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Article

Firm's credit risk in the presence of market structural breaks

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Abstract: Various sudden shifts in financial market conditions over the past decades have demonstrated the significant impact of market structural breaks on firms' credit behavior. To characterize such effect quantitatively, we develop a continuous-time modulated Markov model for firms' credit rating transitions with the possibility of market structural breaks. The model takes a semi-parametric multiplicative regression form, in which the effects of firms' observable covariates and macroeconomic variables are represented parametrically and nonparametrically, respectively, and the frailty effects of unobserved firm-specific and market-wide variables are incorporated via the integration form of the model assumption. We further develop a mixtured-estimating-equation approach to make inference on the effect of market variations, baseline intensities of all firms' credit rating transitions, and rating transition intensities for each individual firm. We then use the developed model and inference procedure to analyze the monthly credit rating of U.S. firms from January 1986 to December 2012, and study the effect of market structural breaks on firms' credit rating transitions.

Keywords: Credit rating transitions; Mixtured estimating equations; Multiplicative intensity model; Structural break

1. Introduction

Various sudden shifts in financial market conditions have been witnessed during the last several decades so that structural breaks seem to be common instead of exceptional in financial market. The most prominent examples of structural breaks in financial market includes the stock market crash of 1987, the credit market turmoil of 1998 contributed by Russia's default, Brazil's currency crisis and the severe disruption of the Long-Term Capital Management's crisis to U.S. commercial paper markets, the dot-com bubble burst and corporate scandals of 2001-02, and the global financial crisis of 2007-08 sparked by the U.S. subprime mortgage crisis. As suggested by these examples, structural breaks in financial market affect a firm's capital structure and her credit risk, so it is important to investigate quantitatively the impact of financial market structural breaks on a firm's credit risk.

This article develops an econometric model that embeds the impact of financial market structural breaks into firms' credit risk and provides a statistical assessment of U.S. firms' credit risk in the presence of unknown market structural breaks. In particular, we characterize the impact of market structural breaks on firms' credit risk as changes of mechanisms, through which a firm's covariates affect her credit rating transitions, and then develop a multiplicative intensity model to extract and aggregate the information of market structural breaks from firms' credit rating and accounting records. We conduct an empirical analysis based on the credit rating and accounting records of U.S. firms during 1986 and 2012, and show the following results. As a source of credit risk that is different from commonly-used risk factors in corporates' credit models, market structural breaks can not be econometrically represented as a macroeconomic covariate in firms' credit analysis. However, since market structural breaks affect a firm's capital structure and then her credit behavior, the information of market structural breaks is hidden in firms' rating and accounting records. Such information can be extracted and aggregated from all firms in credit market, although it is very weak and hidden in each individual firm's rating and accounting records.

Conventionally, credit risk models for corporates assume that a corporate's conditional rating transition (or default) probabilities (or intensities) depend on certain risk factors that can explain the movement and co-movement of credit risk of the obligors (or more generally, the borrowers). As missing or misspecifying an important risk factor to which the obligors are exposed will result in biased estimates of credit risk, the literature has been very careful to identify and measure those risk factors. Depending on whether the risk factors are observable or not, they can be included in credit risk models as explicit covariates or frailty variables. Structural breaks or sudden shifts of financial markets change the environment, within which corporates need to fulfill their financial obligation in the future, and hence influence corporates' credit risk. However, we notice that it is difficult to characterize the instability of financial markets econometrically. One reason is that, although some economic theory has been proposed to discuss financial market instability [1,2], no structural approach has been proposed to characterize or measure such instability. Another reason is that, although various macroeconomic variables or market indices are proposed to characterize the movement of some market fundamentals, none of them is concerned with the effect of market structural changes on firms' credit risk. From this perspective, although instability in financial market is a source of the movement or co-movement of firms' credit risk, it can not be represented as observed or unobserved risk factors, as in commonly-used credit risk models.

This motivates us to consider a model that incorporates market structural breaks for corporates' credit risk analysis. Since channels through which market structural breaks affect firms' credit behavior are too complex, we only consider in this article a statistical approach to model the effect of market structural breaks on firms' credit risk. The basic idea of our approach is to use a functional form of credit rating transition models to represent the market environment so that the time-variation of the functional form depicts the market instability. Specifically, we assume that the intensities of firms' credit rating transitions follow a semi-parametric multiplicative regression with time-varying coefficients, in which the frailty effects are integrated out by the expectation assumption, the nonparametric (or the baseline intensity) and parametric parts represent the effect of macroeconomic variables and observed firm-specific covariates, respectively, and the time-varying parameters represents the market instability. We then develop a mixtured-estimating-equations approach to make inference on the effect of market structural changes on firms' credit behavior (i.e., time-varying coefficients) and the baseline macroeconomic effect for firms' credit rating transitions.

We use the proposed model and developed inference procedure to study the monthly credit ratings of U.S. corporates from January 1985 to December 2012. We show that the market environment, characterized via model parameters, is indeed instable over time, and the estimated market structural breaks are not only statistically significant but economically meaningful as well. The estimated time of structural breaks match the times of several structural changes in the U.S. credit market. We also compute firms' rating transition intensities and probability matrices in the presence of market structural breaks and compare our result with the one without structural breaks assumption. Our comparison indicates that some rating transition types are more sensitive to market structural breaks than others.

The remainder of the article is organized as follows. Section 2 gives an overview of our modeling approach, and places our work in the context of the related literature and clarifies our contribution. Section 3 specifies the statistical model for the intensities of obligors' rating transitions and present the estimation procedure. Section 4 explains our data sources, provides the fitted model, and summarizes some of the implications of the fitted model for rating transitions. Section 5 concludes the paper.

2. Our Modeling Approach and Related Literature

In order to further motivate our approach, we now briefly outline our specification and discuss the connection between our model and existing literatures.

A corporate's credit risk is usually modeled via structural or reduced-form approach. Structural models provide an explicit relationship between a firm's asset structure and its credit risk. Specifically, a standard structural credit risk model assumes that a firm defaults when the market value of its assets drops to a sufficiently low level relative to the firm's liabilities. For instance, [3–7] model the market value of a firm's asset as a geometric Brownian motion, so that a firm's conditional probability of default is determined by the firm's distance to default, or the number of standard deviations of annual asset growth by which the firm's asset level exceeds its liabilities. Extensions of this approach to incorporate other complexities such as assuming jump-diffusion process for asset values or stochastic interest rates are considered by [8–11].

Comparing to the structural approach that directly models the incentives or ability of a corporate to pay its debt, a reduced form approach models the dependence of default probabilities on explanatory variables through an econometric specification. [12] and [13] first used firms' financial accounting data to estimate the likelihoods of firms' default. [14] introduced a duration model of default based on Weibull distributed default times, and [15] extended it to include time-varying covariates. [16–18] further used duration models to predict firm's bankruptcy. However, due to the interpretability issue, the explanatory variables in reduced form models need to be carefully selected to have the spirit of structural default models. [19] modeled the conditional probability of a default time when a firm's distance to default is imperfectly observed, and suggested the existence of a default intensity process depending on firms' distance to default and other covariates may provide more information about the firm's financial condition. [20] considered a joint model of stochastic default intensities and the dynamics of the underlying time-varying covariates, and introduced likelihood estimation of term structures of default probabilities. These models did not discuss the issue of unobservable or missing covariates affecting default probabilities. Assuming that the rating transition intensities depend on a common unobservable factor, [21] introduced dynamic frailty models of default. [22] extended the frailty-based approach to incorporate the variables used by [20]. [23] further discussed the role of frailty in firms' default during the recent financial crisis.

Different from the above literature, our purpose is to understand the role of market structural breaks in firms' credit risk. [24] discussed the effect of market structural breaks on homogeneous firms' rating transition. To study the effect of structural breaks on heterogeneous firms and further distinguish it from other risk factors, we model the market variation though time-varying coefficients in the rating transition intensity processes, and characterize observable and unobservable firm-specific and macroeconomic variables through parametric, nonparametric, and integration forms. The advantage of our specification is the way of handling the effects of unobserved risk factors in credit analysis. To separate the effect of frailty variables from that of market instability, our model assumes the effect of unobserved firm-specific covariates has been integrated out and model the effect of unobserved macroeconomic variables nonparametrically, so that the issue of unobserved covariates is nicely handled. However, such convenience complicates the model inference procedure. First, due to the semiparametric feature of the model, we have to discard the likelihood based inference procedure and consider an estimating equation approach. Second, the inference on the effects of market instability requires us to estimate the path of the point process, or more specifically, the piecewise constant coefficients and their jump locations, numbers and amplitude during the sample period, while the conventional credit analysis doesn't require estimates of the path of default (or rating transition) intensities. To overcome such difficulties, we consider a mixtured estimating equation approach which synthesizes two basic statistical procedures that deal with two "degenerate cases" of the model. One degenerate case assumes the market is stable and hence the time-varying coefficients in the semi-parametric model become constant, and the other decomposes the jump process of coefficients into a series of disjoint events that correspond to sets of jump times of general market conditions with

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probabilities. Then combining these two cases via a mixtured estimating equation yields a inference procedure for the effect of market instability.

Our model extracts and aggregates the information of market structural variation from each obligor's rating transition and accounting records, and the estimated time-varying coefficients demonstrate the extent of market instability and the risk of sudden shifts of general market conditions. Such feature is related to but different from the concept of systemic risk, which refers to the risk of collapse of the entire market caused by the risk exposure of one or a few agents. From this perspective, the model can also be used for regulatory agencies to analyze the risk of financial market instability. Another potential application of the model is to help bank understand the instability risk arising from the "market" that consists of all her own counterparties and exposures. Under the guideline of Basel Accords, banks are allowed to build their internal rating system to assess the risk of all their counterparties and exposures. The proposed model can be extended there to estimate the instability risk of a bank's counterparties and exposures, and hence allows the bank to take necessary actions to mitigate the loss caused by such instability.

3. A modulated semi-Markov model

3.1. Information filtration

To specify an intensity model for firm's rating transitions, we shall discuss econometrician's information filtration first. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a complete information filtration $\{\mathcal{G}_t : t \geq 0\}$. We note that there are three types of information sets in $\mathcal{G}_t$ at time t. The first type, denoted as $\mathcal{M}_t$, is generated by observed and unobserved macroeconomic variables or events. We shall assume that $\mathcal{M}_t$ is also a minimal information set that summarizes events at the macroeconomic level. However, we shall note that $\mathcal{M}_t$ doesn't contain any interactions between macroeconomic and microeconomic variables. The second type, denoted as $\mathcal{B}_t$, is produced by the collection of all firms' (or borrowers') observed and unobserved covariates and events up to time t. This information set still doesn't contain any interactions between macroeconomic and microeconomic variables, and it is independent of $\mathcal{M}_t$. The third type, denoted as $\mathcal{S}_t$, characterizes the time variation of market or economic environment, and summarizes the mechanism that microeconomic variables or events interact with macroeconomic variables or events. One such example of elements in $\mathcal{S}_t$ is that credit rating agencies' rating criteria are not same during different economic situations. Notice that traditional credit risk models assume the existence of this information set implicitly, and they usually specify a functional form with constant coefficient as the only element in $\mathcal{S}_t$, that is, $\mathcal{S}_t = \{\Lambda_{\boldsymbol{\theta}}\}$, where $\boldsymbol{\theta}$ is a parameter vector and Λ is a functional for rating transition intensities. Given $\mathcal{M}_t$, $\mathcal{B}_t$, and $\mathcal{S}_t$, the complete-information filtration $\mathcal{G}_t$ is the σ-algebra generated by these three sets, that is, $\mathcal{G}_t = \sigma\{\mathcal{M}_t \cup \mathcal{B}_t \cup \mathcal{S}_t\}$, and, by the setup itself, $\mathcal{M}_t$, $\mathcal{B}_t$ and $\mathcal{S}_t$ are mutually independent.

3.2. Conventional models for firms' rating transition intensities

For a firm l $(l = 1, \ldots, n)$, we suppose its rating transition process follows a K-state modulated Markov process, that is, the arrival rates of rating transitions among two particular rating categories depend on a vector of covariates. The rating transition process of firm l is allowed to be left-truncated and right-censored, which corresponds to the cases of firm l entering and exiting the rating system respectively. Denote $P_l(s,t)$ $(l = 1, \ldots, n)$ the rating transition probability matrix of firm l over the period (s,t), in which the ij'th element of $P_l(s,t)$ represents the probability that a firm starting in state i at time s is in state j at time t. Let $A_l(t)$ be the rating category of firm l at time t, and $N^*_{ijl}(t)$ the number of transitions from rating category i to rating category j of the firm l that occur over the interval $(0,t]$ for $i, j \in \{1, \ldots, K\}, j \neq i$. If we know the intensity function of $N^*_{ijl}(t)$, then the transition matrices $P_l(s,t)$ can be computed from them; see [25, Section 8.3].

Let $\{\mathbf{X}_l(t)\}$ be a d-dimensional observable firm-specific covariate process during the period $(e_{l,0}, e_{l,1})$, in which $e_{l,0}$ is the first time that covariate $\mathbf{X}(t)$ appears in the data and $e_{l,1}$ is the exit time

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of firm l. Let $\mathcal{B}^{\text{obs}}_{ijl,t}$ be the filtration generated by $\{\mathbf{X}_l(s) : e_{l,0} \le s \le t\}$, $\mathcal{N}_{l,t}$ the filtration generated by $\{N^*_{ijl}(s) : 1 \le i \ne j \le K, e_{l,0} \le s \le t\}$, and $\lambda^{(i,j)}_l(t)$ the intensity function of $N^*_{ijl}(t)$ associated with $\mathcal{B}^{\text{obs}}_{ijl,t} \cup \mathcal{N}_{l,t}$. Note that $\mathcal{B}^{\text{obs}}_t := \cup_{i,j,l}\mathcal{B}^{\text{obs}}_{ijl,t}$ is only a subset of $\mathcal{B}_t$, as it doesn't contain firms' unobserved covariates. To better explain our idea, we assume that $\mathbf{Y}(t)$ is a vector of macroeconomic variables observed at time t and $\mathcal{M}^{\text{obs}}_t$ is the filtration generated by $\mathbf{Y}(t)$. We denote $\mathcal{M}^{\text{unobs}}_t$ the set of unobserved variables or events in $\mathcal{M}_t$, then $\mathcal{M}_t$ is the filtration generated by $\mathcal{M}^{\text{obs}}_t$ and $\mathcal{M}^{\text{unobs}}_t$. Let $\mathcal{F}_{l,t}$ is the information filtration generated by the observed variables $\{\cup_{i,j,l} \mathcal{B}^{\text{obs}}_{ijl,s}; e_{l,0} \le s \le \min(t, e_{l,1})\} \cup \{\mathcal{M}^{\text{obs}}_s; 0 \le s \le t\}$. Then the econometrician's information filtration is the union of $\mathcal{F}_{l,t}$ and firm's transition history $\mathcal{N}_{l,t}$, that is, $\mathcal{F}_{l,t} \cup \mathcal{N}_{l,t}$. When market or economic condition is stable, conventional credit risk models assume the following intensity functions for rating transitions,

$$E\{dN^*_{ijl}(t)|\mathcal{F}_{l,t}, \mathcal{N}_{l,t}, \mathcal{S}\} = \lambda^{(i,j)}(\mathbf{X}_l(t), \mathbf{Y}(t); \boldsymbol{\theta}^{(i,j)})dt, \tag{1}$$

in which $dN^*_{ijl}(t)$ is the increment $N^*_{ijl}\{(t+dt)\} - N^*_{ijl}(t)$ of $N^*_{ijl}(t)$ over the small interval $[t, t+dt)$. We shall note that model (1) assumes that all covariates or risk factors are observable, which introduces a downward biased estimate of tail portfolio losses. To relax such restriction, the frailty correlated model in [22] drops the following assumption in (1) that all the influence of the prior events on future rating transitions (or default) is demonstrated through observed covariates at time t, i.e.,

$$E\{dN^*_{ijl}(t)|\mathcal{F}_{l,t}, \mathcal{N}_{l,t}, \mathcal{S}\} = E\{dN^*_{ijl}(t)|\mathbf{X}_l(t), \mathbf{Y}(t), \mathcal{N}_{l,t}, \mathcal{S}\},$$

and only assumes the following marginal intensity for rating transitions (or default),

$$E\{dN^*_{ijl}(t)|\mathbf{X}_l(t), \mathbf{Y}(t), \mathcal{N}_{l,t}, \mathcal{S}\} = \lambda^{(i,j)}(\mathbf{X}_l(t), \mathbf{Y}(t); \boldsymbol{\theta}^{(i,j)})dt. \tag{2}$$

Furthermore, [22] assume parametric process for $\mathbf{Y}(t)$ and unobserved macroeconomic and firm-specific covariates, and use Markov Chain Monte Carlo (MCMC) methods to perform maximum likelihood estimation and to filter for the conditional distribution of the frailty process.

3.3. Our specification for firms' rating transition intensities

As we only observe firms' covariates $\mathbf{X}_l(t)$, we consider the intensity model based on $E\{dN^*_{ijl}(t)|\mathbf{X}_l(t), \mathcal{S}\}$. To incorporate the effect of unobserved macroeconomic and firm-specific covariates, we consider an approach different from the parametric treatment in [22]. We further relax (2) and allow the frailty effect absorbed into the conditional expectation form. Specifically, we express the model as

$$E\{dN^*_{ijl}(t)|\mathbf{X}_l(t), \mathcal{S}\} = \exp\left[\mathbf{X}_l(t)^T\boldsymbol{\theta}^{(i,j)}\right]d\Lambda^{(i,j)}_0(t), \tag{3}$$

in which $\Lambda^{(i,j)}_0(\cdot)$ is an unknown continuous function and $\boldsymbol{\theta}^{(i,j)}$ is a parameter vector. This specification allows arbitrary dependence structure among rating transitions and is applicable to many process for rating migrations. For example, the unobserved heterogeneity among firms can be characterized through the frailty model

$$\lambda^{(i,j)}(\mathbf{X}_l(t), t) = \exp\left[\eta_l(t) + \mathbf{X}_l(t)^T\boldsymbol{\theta}^{(i,j)}\right]\lambda^{(i,j)}_0(t),$$

in which $\eta_l(t)$ is an unobserved firm-specific random process independent of $\mathbf{X}_l$, and this model falls into the category of (3). Furthermore, assumption (3) merges the effect of observed and unobserved macroeconomic variables into the unspecified function $\Lambda^{(i,j)}_0(\cdot)$.

We are now ready to characterize the effect of market structural breaks on a firm's credit rating transitions econometrically. We extend the constant market environment $\mathcal{S}$ to the time-varying case $\mathcal{S}_t$, which is a set of time varying functional forms. Specifically, we replace the constant coefficient

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$\boldsymbol{\theta}^{(i,j)}$ in (3) by a time-varying vector $\boldsymbol{\theta}^{(i,j)}(t)$. Denoting $E\{dN^*_{ijl}(t)|\mathbf{X}_l(t), \mathcal{S}_t\}$ by $d\Lambda^{(i,j)}_{\mathbf{X}}(t)$, we obtain a specification for firm l's rating transition intensities with market structural breaks

$$E\{dN^*_{ijl}(t)|\mathbf{X}_l(t), \mathcal{S}_t\} = \exp\left[\mathbf{X}_l(t)^T\boldsymbol{\theta}^{(i,j)}(t)\right]d\Lambda^{(i,j)}_0(t), \tag{4}$$

or

$$\Lambda^{(i,j)}_{\mathbf{X}}(t) = \int_0^t \exp\left[\mathbf{X}_l(u)^T\boldsymbol{\theta}^{(i,j)}(u)\right]d\Lambda^{(i,j)}_0(u). \tag{5}$$

in which the baseline rate $\Lambda^{(i,j)}_0(\cdot)$ is an unknown continuous function regarding unobserved macroeconomic and firm-specific covariates (and observed macroeconomic covariates if they are specified). Note that $\Lambda^{(i,j)}_{\mathbf{X}}(t) = E\{N^*_{ijl}(t)|\mathbf{X}_l(t), \mathcal{S}_t\}$ refers to the mean rate function of the transition from rating category i to rating category j, as $\mathbf{X}_l(t)$ here do not involve firms' rating transition history [25, page 281]. Otherwise, they can only be interpreted as the cumulative rates.

3.4. Dynamics of market structural breaks

We now specify a time-varying scheme for parameter vector $\boldsymbol{\theta}^{(i,j)}(t)$. Since market structural changes can be either gradual or abrupt, we assume that $\boldsymbol{\theta}^{(i,j)}(t)$ follows a compounded Poisson process. This assumption best describes the time-varying feature of $\boldsymbol{\theta}^{(i,j)}(t)$, as both the number and locations of structural breaks in $\boldsymbol{\theta}^{(i,j)}(t)$ and the pre- and post-change values of $\boldsymbol{\theta}^{(i,j)}(t)$ are unobserved. Furthermore, this assumption captures abrupt and gradual changes of general market conditions via large and small size jumps of $\boldsymbol{\theta}^{(i,j)}(t)$, respectively. Since the entire path of the jump process $\boldsymbol{\theta}^{(i,j)}(t)$ need to be estimated in our model so that firms' transition intensities or probabilities can be evaluated, we consider the following assumptions for $\boldsymbol{\theta}^{(i,j)}(t)$,

(A1) the number of jumps in $\boldsymbol{\beta}^{(i,j)}(t)$ follows a Poisson process $\{J^{(i,j)}(t); t \geq 0\}$ with rate η and are independent of $\mathbf{X}_l(t)$;

(A2) if a jump occurs at time t, the post-change value of $\boldsymbol{\theta}^{(i,j)}(t)$ is independent of its pre-change value, in particular, denote $\boldsymbol{\theta}^{(i,j)}(t) = \boldsymbol{\omega}^{(i,j)}_{J^{(i,j)}(t)}$, where $\boldsymbol{\omega}^{(i,j)}_0, \boldsymbol{\omega}^{(i,j)}_1, \boldsymbol{\omega}^{(i,j)}_2, \ldots$ are independent and identically distributed (i.i.d.) normal random vectors with mean $\boldsymbol{\mu}^{(i,j)}$ and covariance $\mathbf{V}^{(i,j)}$.

Assumption (A1) implies that the duration between two adjacent jumps in $\boldsymbol{\theta}^{(i,j)}(t)$ follows an exponential distribution with mean $1/\eta$, and $\boldsymbol{\theta}^{(i,j)}(t)$ between two adjacent jumps are constant. The prior assumption with mean $\boldsymbol{\mu}^{(i,j)}$ and covariance $\mathbf{V}^{(i,j)}$ in Assumption (A2) allows econometricians to incorporate their view on rating transmission channel into the model.

Model (4) or (5) with assumptions (A1) and (A2) complete our model specification.

4. Inference procedure

The proposed model has two types of complexities, one is the semiparametric feature of the intensity functions, and the other is the nonlinear dynamics of regression coefficients $\boldsymbol{\theta}^{(i,j)}(t)$. To develop an inference procedure, we borrow the idea of mixtured estimating equations developed by [26]. Specifically, we first consider an estimating equation for the case that there are no structural breaks in $\boldsymbol{\theta}^{(i,j)}(t)$ during the period (t_*, t^*), we then link all estimating-equation-based estimates by mixture weights that can be computed explicitly.

4.1. Inference when no structural breaks exist

When $\boldsymbol{\theta}^{(i,j)}(t)$ is constant and doesn't undergo any structural breaks during the time interval (t_*, t^*), i.e., $\boldsymbol{\theta}^{(i,j)}(t) \equiv \boldsymbol{\theta}^{(i,j)}$, $t \in (t_*, t^*)$, model (4) can be reduced to

$$E\{dN^*_{ijl}(t)|\mathbf{X}_l(t), \mathcal{S}\} = \exp\left[\mathbf{X}_l(t)^T\boldsymbol{\theta}^{(i,j)}\right]d\Lambda^{(i,j)}_0(t), \quad t \in (t_*, t^*),$$

which is same as the Cox's regression model for counting process in [27], except that regression coefficients $\boldsymbol{\theta}^{(i,j)}$ is imposed a Normal prior distribution $\mathrm{N}(\boldsymbol{\mu}^{(i,j)}, \mathbf{V}^{(i,j)})$. Beside the prior mean $\boldsymbol{\mu}^{(i,j)}$ and the prior covariance $\mathbf{V}^{(i,j)}$ can be informative from econometric perspective, they also serve the shrinkage role when not enough data are available when the time interval (t_*, t^*) is too short. As the Cox model without priors can be solved by standard estimating equation procedure, we extend below the procedure by incorporating the prior distribution for $\boldsymbol{\theta}^{(i,j)}$. As $A_l(t)$ represents the rating category of firm l at time t, we denote $Y_{il}(t) = I(A_l(t^-) = i, C_i \geq t)$, i.e., the indicator that the lth obligor is in state i and under observation at time t^-, $i \in \{1, \ldots, K\}$. For the n firms during the time interval (t_*, t^*), we let

$$S^{(k)}(\boldsymbol{\theta}^{(i,j)}, t) = n^{-1} \sum_{l=1}^{n} Y_{il}(t) \mathbf{X}_l(t)^{\otimes k} \exp\{\mathbf{X}_l(t)^T \boldsymbol{\theta}^{(i,j)}\}, \tag{6}$$

$(k = 0, 1, 2)$, where $a^{\otimes 0} = 1, a^{\otimes 1} = a$ and $a^{\otimes 2} = aa^T$. Let $\mathcal{F}_{(t_*, t^*)}$ be the information set generated by the observed variables during (t_*, t^*), i.e., $\{\cup_{i,j,l} \mathcal{B}^{\mathrm{obs}}_{ijl,s}; \max(t_*, e_{l,0}) \leq s \leq \min(t^*, e_{l,1})\}$, and define $\overline{\mathbf{X}}(\boldsymbol{\theta}^{(i,j)}, t) = S^{(1)}(\boldsymbol{\theta}^{(i,j)}, t)/S^{(0)}(\boldsymbol{\theta}^{(i,j)}, t)$. The partial likelihood score function for $\boldsymbol{\theta}^{(i,j)}$ with prior distribution $\mathrm{N}(\boldsymbol{\mu}^{(i,j)}, \mathbf{V}^{(i,j)})$ can be defined as follows

$$U(\boldsymbol{\theta}^{(i,j)}, t | \mathcal{F}_{(t_*, t^*)}) = [\mathbf{V}^{(i,j)}]^{-1}(\boldsymbol{\theta}^{(i,j)} - \boldsymbol{\mu}^{(i,j)}) + \sum_{l=1}^{n} \int_{t^*}^{t} [\mathbf{X}_l(u) - \overline{\mathbf{X}}(\boldsymbol{\theta}^{(i,j)}, u)] dN_l^{(i,j)}(u). \tag{7}$$

Denote the solution to $U(\boldsymbol{\theta}^{(i,j)}, t^* | \mathcal{F}_{(t_*, t^*)}) = 0$ by $\widehat{\boldsymbol{\theta}}^{(i,j)}_{(t_*, t^*)}$. A Newton-Raphson algorithm can be used to calculate $\widehat{\boldsymbol{\theta}}^{(i,j)}_{(t_*, t^*)}$ and we then estimate $\boldsymbol{\theta}^{(i,j)}$ by $\widehat{\boldsymbol{\theta}}^{(i,j)}_{(t_*, t^*)}$. Furthermore, following the method in [28, Section 2], we can show that $n^{1/2}(\widehat{\boldsymbol{\theta}}^{(i,j)}_{(t_*, t^*)} - \boldsymbol{\theta})$ converges in distribution to a d-variate zero-mean normal random vector, whose covariance doesn't depend on the prior as the effect of prior diminishes when $n \longrightarrow \infty$ and can be estimated from data.

4.2. Mixtured estimating equations

We now consider the case that $\boldsymbol{\theta}^{(i,j)}(t)$ have structural breaks, or $\boldsymbol{\theta}^{(i,j)}(t)$ are piecewise constant with the unknown number of jumps, jump times, and jump amplitudes. Since firms' rating and accounting records are in discrete time, we consider an evenly spaced partition for the period $(0, T)$, $0 = t_0 < t_1 < \cdots < t_H = T$, and assume that structural breaks can only happen at times $t_1, \ldots, t_H$. We define the variables $J_1 = 1$ and $J_h = J(t_h-) - J(t_{h-1}-)$ for $h = 2, \ldots, H$ to indicate if $\boldsymbol{\theta}_{ij}(t)$ has a structural break at t_{h-1}, then J_h are independent Bernoulli random variables with success probability $p = 1 - \exp(-\eta T/H)$. We also assume that there is at most one structural break at time t_h. Note that these assumptions are reasonable to identify structural breaks in $\boldsymbol{\theta}^{(i,j)}(t)$ as long as the partition of $(0, T)$ is fine enough.

Let $\boldsymbol{\theta}^{(i,j)}_{(t_m, t_k)}$ be the constant regression coefficient for $t \in (t_m, t_k)$ when t_m and t_k are two adjacent structural breaks around t_h. To estimate $\boldsymbol{\theta}^{(i,j)}(t)$ given $\mathcal{F}_{(0, t_H)}$, we first notice that, for any estimating function $U(\cdot | \mathcal{F}_{(0, t_H)})$,

$$U(\boldsymbol{\theta}^{(i,j)}(t_h) | \mathcal{F}_{(0, t_H)}) = \sum_{1 \leq m \leq h \leq k \leq H} \pi_{mhk} U(\boldsymbol{\theta}^{(i,j)}_{(t_{m-1}, t_k)} | \mathcal{F}_{(t_{m-1}, t_k)}), \tag{8}$$

in which π_{mhk} is the probability that two most recent change-times around t_h are t_m and t_k ($t_m \leq t_h < t_k$). We then compute the mixture probabilities $\{\pi_{mhk}\}$. Let $R_h = \max\{t_{m-1} | J_m = 1, m \leq l\}$ and $\eta_{m,h} = P(R_h = t_{m-1} | \mathcal{F}_{(0, s_h)})$. Then the conditional distribution of $\boldsymbol{\theta}^{(i,j)}(t_l)$ given $\mathcal{F}_{(0, t_h)}$ is expressed as

$$f(\boldsymbol{\theta}^{(i,j)}(t_h) | \mathcal{F}_{(0, t_h)}) = \sum_{m=1}^{l} \eta_{m,l} f(\boldsymbol{\theta}^{(i,j)}_{(t_{m-1}, t_h)} | \mathcal{F}_{(t_{m-1}, t_h)}), \tag{9}$$

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in which $f(\boldsymbol{\theta}^{(i,j)}_{(t_{m-1},t_h)}|\mathcal{F}_{(t_{m-1},t_h)})$ is the conditional distribution of $\boldsymbol{\theta}(t_h)$ given $R_h = t_{m-1}$ and $\mathcal{F}_{(t_{m-1},t_h)}$, and the mixture probabilities are expressed as $\eta_{m,h} = \eta^*_{m,h} / \sum_{u=1}^{h} \eta^*_{u,h}$, and

$$\eta^*_{m,h} = \begin{cases} p\psi_{t_h,t_h} & m = h, \\ (1-p)\eta_{m,h-1}\psi_{t_m,t_h}/\psi_{t_m,t_{h-1}} & m < h. \end{cases} \tag{10}$$

Note that ψ_{t_m,t_h} represents the likelihood of $\mathcal{F}_{(t_{m-1},t_l)}$ given $R_h = t_{m-1}$, for which we replace it by the partial likelihood for observations in (t_{m-1}, t_h) and evaluated at at $\widehat{\boldsymbol{\theta}}^{(i,j)}_{(t_{m-1},t_h)}$.

Denote $\widetilde{R}_{l+1} = \min\{t_k|J_k = 1, k > h\}$ and $\widetilde{\eta}_{k,h+1} = P(\widetilde{R}_{h+1} = t_k|\mathcal{F}_{t_{h+1},t_H})$, then the conditional distribution of $\boldsymbol{\theta}^{(i,j)}(t_h)$ given $\mathcal{F}_{(t_h,t_H)}$ is

$$f(\boldsymbol{\theta}^{(i,j)}(t_h)|\mathcal{F}_{(t_h,t_H)}) = pf(\boldsymbol{\theta}^{(i,j)}(t_h)|\mathcal{F}_0) + (1-p)\sum_{k=h+1}^{H} \widetilde{\eta}_{k,h+1} f(\boldsymbol{\theta}^{i,j)}_{(t_h,t_k)})|\mathcal{F}_{(t_h,t_k)}), \tag{11}$$

in which $f(\boldsymbol{\theta}^{(i,j)}(t_h)|\mathcal{F}_0)$ represents the density of $\boldsymbol{\theta}^{(i,j)}(t_h)$ without any observations, the mixture probabilities $\widetilde{\eta}_{k,h+1} = \widetilde{\eta}^*_{k,h+1} / \sum_{u=h+1}^{H} \widetilde{\eta}^*_{u,h+1}$, and

$$\widetilde{\eta}^*_{k,h+1} = \begin{cases} p\psi_{t_{h+1},t_{h+1}} & k = l+1, \\ (1-p)\eta_{h+2,k}\psi_{t_{h+1},t_k}/\psi_{t_{h+2},t_k} & k > l+1. \end{cases} \tag{12}$$

Finally we use the Bayes theorem to combine functions (9) and (11) to obtain the conditional of $\boldsymbol{\theta}^{(i,j)}(t_h)$ given all observations $\mathcal{F}_{(0,t_H)}$

$$f(\boldsymbol{\theta}^{(i,j)}(t_h)|\mathcal{F}_{(0,t_H)}) = \sum_{1 \le m \le h \le k \le L} \pi_{mhk} f(\boldsymbol{\theta}^{(i,j)}_{(t_{m-1},t_k)}|\mathcal{F}_{(t_{m-1},t_k)}), \tag{13}$$

in which $\pi_{mhk} = \pi^*_{mhk} / \sum_{1 \le u \le h \le v \le H} \pi^*_{uhv}$ and

$$\pi^*_{mhk} = \begin{cases} p\eta_{m,h} & m \le h = k, \\ (1-p)\eta_{m,h}\widetilde{\eta}_{k,h+1}\psi_{t_m,t_k}/(\psi_{t_m,t_h}\psi_{t_{h+1},t_k}) & m \le h < k. \end{cases} \tag{14}$$

As the above procedure provides explicit formulas to compute the mixture weights $\{\pi_{mhk}\}$, we use (8) to construct the estimation procedure as follows. First, we use expressions (10), (12), and (14) to compute the mixture probabilities $\{\pi_{mhk}\}$, then we use observations $\mathcal{F}_{(t_{m-1},t_k)}$ to estimate $\boldsymbol{\theta}^{(i,j)}_{(t_{m-1},t_k)}$ by the procedure in the preceding section and denote the estimate by $\widehat{\boldsymbol{\theta}}^{(i,j)}_{(t_{m-1},t_k)}$. Finally, in the spirit of (8), we construct the estimate of $\boldsymbol{\theta}^{(i,j)}(t_h)$ given $\mathcal{F}_{(0,t_H)}$,

$$\widehat{\boldsymbol{\theta}}^{(i,j)}(t_h) = \sum_{1 \le m \le h \le k \le H} \pi_{mhk} \widehat{\boldsymbol{\theta}}^{(i,j)}_{(t_{m-1},t_k)}. \tag{15}$$

and extend it to the whole sample period by $\widehat{\boldsymbol{\theta}}^{(i,j)}(t) = \widehat{\boldsymbol{\theta}}^{(i,j)}(t_h)$, for $t \in (t_{h-1}, t_h), h = 1, \dots, H$. Estimates for standard errors of $\widehat{\boldsymbol{\theta}}^{(i,j)}(t_h)$ can be constructed in the same spirit. Furthermore, we also obtain a natural estimator for the baseline cumulative intensity $\Lambda_0^{(i,j)}(t)$ which is given by the Aalen-Breslow-type estimator

$$\widehat{\Lambda}_0^{(i,j)}(t) = \int_0^t \frac{d\bar{N}^{(i,j)}(u)}{nS^{(0)}(\widehat{\boldsymbol{\theta}}^{(i,j)}(u), u)}, \tag{16}$$

in which $\bar{N}^{(i,j)}(u) = \sum_{l=1}^{n} N^*_{ijl}(u)$ and $S^{(0)}(\boldsymbol{\theta}^{(i,j)}(t), t)$ is defined via (6).

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4.3. Estimation of informative prior

The preceding estimation procedure contain hyperparameters $\Phi = \{\eta, \boldsymbol{\mu}^{(i,j)}, \mathbf{V}^{(i,j)}; 1 \le i, j \le K, i \ne j\}$. These informative prior represents the information of market structural changes, and can be estimated by a quasi Expectation-Maximization algorithm.

5. An Empirical Study

5.1. Data description

The data are obtained from Compustat and consist of Standard & Poor monthly credit ratings, long-term and short-term debt of U.S. corporates over 23 years starting January 1986 and ending December 2008. As our model involves corporates' credit rating and covariates, our empirical study only focuses on corporates which have both credit rating and debt records in the sample period.

The credit rating data contain ten rating categories, $\mathscr{AAA}$, $\mathscr{AA}$, $\mathscr{A}$, $\mathscr{BBB}$, $\mathscr{BB}$, $\mathscr{B}$, $\mathscr{CCC}$, $\mathscr{CC}$, $\mathscr{C}$ and $\mathscr{D}$ (default), and 25 rating subcategories. Subcategories are obtained by possibly adding "+" or "-" to the letter grade of categories, which shows relative standing within the major rating categories. We then clean the data as follows. We first group $\mathscr{C}$ and $\mathscr{CC}$ into $\mathscr{CCC}$ as the records in the former two rating categories are relatively few, and then remove rating records of two invalid ratings "N.M." and "Suspended". After the above data-cleaning process, we extract the initial rating and transition information from the rating records. Then we obtain 1814 initial rating and 2926 transition records covering 1172 firms, and eight rating categories, $\mathscr{AAA}$, $\mathscr{AA}$, $\mathscr{A}$, $\mathscr{BBB}$, $\mathscr{BB}$, $\mathscr{B}$, $\mathscr{CCC}$, and $\mathscr{D}$. For observable firms-specific covariates, we follow [22] and adopt the firm's distance to default and trailing 1-year stock return as $X_{l,1}(t)$ and $X_{l,2}(t)$, respectively. The distance to default is a volatility-adjusted measure of leverage and has theoretical underpinnings in the Black-Scholes-Merton structural model of default probabilities. We make use of the market equity data, Compustat book liability data (current liabilities, long-term debt, common shares outstanding, total current liabilities, stock price closed), and 1-year Treasury bill rate to construct this covariate. The construct method follows the lines of that used by [18,19,22]. The firm's trailing 1-year stock return is a covariate of forecasting bankruptcy suggested by [16].

5.2. Estimates of regression coefficients and baseline cumulative intensities

We use the inference procedure developed in Section 4 to first estimate the hyperparameters Φ and then the time-varying coefficients $\boldsymbol{\theta}^{(i,j)}(t)$. Figures 1 and 2 show the estimated regression coefficients, i.e., $\widehat{\theta}_1^{(i,j)}(t)$ and $\widehat{\theta}_2^{(i,j)}(t)$, and their 95% confidence bands, respectively. The estimated coefficients show clearly the market instability over time, and in particular, big changes around October 1994, March 2001, April 2007, and January 2010. It is intriguing to notice that the credit market in U.S. did experience big change around those periods. During February 1994 to February 1995, the U.S. Federal Reserve doubled short-term interest rates to 6% in a year, which make the US bond market suffered a major shock. Around the beginning of 2001, the collapse of the Internet bubble reaches its peak. Furthermore, the U.S financial market experiences a severe crisis starting from the housing bubble burst in the beginning of 2007, and seemingly beginning to recover in the second half of 2009.

Different from [22] who found firm's trailing returns provide a significant incremental explanatory power, we find that all the 95% confidence bands of $\widehat{\theta}_2^{(i,j)}(t)$ in Figure 2 includes the value 0, indicating the effect of firms' trailing 1-year stock return is not significant. This may be due to the fact our model specification integrates out all the frailty effects, while [20] only considered a specific dynamics as the frailty effect in the model.

Figure 3 shows the estimated baseline cumulative intensities (solid lines) based on (16). To see the effect of structural breaks, we also plot the estimated baseline cumulative intensities (dotted lines) when no structural breaks are assumed during the sample period. We see that for some rating transitions such as $\mathcal{AAA} \to \mathcal{AA}$, $\mathcal{AA} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{AA}$, the cumulative intensities with structural

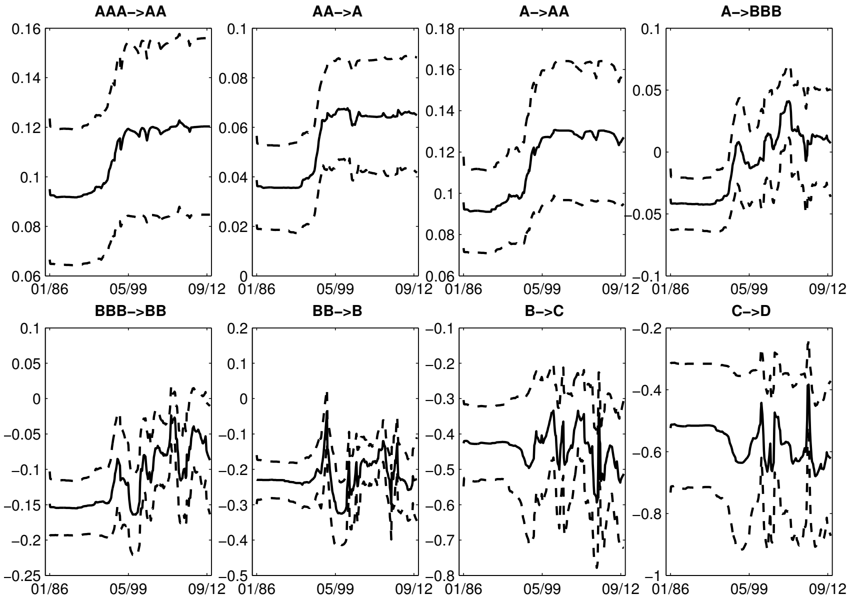

Figure 1. Estimated coefficients (solid) and their 95% confidence bands (dotted) for firms' distance to default.

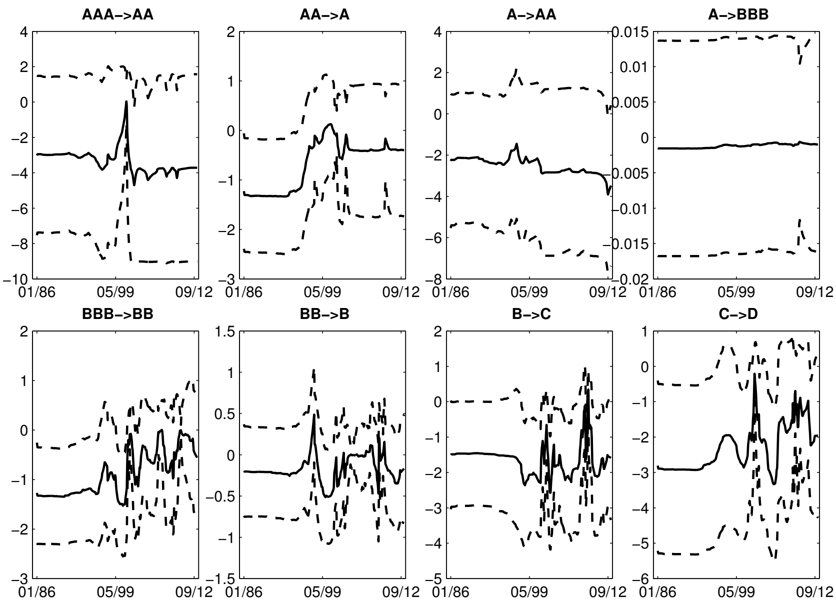

Figure 2. Estimated coefficients (solid) and their 95% confidence bands (dotted) for firms' trailing returns.

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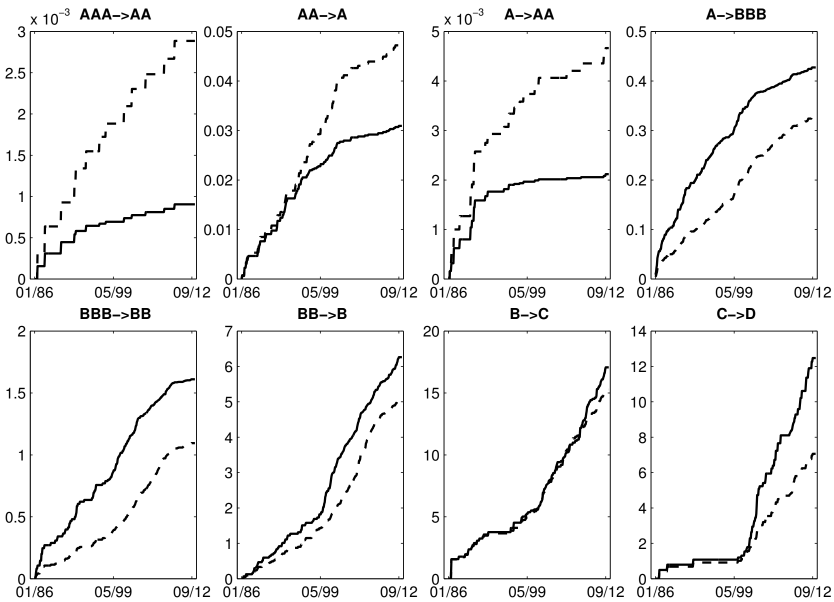

Figure 3. Estimated baseline cumulative intensities with (solid) and without (dashed) structural breaks.

break assumption are more steep than those without structural break assumption, while the other way around for other cases such as $\mathcal{A} \to \mathcal{BBB}$, $\mathcal{BBB} \to \mathcal{B}$ and $\mathcal{C} \to \mathcal{D}$. This indicates that some rating transitions are more sensitive to market structural breaks than others.

5.3. Firms' rating transition intensities and probabilities

With the estimated $\boldsymbol{\theta}^{(i,j)}(t)$ and baseline intensities, we can use (4) and (5) to compute all types of rating transition intensities for each firm and furthermore the rating transition probabilities. Note that the firm's intensities given by (4) and (5) are the mean functions after integrating all random effects. Take the Costco Wholesale Corporation for example, Figure 4 plots the estimated mean functions of cumulative intensities for different rating transitions with (solid) and without (dashed) the assumptions of structural breaks. We notice that for some rating transition types such as $\mathcal{AAA} \to \mathcal{AA}$, $\mathcal{AA} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{AA}$, the cumulative intensities of Costco under structural breaks assumption are smaller than the ones without structural breaks assumption, while larger for other transitions types such as $\mathcal{BB} \to \mathcal{B}$ and $\mathcal{C} \to \mathcal{D}$, which is contrary to the finding for baseline cumulative intensities. This further confirms the significant effect of firms' covariates on firms' rating transitions.

We further compute the Costco's transition probability matrices for different periods and with different assumptions. The first panel of Table 1 shows the estimated transition probability matrix for the whole sample period without structural break assumption, and the second and third panels show the estimated matrices for two periods with the structural break assumption. We choose these two periods because both the estimated baseline and the Costco's cumulative intensities show big shifts around these periods. We find that the transition probabilities from non-default ratings to the default state are much smaller when the assumption of market structural break is incorporated.

6. Concluding remarks

To incorporate the impact of market structural breaks on firm's credit risk, we have developed a modulated semi-Markov model with unknown number, locations and magnitude of market structural breaks for firms' credit rating transition intensities. The model allows a mixtured estimating equation approach to make inference on the time-varying regression coefficients that represent the effect of market structural breaks, baseline intensities of rating transitions for all firms and rating transition intensities for each individual firm.

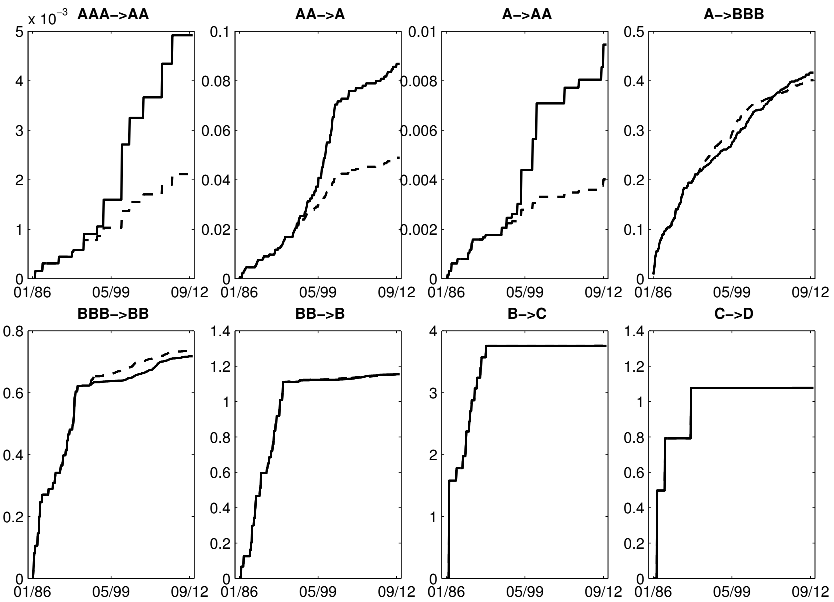

Figure 4. Mean functions of the Costco Wholesale Corporation's cumulative intensities.

As market structural breaks cannot be simply represented as a risk factor in credit risk modeling, the proposed model aggregates firm's rating and accounting records, and extracts the market structural break information effectively. The model also characterizes the impact of market structural break on firm's rating transitions, and allow explicit computation of firms' rating transition probabilities in the presence of unknown market structural breaks. The proposed model has potential applications for the regulatory authority to monitor the market movement. It can also be used as a tool of risk analysis for banks to monitor the risk of structural change faced themselves.

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Table 1. Estimated transition probability matrices of the AT&T Inc.

	$\mathscr{AAA}$	$\mathscr{AA}$	$\mathscr{A}$	$\mathscr{BBB}$	$\mathscr{BB}$	$\mathscr{B}$	$\mathscr{CCC}$	$\mathscr{D}$
January 1986—September 2012 (without structural break assumption)								
$\mathscr{AAA}$	.9993	7e-4	3e-6	2e-8	2e-10	2e-11	1e-12	2e-14
$\mathscr{AA}$	1e-4	0.9927	7e-3	9e-5	8e-7	1e-7	9e-9	1e-10
$\mathscr{A}$	9e-8	.0012	.9742	0.0241	3e-4	5e-5	5e-6	1e-7
$\mathscr{BBB}$	3e-10	6e-6	0.0098	0.9577	0.0274	0.0043	6e-4	2e-5
$\mathscr{BB}$	1e-12	4e-8	9e-5	0.0183	0.9173	0.0552	0.0087	2e-4
$\mathscr{B}$	7e-15	3e-10	8e-7	2e-4	0.0219	0.7215	0.2463	0.0100
$\mathscr{CCC}$	3e-17	1e-12	5e-9	2e-6	3e-4	0.0208	0.9066	0.0722
October 1994—March 2001 (with structural break assumption)								
$\mathscr{AAA}$	0.9998	2e-4	3e-7	6e-10	2e-13	1e-14	2e-20	5e-24
$\mathscr{AA}$	2e-5	0.9968	0.0031	1e-5	5e-9	3e-10	5e-16	2e-19
$\mathscr{A}$	3e-9	3e-4	0.9933	0.0063	4e-6	2e-7	6e-13	2e-16
$\mathscr{BBB}$	4e-12	7e-7	0.0041	0.9944	0.0013	8e-5	3e-10	9e-14
$\mathscr{BB}$	5e-15	1e-9	1e-5	0.0055	0.9935	0.0010	4e-9	1e-12
$\mathscr{B}$	3e-18	9e-13	1e-8	8e-6	0.0028	0.9971	8e-6	2e-9
$\mathscr{CCC}$	2e-23	7e-18	1e-13	1e-10	6e-8	4e-5	0.9999	4e-9
April 2007—January 2010 (with structural break assumption)								
$\mathscr{AAA}$	.9999	1e-4	1e-8	7e-12	3e-15	3e-17	5e-23	9e-27
$\mathscr{AA}$	7e-6	.9998	.0002	2e-7	1e-10	1e-12	2e-18	4e-22
$\mathscr{A}$	9e-11	3e-5	0.9981	0.0018	2e-6	1e-8	4e-14	8e-18
$\mathscr{BBB}$	4e-14	1e-8	0.0011	0.9972	0.0017	2e-5	6e-11	1e-14
$\mathscr{BB}$	2e-17	1e-11	1e-6	0.0023	0.9967	9e-4	4e-9	8e-13
$\mathscr{B}$	7e-21	5e-15	7e-10	2e-6	0.0016	0.9984	8e-6	2e-9
$\mathscr{CCC}$	1e-25	8e-20	2e-14	6e-11	7e-8	9e-5	0.9999	1e-5

Appendix A quasi EM approach to estimate hyperparameters

We consider a quasi EM approach to estimate Φ. We first note that the partial likelihood $l_c(\Phi)$ of the complete data, which consists of all observations and time varying parameters $\{\boldsymbol{\theta}^{(i,j)}(t); 1 \le i \neq j \le K, 0 < t < T\}$, can be written as

$$\begin{aligned} l_c(\Phi) = &\sum_{h=1}^{H}\sum_{i=1}^{K}\sum_{j\neq i}\sum_{l=1}^{n}\Big\{\log P(dN_l^{(i,j)}(t_h)=1|dN_\cdot^{(i,j)}(t_h)\ge 1, \mathcal{F}_{t_h-})\Big\} \\ &-\frac{1}{2}\sum_{h=1}^{H}\sum_{i=1}^{K}\sum_{j\neq i} l_c^{(i,j)}(\Phi)\Big\{(\boldsymbol{\theta}^{(i,j)}(t_h)-\boldsymbol{\mu}^{(i,j)})^T\big[\mathbf{V}^{(i,j)}\big]^{-1}(\boldsymbol{\theta}^{(i,j)}(t_h) \\ &\qquad -\boldsymbol{\mu}^{(i,j)}) + \log|\mathbf{V}^{(i,j)}| + d\log(2\pi)\Big\}\mathbf{1}_{\{\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1})\}} \\ &+\sum_{h=1}^{H}\Big\{\big[\log(1-p)\big]\mathbf{1}_{\{\boldsymbol{\theta}^{(i,j)}(t_h)=\boldsymbol{\theta}^{(i,j)}(t_{h-1});1\le i\neq j\le K\}} \\ &\qquad + (\log p)\mathbf{1}_{\{\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1});1\le i\neq j\le K\}}\Big\}. \end{aligned} \tag{A1}$$

Note that the E-step of the EM algorithm involves the following conditional probabilities or expectations,

(a) $P\big(\boldsymbol{\theta}^{(i,j)}(t_h) \neq \boldsymbol{\theta}^{(i,j)}(t_{h-1})\big)|\mathcal{F}_{(0,t_H)}\big)$,

(b) $E\big(\log P(dN_l^{(i,j)}(t_h)=1|dN_\cdot^{(i,j)}(t_h)=1, \mathcal{G}_{t_h-})|\mathcal{F}_{(0,t_H)}\big)$,

(c) $E\big((\boldsymbol{\theta}^{(i,j)}(t_h)-\boldsymbol{\mu}^{(i,j)})^T\big[\mathbf{V}^{(i,j)}\big]^{-1}(\boldsymbol{\theta}^{(i,j)}(t_h)-\boldsymbol{\mu}^{(i,j)})|\mathcal{F}_{(0,t_H)}\big)$.

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then in view of the above complete log partial likelihood, the M-step of the EM algorithm yields the closed-form updating formulas

$$\begin{aligned}
\widehat{\boldsymbol{\mu}}^{(i,j)}_{\text{new}} &= \frac{\sum_{h=1}^{H} E\big(\boldsymbol{\theta}^{(i,j)}(t_h)\mathbf{1}_{\{\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1})\}}\big|\mathcal{F}_{(0,t_H)},\widehat{\Phi}_{\text{old}}\big)}{\sum_{h=1}^{H} P\big(\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1})\big|\mathcal{F}_{(0,t_H)},\widehat{\Phi}_{\text{old}}\big)},\\
\widehat{\mathbf{V}}^{(i,j)}_{\text{new}} &= \frac{\sum_{h=1}^{H} E\big((\boldsymbol{\theta}^{(i,j)}(t_h)-\widehat{\boldsymbol{\mu}}^{(i,j)}_{\text{old}})^{\otimes 2}\mathbf{1}_{\{\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1})\}}\big|\mathcal{F}_{(0,t_H)},\widehat{\Phi}_{\text{old}}\big)}{\sum_{h=1}^{H} P\big(\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1})\big|\mathcal{F}_{(0,t_H)},\widehat{\Phi}_{\text{old}}\big)},\\
\widehat{p}_{\text{new}} &= \sum_{h=1}^{H} P\big(\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1})\big|\mathcal{F}_{(0,t_H)},\widehat{\Phi}_{\text{old}}\big)\Big/H.
\end{aligned}$$

For the updating formulas above, we can show that

$$P\big(\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1})\big|\mathcal{F}_{(0,t_H)}\big)=\sum_{h\le k\le H}\pi_{hkh}.$$

$$E\big(\boldsymbol{\theta}^{(i,j)}(t_h)\mathbf{1}_{\{\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1})\}}\big|\mathcal{F}_{(0,t_H)}\big)=\sum_{h\le k\le H}\pi_{hkh}E\big(\boldsymbol{\theta}^{(i,j)}_{(t_h,t_k)}\big|\mathcal{F}_{(t_h,t_k)}\big),$$

and

$$\begin{aligned}
&E\big((\boldsymbol{\theta}^{(i,j)}(t_h)-\widehat{\boldsymbol{\mu}}^{(i,j)}_{\text{old}})^{\otimes 2}\mathbf{1}_{\{\boldsymbol{\theta}^{(i,j)}(t_h)\neq\boldsymbol{\theta}^{(i,j)}(t_{h-1})\}}\big|\mathcal{F}_{(0,t_H)}\big)\\
&=\sum_{h\le k\le H}\pi_{hkh}E\big((\boldsymbol{\theta}^{(i,j)}_{(t_h,t_k)}-\widehat{\boldsymbol{\mu}}^{(i,j)}_{\text{old}})^{\otimes 2}\big|\mathcal{F}_{(t_h,t_k)}\big).
\end{aligned}$$

We then approximate $E\big(\boldsymbol{\theta}^{(i,j)}_{(t_h,t_k)}\big|\mathcal{F}_{(t_h,t_k)}\big)$ and $E\big[\boldsymbol{\theta}^{(i,j)}_{(t_h,t_k)})^{\otimes 2}\big|\mathcal{F}_{(t_h,t_k)}\big]$ by the first and second moments of the asymptotic distributions of the estimate $\widehat{\boldsymbol{\theta}}^{(i,j)}_{(t_h,t_k)}$.

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