 doi:10.20944/preprints201809.0562.v1

Approximation operator based on neighborhood systems

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Abstract

In this paper, we propose a new covering-based set in which the lower and the upper approximation operation are defined by neighborhood systems. We discuss this new type of covering-based set systematically in two steps. First, we study the basic properties of this covering-based set, such as the properties of normality, contraction, and monotone. Second, we discuss the relationship between the new type of covering-based set and the other ten sets proposed.
Keywords: rough sets, Covering approximation space, Neighborhood system, approximation operation, Partition.

1. Introduction

The concept of rough set was first propsed by Pawlak[11]. It is a useful tool for handing uncertain things . Comparing with other methods, the rough set theory has its advantages. For example it does not need any additional information about data in the process of dealing with uncertain data. It has been applied successfully in process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis,and so on. From then on, many researchers have made some significant contributions to developing the rough theory [11, 13, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. However, a problem with Pawlak's rough set theory is that partition or equivalence relation is explicitly used in the definition of the lower and upper approximations. Such a partition or equivalence relation is too restrictive for many applications because it can only deal with complete information systems. Zakowski generalized the classical rough set theory using coverings of a universe instead of partition[27]. Such generalization leads to various covering approximation operators that are both theoretical and practical importance[28, 29, 30]. The relationships between properties of covering-based approximation and their corresponding coverings have attracted intensive research. How to obtain this useful information and deal with uncertain data has become a widely studied problem. In the process of solving the problem, many methods have been proposed, such as statistical methods, fuzzy set theory [9, 16], computing word [16], and rough set theory [12, 14].

In the following, Bonikowski et al. studied this type of covering-based rough sets from the viewpoint of formal concepts. At this time, covering-based rough sets theory was proposed. As a

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Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 September 2018 doi:10.20944/preprints201809.0562.v1

more powerful tool,it can be used to handled the problems which can not be solved by classical Rough set theory such as the granularity problems in information systems. Up to now, about ten types of covering-based rough sets have been proposed and studied [10, 19, 21].

In this paper, a new type of covering-based rough set is proposed. This paper is arranged as follow: In section 2, the properties such as normality, contraction, and monotone are studied. If a property does not hold, the necessary and sufficient condition about neighborhood system in which this property holds is researched. In section 3, the condition that the type of covering-based rough set equals the other ten sets proposed by other scholars is evaluated and discussed.

2. The definition and properties of covering-based approximation operators

Let U be a finite set, called an universe, R be a equivalence relation on U, then the partition induced by R is denoted by $U/R = \{X_1, X_2, \cdots X_n\}$. For any $X \subseteq U$, two subsets of U are given as follows:

$$\underline{R}(X) = \cup\{K \in C : K \subseteq X\}$$
$$\overline{R}(X) = \cup\{K \in C : K \cap X \neq \emptyset\}$$

The first subset $\underline{R}(X)$ and the second $\overline{R}(X)$ are called lower approximation and upper approximation of X respectively.

Obviously, a partition of U is a covering of U, but a covering of U is not necessarily a partition of U, so the definition of covering approximation space has been introduced. Before definite the new type of covering-based rough set, it is necessary for us to give some basic definitions about covering approximation space.

Definition 2.1 (**Covering**[25]). Let U be an universe, a set of non-empty subsets $C = \{K_i \subseteq U : i \in I\}$ is called a covering of U if it satisfies $\cup C = U$, and $K_i \neq \varnothing$ for each $i \in I$.

Definition 2.2 (**Covering approximation space**[23]). Let U be an universe, C a covering of U , then we call U together with covering C a covering approximation space, denoted by (U, C).

Definition 2.3 (**The neighborhood of a point** x[24]). Let (U, C) be a Covering approximation space. For $x \in U$, we call $N(x) = \cap\{K \in C : x \in K\}$ the neighborhood of point x.

Definition 2.4 (**Neighborhood system**[24]). Let (U, C) be a Covering approximation space. We call $N = \{N(x) : x \in U\}$ the neighborhood system induced by (U, C).

From now on, symbol (U, C, N) is used to represent a Covering approximation space (U, C), N is the neighborhood system induced by (U, C).

Lemma 2.1. [24] Let (U, C, N) be a Covering approximation space. If $x, y \in U$ such that $x \in N(y)$, then $N(x) \subseteq N(y)$.

Proposition 2.1. Let (U, C, N) be a Covering approximation space. N froms a partition of U $\Leftrightarrow$ there does not exist a pair $x, y \in U$ such that $x \in N(y)$ and $y \notin N(x)$.

Proof. Necessity is simple, we only need to prove the sufficiency. Suppose there does not exist a pair $x, y \in U$ such that $x \in N(y)$ and $y \notin N(x)$, but N is not a partition of U. We take two conditions into consideration: (1) $\exists x_0, y_0 \in U$ such that $x_0 \in N(y_0)$ and $y_0 \notin N(x_0)$. It is a contradiction to the assumption. (2) $\exists x_1, y_1 \in U$ such that $N(x_1) \cap N(y_1) \neq \emptyset$, $x_1 \notin N(y_1)$ and $y_1 \notin N(x_1)$. Select $z_1 \in N(x_1) \cap N(y_1)$, since $y_1 \notin N(x_1)$, by lemma 2.1, we obtain a pair $y_1, z_1 \in U$ such that $z_1 \in N(y_1)$ and $y_1 \notin N(z_1)$. It is also a contradiction to the assumption. From condition (1)(2), the proof of the sufficiency is completed. □

Definition 2.5 (**Membership of a point** x). [23] Let (U, C, N) be a Covering approximation space. For a point $x \in U$, $FM(x) = \{K \in C : x \in K\}$ is called the membership of x.

Definition 2.6 (**Minimal description of a point** x). [23] Let (U, C, N) be a Covering approximation space. The minimal description of a point x is defined as

$$Md(x) = \{K \in C : x \in K \in C \wedge (\forall S \in C \wedge x \in S \subseteq K \Rightarrow K = S)\}$$

Definition 2.7. Let (U, C, N) be a Covering approximation space. For $X \subseteq U$, the covering-based lower approximation operation $N : 2^U \to 2^U$ is defined as

$$\underline{N}(X) = \cup\{N(x) : N(x) \subseteq X\}$$

And the covering-based upper approximation operation $N : 2^U \to 2^U$ is defined as

$$\overline{N}(X) = \underline{N}(X) \cup \{x \in U : N(x) \cap (X - \underline{N}(X)) \neq \emptyset\}$$

Definition 2.8. Let (U, C, N) be a Covering approximation space. For $X \subseteq U$,
(1) If $\underline{N}(X) = X$, then X is called an inner definable subset.
(2) If $\overline{N}(X) = X$, then X is called an outer definable subset.
(3) If $\overline{N}(X) = X = \underline{N}(X)$, then X is called a definable subset.

Theorem 2.1 below describe what is the essence of inner definable subset, outer definable subset and definable subset.

Theorem 2.1. *Let (U, C, N) be a Covering approximation space. For $X \subseteq U$,*
(1) *X is an inner definable subset $\Leftrightarrow \exists A \subseteq U$ such that $X = \cup\{N(x) : x \in A\}$.*
(2) *X an outer definable subset $\Leftrightarrow \forall x \notin X \Rightarrow (N(x) \cap X) \subseteq \underline{N}(X)$.*
(3) *X is a definable subset $\Leftrightarrow X$ is an inner definable subset.*

Proof. The proof is simple. □

Remark 2.1. X is a definable subset $\Leftrightarrow$ X is an inner definable subset $\Rightarrow$ X is an outer definable subset, but X is an outer definable subset$\not\Rightarrow$ X is an inner definable subset.

Example 2.1. If let $U = \{1, 2, 3, 4, 5\}, C = \{\{1, 2\}, \{3, 4\}, \{4\}, \{5\}\}, X_0 = \{1, 2, 3\}$, then $\underline{N}(X_0) = \{1, 2\} \neq X_0$, but $\overline{N}(X_0) = \{1, 2, 3\} = X_0$.

Proposition 2.2. Let (U, C, N) be a Covering approximation space. $\forall X, Y \subseteq U$, we have:

$$
\begin{aligned}
&(1)\underline{N}(U) = U(Co-normality)\\
&(2)\overline{N}(U) = U(Co-normality)\\
&(3)\underline{N}(\emptyset) = \emptyset(Normality)\\
&(4)\overline{N}(\emptyset) = \emptyset(Normality)\\
&(5)\underline{N}(X) \subseteq X \subseteq \overline{N}(X)(Contraction-Extension)\\
&(6)\underline{N}(X \cap Y) = \underline{N}(X) \cap \underline{N}(Y)(Multiplication)\\
&(7)X \subseteq Y \Rightarrow \underline{N}(X) \subseteq \underline{N}(Y)(Monotone)\\
&(8)\underline{N}(\underline{N}(X)) = \underline{N}(X)(Idempotency)\\
&(9)\overline{N}(\overline{N}(X)) = \overline{N}(X)(Idempotency)\\
&(10)\underline{N}(X) \cup \underline{N}(Y) \subseteq \underline{N}(X \cup Y)\\
&(11)\overline{N}(X \cup Y) \subseteq \overline{N}(X) \cup \overline{N}(Y).
\end{aligned}
$$

Proof. The proofs of $(1)-(7),(10)(11)$ are obvious. We only prove (8),(9).

Firstly, we prove (8). From proposition 2.2 property (5), $\underline{N}(\underline{N}(X)) \subseteq \underline{N}(X)$ holds. $\forall y \in \underline{N}(X)$, since $\underline{N}(X) = \cup\{N(x) : N(x) \subseteq X\}$, so $N(y) \subseteq \underline{N}(X)$. By the definition of $\underline{N}(\underline{N}(X))$, we have $y \in \underline{N}(\underline{N}(X))$. This means $\underline{N}(X) \subseteq \underline{N}(\underline{N}(X))$, combining $\underline{N}(\underline{N}(X)) \subseteq \underline{N}(X)$, the proof of property (8) is completed.

Secondly, we prove property (9). From proposition 2.2 property (5), $\overline{N}(X) \subseteq \overline{N}(\overline{N}(X))$ holds. $\forall x \in \overline{N}(\overline{N}(X))$, we take two conditions into consideration: (a) $x \in \underline{N}(\overline{N}(X))$, we have $x \in \overline{N}(X))$. (b) $N(x) \cap (\overline{N}(X) - \underline{N}(\overline{N}(X))) \neq \emptyset$, select $x_0 \in N(x) \cap (\overline{N}(X) - \underline{N}(\overline{N}(X)))$. Since $x_0 \in \overline{N}(X) - \underline{N}(\overline{N}(X))$, so $x_0 \notin \underline{N}(X)$ and $N(x_0) \cap (X - \underline{N}(X)) \neq \emptyset$. On the other hand, from the condition that $x_0 \in N(x)$ and Lemma 2.1, we have $N(x) \cap (X - \underline{N}(X)) \neq \emptyset$. This means $x \in \overline{N}(X)$. According to $(a)(b)$, the proof of (9) is completed. □

Generally speaking, suppose (U, C, N) be a Covering approximation space. $(\natural) X \subseteq Y \subseteq U \nRightarrow \overline{N}(X) \subseteq \overline{N}(Y)$, $(\sharp)$ $\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)$ does not always hold.

Example 2.2. Let $U = \{1,2,3,4,5,6\}, C = \{\{1\},\{2\},\{3\},\{1,2,3,4,5\},\{1,2,3,4,5,6\}\}, X_0 = \{1,2,3,4\}, Y_0 = \{1,2,3,4,5\}$, then $X_0 \subseteq Y_0, \overline{N}(X_0) = \{1,2,3,4,5,6\} \nsubseteq \overline{N}(Y_0) = \{1,2,3,4,5\}$ and $\overline{N}(X_0 \cup Y_0) = \{1,2,3,4,5\} \neq \overline{N}(X_0) \cup \overline{N}(Y_0) = \{1,2,3,4,5,6\}$.

Theorem 2.2. *Let (U, C, N) be a Covering approximation space.* $\forall X \forall Y[(X \subseteq Y \subseteq U) \Rightarrow (\overline{N}(X) \subseteq \overline{N}(Y))] \Leftrightarrow \forall X \forall Y(\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y))$

Proof. "$\Rightarrow$". $\forall X, Y \subseteq U$, since $X, Y \subseteq (X \cup Y)$, so $\overline{N}(X) \cup \overline{N}(Y) \subseteq \overline{N}(X \cup Y)$. By proposition 2.2(11), we have $\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)$.

"$\Leftarrow$". $\forall X \subseteq Y \subseteq U$, since $X \cup Y = Y$, so $\overline{N}(Y) = \overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)$. This means $\overline{N}(X) \subseteq \overline{N}(Y))$. □

Theorem 2.3. *Let (U, C, N) be a Covering approximation space.* $\forall X \forall Y[(X \subseteq Y \subseteq U) \Rightarrow (\overline{N}(X) \subseteq \overline{N}(Y))] \Leftrightarrow$ *There does not exist a pair* $x, y[(|N(x)| > 1) \wedge (|N(y)| > 1) \wedge (x \in N(y)) \wedge (y \notin N(x))]$.

Proof. "⇒", proof by contradiction. Suppose $\exists x_0 \exists y_0[(|N(x_0)| > 1) \wedge (|N(y_0)| > 1) \wedge (x_0 \in N(y_0)) \wedge (y_0 \notin N(x_0))]$. Select $z_0 \in N(x_0)$ and $x_0 \neq z_0$, let $X_0 = N(x_0) - \{z_0\}$ and $Y_0 = N(x_0)$. We can obtain the fact that $X_0 \subseteq Y_0, y_0 \in \overline{N}(X_0)$ and $y_0 \notin \overline{N}(Y_0)$. This means $\overline{N}(X_0) \nsubseteq \overline{N}(Y_0)$, contradicts the assumption of necessity.

"⇐", proof by contradiction. Suppose $\exists X_0 \exists Y_0 \exists p_0[(X_0 \subseteq Y_0) \wedge (p_0 \in \overline{N}(X_0)) \wedge (p_0 \notin \overline{N}(Y_0))]$. Since $p_0 \notin \overline{N}(Y_0)$, so $p_0 \notin Y_0$ and $p_0 \notin X_0$. From the fact that $p_0 \in \overline{N}(X_0)$, we have $N(p_0) \cap (X_0 - \underline{N}(X_0)) \neq \emptyset$. Select $q_0 \in N(p_0) \cap (X_0 - \underline{N}(X_0))$, take the conditions $p_0 \notin X_0$ and $p_0 \notin \overline{N}(Y_0)$ into consideration, we have $p_0 \neq q_0, q_0 \in N(p_0), |N(q_0)| > 1$ and $p_0 \notin N(q_0)$. This means $p_0, q_0[(|N(p_0)| > 1) \wedge (|N(q_0)| > 1) \wedge (q_0 \in N(p_0)) \wedge (p_0 \notin N(q_0))]$, contradicts the assumption of sufficiency. □

Corollary 2.1. By using theorem 2.2, 2.3, we obtain the fact that $\forall X \forall Y(\overline{N}(X \cup Y) = \overline{N}(X) \cup \overline{N}(Y)) \Leftrightarrow$ There does not exist a pair $x, y[(|N(x)| > 1) \wedge (|N(y)| > 1) \wedge (x \in N(y)) \wedge (y \notin N(x))]$.

Proposition 2.3. Let (U, C, N) be a Covering approximation space. The properties below hold.

$$(1)\underline{N}(U - \underline{N}(X)) \subseteq U - \underline{N}(X)$$
$$(2)\underline{N}(U - X) \subseteq U - \overline{N}(X)$$

Generally speaking, equality $\underline{N}(U - \underline{N}(X)) = U - \underline{N}(X)$ and $\underline{N}(U - X) = U - \overline{N}(X)$ do not always hold.

Example 2.3. Let $U = \{1, 2, 3, 4\}, C = \{\{1\}, \{2\}, \{3\}, \{3, 4\}\}, X_0 = \{1, 2, 3\}$. We have $\underline{N}(U - \underline{N}(X_0)) = \emptyset \neq U - \underline{N}(X_0) = \{4\}$

Example 2.4. Let $U = \{1, 2, 3, 4, 5\}, C = \{\{1\}, \{2\}, \{3\}, \{1, 2, 3, 4\}, \{5\}\}, X_0 = \{1, 2, 3\}$. We have $\underline{N}(U - X_0) = \{5\} \neq U - \overline{N}(X_0) = \{4, 5\}$.

Theorem 2.4. *Let (U, C, N) be a Covering approximation space. $\forall X[\underline{N}(U - \underline{N}(X)) \subseteq (U - \underline{N}(X))] \Leftrightarrow N$ forms a partition of U.*

Proof. "⇒", proof by contradiction. Suppose N does not form a partition of U. From proposition 2.1, we can obtain $x_0, y_0 \in U$ such that $x_0 \in N(y_0)$ and $y_0 \notin N(x_0)$. If we choose $X_0 = N(x_0)$, then $y_0 \in U - \underline{N}(X_0)$ and $y_0 \notin \underline{N}(U - \underline{N}(X_0))$. This means $\underline{N}(U - \underline{N}(X_0)) \neq U - \underline{N}(X_0)$, contradicts the assumption of necessity.

"⇐" is simple. □

Theorem 2.5. *Let (U, C, N) be a Covering approximation space.*
(1)$\forall X[\underline{N}(U - \underline{N}(X)) = U - \underline{N}(X)] \Leftrightarrow N$ forms a partition of U.
(2)$\forall X[\underline{N}(U - X) = (U - \overline{N}(X))] \Leftrightarrow N$ forms a partition of U.

Proof. (1)"⇒", proof by contradiction. Suppose N does not form a partition of U. From proposition 2.1, we can obtain $x_0, y_0 \in U$ such that $x_0 \in N(y_0)$ and $y_0 \notin N(x_0)$. If we choose $X_0 = N(x_0)$, then $y_0 \in U - \underline{N}(X_0)$ and $y_0 \notin \underline{N}(U - \underline{N}(X_0))$. This means $\underline{N}(U - X_0) \neq \underline{N}(U - \underline{N}(X_0))$, contradicts the assumption of necessity.

"⇐" is simple.

(2)"⇒", proof by contradiction. Suppose N does not form a partition of U. From proposition 2.1, we can obtain $x_0, y_0 \in U$ such that $x_0 \in N(y_0)$ and $y_0 \notin N(x_0)$. If we choose $X_0 = N(x_0)$, then $y_0 \in U - \overline{N}(X_0)$ and $y_0 \notin \underline{N}(U - X_0)$. This means $\underline{N}(U - X_0) \neq (U - \overline{N}(X_0))$, contradicts the assumption of necessity.

"$\Leftarrow$" is simple. □

3. Relationships between the new type lower and upper approximation operations and the other types

For a covering of U, there are about ten types of lower approximation operations and upper approximation operations. A natural question is what are the Relationships between the new type of lower approximation and upper approximation operation and the other types. To answer this question , we need to narrate the definitions of ten types of lower approximation operations and upper approximation operations.

Definition 3.1. Let (U, C, N) be a Covering approximation space. For each $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $\underline{C}_n$ and $\overline{C}_n$ are called the $n - th$ lower approximation operation and upper approximation operation respectively, defined as follow:

$$
\begin{array}{ll}
(1) & \underline{C}_1(X) = \cup\{K : K \in C \wedge K \subseteq X\}, \\
 & \overline{C}_1(X) = \underline{C}_1(X) \cup (\cup\{\cup Md(x) : x \in X - \underline{C}_1(X)\}). \\
(2) & \underline{C}_2(X) = \cup\{K : K \in C \wedge K \subseteq X\}, \\
 & \overline{C}_2(X) = \cup\{K : K \in C \wedge K \cap X \neq \emptyset\}. \\
(3) & \underline{C}_3(X) = \cup\{K : K \in C \wedge K \subseteq X\}, \\
 & \overline{C}_3(X) = \cup\{\cup Md(x) : x \in X\}. \\
(4) & \underline{C}_4(X) = \cup\{K : K \in C \wedge K \subseteq X\}, \\
 & \overline{C}_4(X) = \underline{C}_4(X) \cup (\cup\{K : K \in C \wedge K \cap (X - \underline{C}_4(X) \neq \emptyset\}). \\
(5) & \underline{C}_5(X) = \cup\{K : K \in C \wedge K \subseteq X\}, \\
 & \overline{C}_5(X) = \underline{C}_5(X) \cup (\cup\{N(x) : x \in X - \underline{C}_5(X)\}). \\
(6) & \underline{C}_6(X) = \{x \in U : N(x) \subseteq X\}, \\
 & \overline{C}_6(X) = \{x \in U : N(x) \cap X \neq \emptyset\}. \\
(7) & \underline{C}_7(X) = \{x \in U : \forall K \in C(x \in K \Rightarrow K \subseteq X)\}, \\
 & \overline{C}_7(X) = \cup\{K : K \in C \wedge K \cap X \neq \emptyset\}. \\
(8) & \underline{C}_8(X) = \cup\{K : K \in C \wedge K \subseteq X\}, \\
 & \overline{C}_8(X) = U - \underline{C}_8(U - X). \\
(9) & \underline{C}_9(X) = \{x \in U : \forall u(x \in N(u) \Rightarrow N(u) \subseteq X)\}, \\
 & \overline{C}_9(X) = \cup\{N(x) : x \in U \wedge N(x) \cap X \neq \emptyset\}. \\
(10) & \underline{C}_{10}(X) = \{x \in U : \forall u(x \in N(u) \Rightarrow u \in X)\}, \\
 & \overline{C}_{10}(X) = \cup\{N(x) : x \in X\}.
\end{array}
$$

Remark 3.1. $\underline{C}_n$ and $\overline{C}_n(n = 1, 2, 3)$ can be found from [19], $\underline{C}_4$ and $\overline{C}_4$ can be found from [21], $\underline{C}_5$ and $\overline{C}_5$ can be found from [18], $\underline{C}_6$ and $\overline{C}_6$ can be found from [21], $\underline{C}_7$ and $\overline{C}_7$ can be found from [25], $\underline{C}_n$ and $\overline{C}_n(n = 8, 9, 10)$ can be found from [10].

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Proposition 3.1. Let (U, C, N) be a Covering approximation space. The properties below hold, but all the symbol"$\subseteq$" can not be replaced by symbol"=".

$$\begin{array}{ll} (1) & \forall X(\underline{C}_1(X) \subseteq \underline{N}(X)), \\ (2) & \forall X(\overline{N}(X)) \subseteq \overline{C}_2(X)), \\ (3) & \forall X(\overline{N}(X)) \subseteq \overline{C}_4(X)). \end{array}$$

Example 3.1. Let $U = \{1,2,3,4,5\}, C = \{\{1,2\},\{3,4\},\{5\},\{3,5\}\}, X_0 = \{1,2,3\}$. We have $\underline{N}(X_0) = \{1,2,3\} \neq \underline{C}_1(X_0) = \{1,2\}$.

Example 3.2. Let $U = \{1,2,3,4,5\}, C = \{\{1,2,3\},\{3,4,5\}\}, X_0 = \{3\}$. We have $\overline{N}(X_0) = \{3\} \neq \overline{C}_2(X_0) = \{1,2,3,4,5\} = \overline{C}_4(X_0)$.

Theorem 3.1. *Let (U, C, N) be a Covering approximation space.*
(1)$\forall X(\underline{C}_1(X) = \underline{N}(X)) \Leftrightarrow \forall x(|Md(x)| = 1)$,
(2)$\forall X(\overline{C}_1(X) = \overline{N}(X)) \Leftrightarrow [\forall x(|Md(x)| = 1) \wedge N$ forms a partition of $U]$,
(3)$\forall X(\overline{C}_2(X) = \overline{N}(X)) \Leftrightarrow C$ forms a partition of U,
(4)$\forall X(\overline{C}_3(X) = \overline{N}(X)) \Leftrightarrow [\forall x(|Md(x)| = 1) \wedge N$ forms a partition of $U]$,
(5)$\forall X(\overline{C}_4(X) = \overline{N}(X)) \Leftrightarrow \{[\forall x(|Md(x)| = 1)] \wedge \forall K \in C[\forall y \in K(\{y\} \in C) \vee \forall z \in K(Md(z) = \{K\})]\}$.

Proof. (1)"$\Rightarrow$", proof by contradiction. Suppose $\exists x_0(|Md(x_0)| > 1)$, we can find $K_1, K_2 \in Md(x_0)$ such that $x_0 \in K_1 \cap K_2, K_1 \cap K_2 \subsetneq K_1$ and $K_1 \cap K_2 \subsetneq K_2$. If we choose $X_0 = K_1 \cap K_2$, then $\underline{N}(X_0) = K_1 \cap K_2 \neq \underline{C}_1(X_0) = \emptyset$. This contradicts the assumption of necessity.

"$\Leftarrow$" is simple.

(2)"$\Rightarrow$", proof by contradiction. Firstly, we prove $\forall x(|Md(x)| = 1)$. Suppose $\exists x_0(|Md(x_0)| > 1)$, select $K_1, K_2 \in Md(x_0)$ such that $x_0 \in K_1 \cap K_2, K_1 \cap K_2 \subsetneq K_1$ and $K_1 \cap K_2 \subsetneq K_2$. Without loss of generality, if we choose $y_0 \in K_2$ and $y_0 \notin K_1$, then $y_0 \notin \overline{N}(K_1 \cap K_2)$. Since there does not exist $K \in C$ such that $x_0 \in K \subseteq K_1 \cap K_2$, so $x_0 \in K_1 \cap K_2 - \underline{C}_1(K_1 \cap K_2)$ and $y_0 \in \overline{C}_1(K_1 \cap K_2)$. This means $\overline{N}(K_1 \cap K_2) \neq \overline{C}_1(K_1 \cap K_2)$, contradicts the assumption of necessity.

Secondly, we prove that N forms a partition of U. Suppose N is not a partition of U, by proposition 2.1, $\exists x_1, y_1 \in U$ such that $x_1 \in N(y_1)$ and $y_1 \notin N(x_1)$. If we choose $X_0 = N(y_1) - N(x_1)$, then $x_1 \notin \overline{N}(X_0)$. Since $x_1 \in N(y_1)$, so there dose not exist $K \in C$ such that $y_1 \in K \subseteq X_0$. Thus $y_1 \in X_0 - \underline{C}_1(X_0)$ and $x_1 \in \cup Md(y_1) \subseteq \overline{C}_1(X_0)$. This means $\overline{C}_1(X_0) \neq \overline{N}(X_0)$, contradicts the assumption of necessity.

"$\Leftarrow$", $\forall X \subseteq U$, by theorem 3.1(1) and $\forall x(|Md(x)| = 1)$, we have $\underline{C}_1(X) = \underline{N}(X)$ and $\forall y(N(y) = \cup Md(y))$. $\forall z \in \overline{C}_1(X)$, we take two conditions into consideration: ($\sharp$) $z \in \underline{C}_1(X) \subseteq X$, we have $z \in \overline{N}(X)$. ($\natural$) $\exists z_0 \in X - \underline{C}_1(X)$ such that $z \in \cup Md(z_0) = N(z_0)$. Since N is a partition of U, so $N(z) = N(z_0)$. This means $z_0 \in N(z) \cap (X - \underline{C}_1(X)) = N(z) \cap (X - \underline{N}(X) \neq \emptyset$. By the definition of $\overline{N}(X)$, We have $z \in \overline{N}(X)$. Coming ($\sharp$) with ($\natural$), $\overline{C}_1(X) \subseteq \overline{N}(X)$. On the other hand, $\forall p \in \overline{N}(X)$, We also take two conditions into consideration:($\sharp\sharp$)$p \in \underline{N}(X) \subseteq X$,we have $p \in \overline{C}_1(X)$. ($\natural\natural$)$N(p) \cap (X - \underline{N}(X)) \neq \emptyset$, We can choose $p_0 \in N(p) \cap (X - \underline{N}(X)) = N(p) \cap (X - \underline{C}_1(X))$, consider that N is a partition of U, thus $p \in N(p_0) = \cup Md(p_0)$. By the definition of $\overline{C}_1(X)$, we have $p \in \overline{C}_1(X)$. Coming ($\sharp$)($\natural$) with ($\sharp\sharp$)($\natural\natural$), $\overline{N}(X) = \overline{C}_1(X)$ holds.

(3), the proof of (3) is simple.

(4), the proof of (4) is similar to (2) .

(5), "$\Rightarrow$", proof by contradiction. Firstly, we prove $\forall x(|Md(x)| = 1)$. Suppose $\exists x_0(|Md(x_0)| > 1)$, select $K_1, K_2 \in Md(x_0)$ such that $x_0 \in K_1 \cap K_2, K_1 \cap K_2 \subsetneq K_1$ and $K_1 \cap K_2 \subsetneq K_2$. Without loss of generality, if we choose $y_0 \in K_2$ and $y_0 \notin K_1$, then $y_0 \notin \overline{N}(K_1 \cap K_2)$. Since there does not exist $K \in C$ such that $x_0 \in K \subseteq K_1 \cap K_2$, so $x_0 \in K_1 \cap K_2 - \underline{C}_4(K_1 \cap K_2)$ and $y_0 \in \overline{C}_4(K_1 \cap K_2)$. This means $\overline{N}(K_1 \cap K_2) \neq \overline{C}_4(K_1 \cap K_2)$, contradicts the assumption of necessity.

Secondly, we prove $\forall K \in C[\forall y \in K(\{y\} \in C) \vee \forall z \in K(Md(z) = \{K\})]$. $\forall K \in C$, we take two conditions into consideration: ($\sharp$) $\exists p_0 \in K(\{p_0\} \in C) \Rightarrow \forall y \in K(\{y\} \in C)$. Otherwise, $\exists q_0 \in K(Md(q_0) = \{K_3\} \wedge |K_3| > 1)$. If select $q' \in K_3, q' \neq q_0$ and let $Y_0 = K_3 - \{q', p_0\}$, then $p_0 \notin \overline{N}(Y_0)$ and $p_0 \in \overline{C}_4(Y_0)$. This means $\overline{N}(Y_0) \neq \overline{C}_4(Y_0)$, contradicts the assumption of necessity. ($\natural$) $\forall m \in K(\{m\} \notin C) \Rightarrow \forall n \in K(Md(n) = \{K\})$. Otherwise, $\exists m_0 \in K$ such that $Md(m_0) = \{K_4\}, |K_4| > 1$ and $K_4 \subsetneq K$. Select $n_0 \in K - K_4, m' \in K_4, m' \neq m_0$ and let $Z_0 = \{n_0\}$, we obtain that $m_0 \notin \overline{N}(Z_0)$ and $m_0 \in \overline{C}_4(Z_0)$. This means $\overline{C}_4(Z_0) \neq \overline{N}(Z_0)$, contradicts the assumption of necessity.

"$\Leftarrow$" is simple. □

Proposition 3.2. Let (U, C, N) be a Covering approximation space. The properties below hold, but all the symbol"$\subseteq$" can not be replaced by symbol"$=$".

$$\begin{array}{ll} (1) & \forall X(\underline{C}_7(X) \subseteq \underline{N}(X)), \\ (2) & \forall X(\overline{N}(X)) \subseteq \overline{C}_6(X)). \end{array}$$

Example 3.3. Let $U = \{1,2,3,4\}, C = \{\{1,2,3\},\{1,2,4\}\}, X_0 = \{1,2\}$. We have $\underline{N}(X_0) = \{1,2\} \neq \underline{C}_7(X_0) = \emptyset$.

Example 3.4. Let $U = \{1,2,3,4\}, C = \{\{1\},\{2\},\{3,4\},\{4,5\}\}, X_0 = \{4\}$. We have $\overline{N}(X_0) = \{4\} \neq \overline{C}_6(X_0) = \{3,4,5\}$.

Theorem 3.2. *Let (U, C, N) be a Covering approximation space.*
(1)$\forall X(\underline{C}_7(X) = \underline{N}(X)) \Leftrightarrow C$ forms a partition of U,
(2)$\forall X(\overline{C}_5(X) = \overline{N}(X)) \Leftrightarrow N$ forms a partition of U,
(3)$\forall X(\overline{C}_6(X) = \overline{N}(X)) \Leftrightarrow N$ forms a partition of U.

Proof. (1) the proof of (1) is simple.

(2)"$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by proposition 2.1, $\exists x_0 \exists y_0(x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(y_0) - N(x_0)$, then $x_0 \notin \overline{N}(X_0)$ and $x_0 \in \overline{C}_5(X_0)$. This means $\overline{C}_5(X_0) \neq \overline{N}(X_0)$, contradicts the assumption of necessity.

"$\Leftarrow$", $\forall X \subseteq U$. Firstly, we prove $\overline{N}(X) \subseteq \overline{C}_5(X)$. $\forall x \in \overline{N}(X)$, we take two conditions into consideration: ($\sharp$), $x \in \underline{N}(X)$, we have $x \in X \subseteq \overline{C}_5(X)$. ($\natural$)$N(x) \cap (X - \underline{N}(X)) \neq \emptyset$, take $x_0 \in N(x) \cap (X - \underline{N}(X))$, from proposition 3.1 (1), $x_0 \in X - \underline{N}(X) \subseteq X - \underline{C}_5(X)$ holds. By the assumption that N is a partition of U, we have $N(x_0) = N(x)$. According to the definition of $\overline{C}_5(X)$, we have $x \in \overline{C}_5(X)$, this means $\overline{N}(X) \subseteq \overline{C}_5(X)$. Secondly, we prove $\overline{C}_5(X) \subseteq \overline{N}(X)$. $\forall y \in \overline{C}_5(X)$, we also take two conditions into consideration: ($\sharp\sharp$) $y \in X$, we have $y \in \overline{N}(X)$. ($\natural\natural$) $y \in \overline{C}_5(X) - X$, $\exists y_0 \in X - \underline{C}_5(X)$ such that $y \in N(y_0)$. By the assumption that N is a partition of U, we have $N(y) = N(y_0)$. That is to say $y_0 \in X - \underline{N}(X)$ and $y_0 \in N(y) \cap (X - \underline{N}(X)) \neq \emptyset$. By the definition of $\overline{N}(X)$, we have $y \in \overline{N}(X)$. This means $\overline{C}_5(X) \subseteq \overline{N}(X)$. Therefore $\overline{C}_5(X) = \overline{N}(X)$ holds.

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(3)"$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by proposition 2.1, $\exists x_0 \exists y_0 (x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(x_0)$, then $y_0 \in \overline{C}_6(X_0)$ and $y_0 \notin \overline{N}(X_0)$. this means $\overline{C}_6(X_0) \neq \overline{N}(X_0)$, contradicts the assumption of necessity.

"$\Leftarrow$" is simple. □

Proposition 3.3. Let (U, C, N) be a Covering approximation space. The properties below hold, but all the symbol"$\subseteq$" can not be replaced by symbol"$=$".

$$\begin{array}{ll} (1) & \forall X(\underline{C}_9(X) \subseteq \underline{N}(X)), \\ (2) & \forall X(\overline{N}(X)) \subseteq \overline{C}_8(X)), \\ (3) & \forall X(\overline{N}(X)) \subseteq \overline{C}_9(X)). \end{array}$$

Example 3.5. Let $U = \{1, 2, 3, 4, 5\}, C = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 3, 4, 5\}\}, X_0 = \{1, 2, 3\}$. We have $\underline{N}(X_0) = \{1, 2, 3\} \neq \underline{C}_9(X_0) = \emptyset$.

Example 3.6. Let $U = \{1, 2, 3, 4, 5\}, C = \{\{1, 2\}, \{1, 2, 3, 4, 5\}\}, X_0 = \{1, 2\}$. We have $\overline{N}(X_0) = \{1, 2\} \neq \overline{C}_8(X_0) = \{1, 2, 3, 4, 5\}$.

Example 3.7. Let $U = \{1, 2, 3, 4\}, C = \{\{1\}, \{2\}, \{3, 4\}, \{4\}\}, X_0 = \{3\}$. We have $\overline{N}(X_0) = \{3\} \neq \overline{C}_9(X_0) = \{3, 4\}$.

Theorem 3.3. *Let (U, C, N) be a Covering approximation space.*
(1)$\forall X(\underline{C}_9(X) = \underline{N}(X)) \Leftrightarrow N$ forms a partition of U,
(2)$\forall X(\overline{C}_8(X) = \overline{N}(X)) \Leftrightarrow [\forall x(|Md(x)| = 1) \wedge N$ forms a partition of $U]$,
(3)$\forall X(\overline{C}_9(X) = \overline{N}(X)) \Leftrightarrow N$ forms a partition of U,
(4)$\forall X(\overline{C}_{10}(X) = \overline{N}(X)) \Leftrightarrow N$ forms a partition of U.

Proof. (1), "$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by proposition 2.1, $\exists x_0 \exists y_0 (x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(x_0)$, then $x_0 \in \underline{N}(X_0)$ and $x_0 \notin \underline{C}_9(X_0)$. This means $\underline{N}(X_0) \neq \underline{C}_9(X_0)$, contradicts the assumption of necessity.

"$\Leftarrow$" is simple.

(2), "$\Rightarrow$", proof by contradiction. Firstly, we prove $\forall x(|Md(x)| = 1)$. Suppose $\exists x_0(|Md(x_0)| > 1)$, we can find out $K_1, K_2 \in Md(x_0)$ such that $x_0 \in K_1 \cap K_2, K_1 \cap K_2 \subsetneq K_1$ and $K_1 \cap K_2 \subsetneq K_2$. By the assumption that $\forall X(\overline{C}_8(X) = \overline{N}(X))$ and the fact that $\overline{N}(K_1 \cap K_2) = K_1 \cap K_2$, $\exists L_1, L_2, \cdots L_n \in C$ such that $U - (K_1 \cap K_2) = L_1 \cup L_2 \cup \cdots \cup L_n$. Since $\overline{N}(L_1 \cup L_2 \cup \cdots \cup L_n) = L_1 \cup L_2 \cup \cdots \cup L_n = U - (K_1 \cap K_2)$, so $\exists L^1, L^2, \cdots L^m \in C$ such that $K_1 \cap K_2 = L^1 \cup L^2 \cup \cdots \cup L^m$. This means $\exists i_0 \in \{1, 2, \cdots, m\}$ such that $x_0 \in L^{i_0} \subseteq K_1 \cap K_2 \subsetneq K_2$, contradicts the fact that $K_2 \in Md(x_0)$. Secondly, we prove that N is a partition of U. Otherwise, by proposition 2.1, $\exists y_0 \exists z_0 (y_0 \in N(z_0) \wedge z_0 \notin N(y_0))$. If we let $X_0 = N(y_0)$, then $z_0 \in \overline{C}_8(X_0)$ and $z_0 \notin \overline{N}(X_0)$. This means $\overline{C}_8(X_0) = \overline{N}(X_0)$, contradicts the assumption of necessity.

"$\Leftarrow$". $\forall X \subseteq U$, by proposition 3.3(2), we only need to prove $\overline{C}_8(X) \subseteq \overline{N}(X)$. $\forall x \in \overline{C}_8(X)$, we take two conditions into consideration: ($\sharp$) $x \in X$, we have $x \in \overline{N}(X)$. ($\natural$) $x \in \overline{C}_8(X) - X$, since $\forall y(|Md(y)| = 1)$, so $\cup Md(x) = N(x)$ and $N(x) \cap X \neq \emptyset$. We can select $x_0 \in N(x) \cap X$, by the condition that N is a partition of U, we have $N(x) = N(x_0)$ and $N(x) = N(x_0) \nsubseteq X$. This means $x_0 \notin \underline{N}(X)$ and $x_0 \in N(x) \cap (X - \underline{N}(X)) \neq \emptyset$. From the definition of $\overline{N}(X)$, we have $x \in \overline{N}(X)$. According to ($\sharp$)($\natural$), we finally have $\overline{C}_8(X) \subseteq \overline{N}(X)$.

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(3), "$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by proposition 2.1, $\exists x_0 \exists y_0 (x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(x_0)$, then $y_0 \in \overline{C}_9(X_0)$ and $y_0 \notin \overline{N}(X_0)$. This means $\overline{C}_9(X_0) \neq \overline{N}(X_0)$, contradicts the assumption of necessity.

"$\Leftarrow$" is simple.

(4), "$\Rightarrow$", proof by contradiction. Suppose N is not a partition of U, by proposition 2.1, $\exists x_0 \exists y_0 (x_0 \in N(y_0) \wedge y_0 \notin N(x_0))$. If we let $X_0 = N(y_0) - N(x_0)$, then $x_0 \in \overline{C}_{10}(X_0)$ and $x_0 \notin \overline{N}(X_0)$. This means $\overline{C}_{10}(X_0) \neq \overline{N}(X_0)$, contradicts the assumption of necessity.

"$\Leftarrow$" is simple. □

In order to show the structures of $\underline{N}(X)$ and $\overline{N}(X)$ more clearly, we introduce the conception of Alexander topological space. Let (U, C, N) be a covering approximation space. As a topological base, N can induce a topology T on U. the topological space (U, T) is called Alexander topological space.

$\forall X \subseteq U$, let symbol $int(X)$ represent the interior of X and $cl(X)$ represent the closure of X , then

$$\begin{aligned} &\underline{N}(X) = int(X), \\ &\overline{N}(X) = int(X) \cup cl(X - int(X)) \end{aligned}$$

As the end,we introduce definitions of $n-th$ inner and outer accuracy to show the reason why we introduce the type of covering-based generalized rough set.

Definition 3.2. Let (U, C, N) be a Covering approximation space. For a subset X of U,denote $\rho_i(X) = \frac{|\underline{C}_i(X)|}{|X|} (i \in \{1, 2, \cdots, 10\})$ the $n-th$ inner accuracy of X,and $\rho^i(X) = \frac{|\overline{C}^i(X)|}{|X|} (i \in \{1, 2, \cdots, 10\})$ the $n-th$ outer accuracy of X, where symbol $|.|$ represents the cardinality of a set. For $i = 0$, denote $\rho_0(X) = \frac{|\underline{N}_(X)|}{|X|}$ and $\rho^0(X) = \frac{|\overline{N}_(X)|}{|X|}$.

From the definition 3.2, we easily see that $\rho_i(X) \leqslant 1$ for each i and X, and $\rho^i(X) \geqslant 1$ for each i and X. For a fixed subset X of U, if $\rho_i(X) \geqslant \rho_j(X)$,we say the $i-th$ inner accuracy of X is higher than the $j-th$ inner accuracy of X, similarly, if $\rho^i(X) \leqslant \rho^j(X)$,we say the $i-th$ outer accuracy of X is higher than the $j-th$ outer accuracy of X.

Theorem 3.4. *Let (U, C, N) be a Covering approximation space.*

$$\begin{aligned} &(1) \quad \forall X(\rho_0(X) \geqslant \rho_7(X)), \\ &(3) \quad \forall X(\rho_0(X) \geqslant \rho_9(X)), \\ &(4) \quad \forall X(\rho^0(X) \leqslant \rho^2(X)), \\ &(5) \quad \forall X(\rho^0(X) \leqslant \rho^4(X)), \\ &(6) \quad \forall X(\rho^0(X) \leqslant \rho^6(X)), \\ &(7) \quad \forall X(\rho^0(X) \leqslant \rho^8(X)), \\ &(8) \quad \forall X(\rho^0(X) \leqslant \rho^9(X)). \end{aligned}$$

Proof. Straightforwardly by proposition 3.1,proposition 3.2,and proposition 3.3. □

Definition 3.2 and Theorem 3.4 indicate that the type of covering-based rough set possess a well inner and outer accuracy, this is the meaning we propose this kind of covering-based rough set.

4. Conclusions

In this paper,we have presented a new type of covering-based generalized rough set, and proved some properties of $\underline{N}(X)$ and $\overline{N}(X)$. Here, we could not obtain the sufficient and necessary condition for $\forall X(\overline{N}(U - \overline{N}(X)) = U - \overline{N}(X))$. We mainly discussed the sufficient and necessary conditions for $\forall X(\underline{C}_i(X) = \underline{N}(X))$ and $\forall X(\overline{C}_i(X) = \overline{N}(X))(i = \{1, 2, \cdots, 10\})$. The most important sufficient and necessary condition is that N forms a partition of U. This article introduces two interesting questions : (1) What conditions of C should be satisfied can infer that N is a partition of U ? (2) What conditions $\underline{N}(X)$ or $\overline{N}(X)$ should be satisfied can infer that N is a partition of U ? Solving problem (1) and (2) will be our future work.

Acknowledgment

This work is supported by (No.G2018004)

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