begin{aligned}
&\left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\
&=\left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}\right)^{-r}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\
&=\sum_{n=0}^{\infty}B_n^{(n+r+1)}(x+1)\frac{1}{n!}\left((1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1\right)^{n}\\
&=\left(\sum_{n=0}^{\infty}B_n^{(n+r+1)}(x+1)\right)\left(\sum_{m=n}^{\infty}S_{2,\lambda}(m,n)\frac{(\log(1+t))^m}{m!}\right)\\
&=\sum_{p=0}^{\infty}\sum_{n=0}^{p}B_n^{(n+r+1)}(x+1)S_{2,\lambda}(p,n)\frac{1}{p!}\left(\log(1+t)\right)^{p}\\
&=\sum_{q=0}^{\infty}\left(\sum_{p=0}^{q}\sum_{n=0}^{p}B_n^{(n+r+1)}(x+1)S_{2,\lambda}(q-l,n)S_1(q,p)\right)\frac{t^q}{q!}.
\end{aligned} \tag{3.7}$$

By (3.5) and (3.7), we obtain the following theorem.

Theorem 3.2. *For each $n, q \geq 0$, we have*

$$D_n^{(r)}(x|\lambda)=\sum_{m=0}^{n}\binom{n}{m}D_m^{(k)}(\lambda)D_{n-m}^{(r-k)}(x|\lambda),$$

and

$$D_q^{(r)}(x|\lambda)=\sum_{p=0}^{q}\sum_{n=0}^{p}B_n^{(n+r+1)}(x+1)S_{2,\lambda}(q-l,n)S_1(q,p).$$

Note that, by (1.3),

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{x+x_1+\cdots+x_r}{\lambda}}\,d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\left(\frac{\frac{1}{\lambda}\log\left(1+\lambda\log(1+t)\right)}{(1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1}\right)^{r}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}\\
&=\sum_{n=0}^{\infty}D_n^{(r)}(x|\lambda)\frac{t^n}{n!}.
\end{aligned} \tag{3.8}$$

By (1.14) and (3.8), we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{x+x_1+\cdots+x_r}{\lambda}}\,d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\binom{\frac{x+x_1+\cdots+x_r}{\lambda}}{n}\lambda^n\left(\log(1+t)\right)^n d\mu(x_1)\cdots d\mu_0(x_r)\\
&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_{n,\lambda}\frac{1}{n!}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{x^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_{m,\lambda}S_1(n,m)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_1(n,m)\beta_m^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{3.9}$$

Thus, by (3.8) and (3.9), we obtain the following theorem.

Theorem 3.3. *For each nonnegative integer n and each integer r,*

$$D_n^{(r)}(x|\lambda)=\sum_{m=0}^{n}S_1(n,m)\beta_m^{(r)}(x).$$

By (1.7) and (1.17), we have

$$\begin{aligned}
&(1+\lambda\log(1+t))^{\frac{x+x_1+\cdots+x_r}{\lambda}}\\
&=\left((1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1+1\right)^{x+x_1+\cdots+x_r}\\
&=\sum_{m=0}^{\infty}\binom{x+x_1+\cdots+x_r}{m}\left((1+\lambda\log(1+t))^{\frac{1}{\lambda}}-1\right)^m\\
&=\sum_{m=0}^{\infty}(x+x_1+\cdots+x_r)_m\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{(\log(1+t))^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(x+x_1+\cdots+x_r)_mS_{2,\lambda}(n,m)\frac{(\log(1+t))^n}{n!}\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(x+x_1+\cdots+x_r)_mS_{2,\lambda}(n,m)\frac{1}{n!}n!\sum_{l=n}^{\infty}S_1(l,n)\frac{t^l}{l!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}(x+x_1+\cdots+x_r)_mS_{2,\lambda}(k,m)S_1(n,k)\right)\frac{t^n}{n!},
\end{aligned}\tag{3.10}$$

and so by (1.16), (3.8)and (3.10) , we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{x+x_1+\cdots+x_r}{\lambda}}d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}S_{2,\lambda}(k,m)S_1(n,k)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x+x_1+\cdots+x_r)_m d\mu_0(x_1)\cdots d\mu_0(x_r)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}S_{2,\lambda}(k,m)S_1(n,k)D_m^{(r)}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.11}$$

By (3.8) and (3.11), we obtain the following theorem.

Theorem 3.4. *For each nonnegative integer m,*

$$D_n^{(r)}(x|\lambda)=\sum_{k=0}^{n}\sum_{m=0}^{k}S_{2,\lambda}(k,m)S_1(n,k)D_m^{(r)}(x).$$

4. Conclusion

In the past years, the degenerate of special functions and the applications of those functions to new areas of mathematics have been studied by many researchers. In this paper, we considered the degenerate Daehee numbers and polynomials by using p-adic invariant integral on $\mathbb{Z}_p$ which are different from the Kim's degenerate Daehee polynomials. As a presentation, we derive some new and interesting properties of those polynomials.

Next, we defied the higher-order degenerate Daehee numbers and polynomials of the second kind, and found the relationship between the degenerate Bernoulli polynomials, the first and second Stirling numbers, the Bernoulli polynomials, degenerate Stirling numbers of the second kind and those numbers and polynomials.

5. Competing interests

The authors declare that they have no competing interests.

6. Author's Contributions

All authors contributed equally to this work. All authors read and approved the final manuscript.

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