 doi:10.20944/preprints201809.0514.v1

Almost Global Stability of Nonlinear Switched Systems with Time-Dependent Switching

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***Abstract*—For a dynamical system, it is known that the existence of a Lyapunov density implies almost global stability of an equilibrium. It is then natural to ask whether the existence of a common Lyapunov density for a nonlinear switched system implies almost global stability, in the same way as a common Lyapunov function implies global stability for nonlinear switched systems. In this work, the answer to this question is shown to be affirmative as long as switchings satisfy a dwell-time constraint with an arbitrarily small dwell time. As a straightforward extension of this result, we employ multiple Lyapunov densities in analogy with the role of multiple Lyapunov functions for the global stability of switched systems. This gives rise to a minimum dwell time estimate to ensure almost global stability of nonlinear switched systems, when a common Lyapunov density does not exist.**

The results obtained for continuous-time switched systems are based on some sufficient conditions for the almost global stability of discrete-time non-autonomous systems. These conditions are obtained using the duality between Frobenius-Perron operator and Koopman operator.

***Index Terms*—Almost global stability, nonlinear switched systems, common Lyapunov density, multiple Lyapunov density.**

I. Introduction

THERE exist many examples of dynamical systems (see for example [1] and [2]) that are not globally stable but almost globally stable. For such systems, there is a non-empty set of initial states that do not converge to the origin, but this set is negligible, as it has zero Lebesgue measure. Almost global stability, namely, convergence of almost all solutions to an invariant set, has been first considered by Milnor [3] as a candidate of a useful notion of an attractor. Almost global stability has proved to be useful in the systems and control community after the work of Rantzer [1]. Rantzer showed that almost globally stability of an equilibrium can be verified by the existence of a density function, which is now called Lyapunov density by many researchers. Since then, Lyapunov densities have been used for the analysis of dynamical systems [4], [5], [6], [7], [8], [9], [10], [11] and for the design of control systems [12], [13], [14], [15].

Various extensions of Rantzer's result on almost global stability have appeared in literature. To mention a few, almost global stability has been studied for discontinuous vector fields (switched systems with state dependent switching) [16], for smoothly time-varying systems [17], [18] and for discrete-time nonlinear stochastic systems [19]. However, to the best of our knowledge, almost global stability of nonlinear switched systems with time-dependent switching has not been considered in the literature yet, and is the subject of study of this paper.

As the main results on the global stability of nonlinear switched systems are formulated in terms of a common Lyapunov function, it is natural to investigate the consequences of the existence of a common Lyapunov density for a nonlinear switched system. Consequently, we pose the following question: Does the existence of a common Lyapunov density imply almost global stability of a switched system?

In the sequel, we provide an affirmative answer to this question for systems with (arbitrary small) dwell time switching. As a straightforward extension of this result, we show how multiple Lyapunov densities can be used to ensure almost global stability of a switched system. This leads to an estimation of the minimum dwell time that guarantees almost global stability. To summarize, this paper has the following main contributions on the almost global stability of continuous-time switched systems with time dependent switching:

- A sufficient condition based on a common Lyapunov density (Theorem 2),
- A sufficient condition based on multiple Lyapunov densities (Theorem 3),
- An estimate for the minimum dwell time (Corollary 2).

To prove the above-mentioned results for a continuous-time switched system, we fix a switching signal leading to a time-varying system, discretize the time-varying system (with a fixed but arbitrarily small sampling time) giving rise to a discrete-time non-autonomous system, and finally lean upon the almost global stability of the latter. To this end, we obtain sufficient conditions for almost global stability of discrete-time non-autonomous systems (Theorem 4 and Corollary 3). All findings of the paper and the relations between them are depicted in Fig. 1. For simplicity, we consider almost global stability of a common equilibrium, however all results in this paper also hold when the common equilibrium is replaced by a common compact invariant set.

Our approach on almost global stability is based on linear transfer operators, Frobenius-Perron operator and Koopman operator, which are used to capture the global dynamics of a system (see [20], [21], [22]). This approach was first used for almost global stability by Vaidya and Mehta in [6], where they give a sufficient condition for the almost global stability of an invariant set for discrete-time, autonomous systems with compact state space using a local attraction assumption. This result is extended in [11] to systems with non-compact state space without using any local stability assumption and in [23] to the problem of finite-time stability. Our results on almost

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Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 September 2018 doi:10.20944/preprints201809.0514.v1

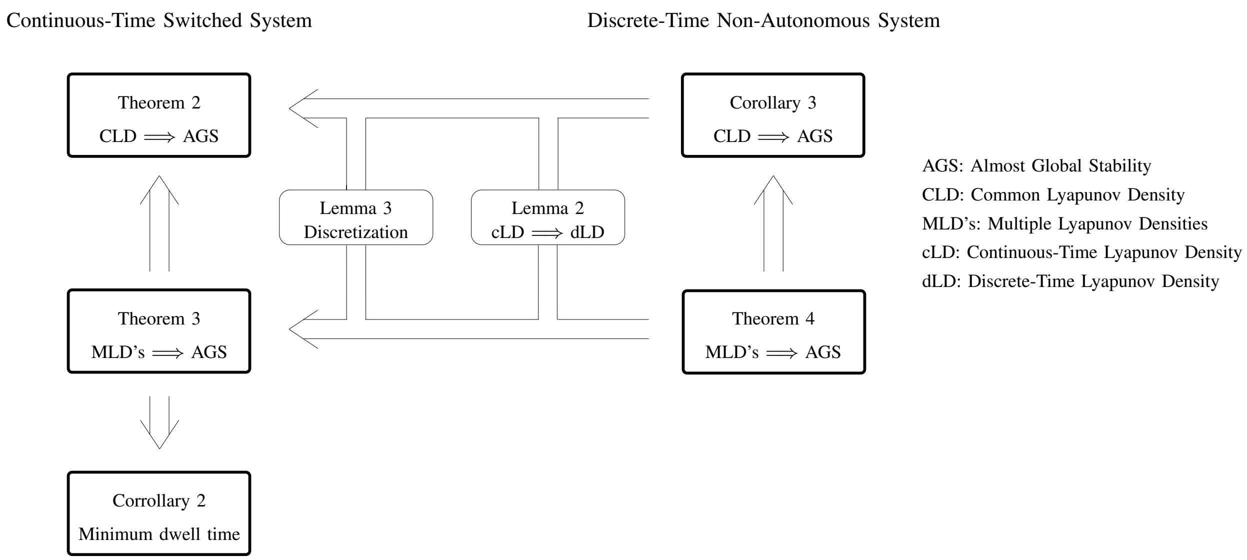

Fig. 1: Relations between the results obtained in the paper.

global stability of discrete-time non-autonomous systems are in the spirit of [6], [23] and [11].

To derive the continuous-time results from the discrete-time results, we use two technical lemmas (Lemma 2 and Lemma 3). The first lemma shows that Lyapunov densities for continuous-time systems can be used as Lyapunov densities for their discretizations. This lemma generalizes a similar result in [11, Lemma 4.4] (restated in Appendix as Lemma 5), both being derived from Rantzer's lemma [1, Lemma A.1] (restated in Appendix as Lemma 4). The second technical lemma shows that almost global stability of a continuous-time system follows from almost global stability of its discretization for every sufficiently small sampling time. This latter result has already appeared in literature as a part of the proof of [11, Theorem 4.2].

The outline of the paper is the following: The main results on almost global stability of continuous-time nonlinear switched systems via a common Lyapunov density and multiple Lyapunov densities are presented in Section II. In this section, we also present examples illustrating applications of the main theorems. In Section III, we give preliminaries for transfer operators and present sufficient conditions based on transfer operators, for almost global stability of discrete-time non-autonomous systems. Section IV contains the proofs of the main theorems in Section II. Even though the methods in the paper are developed for nonlinear systems, they generalize some already known linear techniques. Specifically, in Section V, we show that, for linear switched systems, the presented sufficient conditions are not more conservative than the LMI conditions in [24], [25], which are based on multiple quadratic Lyapunov functions.

Notation. $\mathbb{R}(\mathbb{Z}), \mathbb{R}_{>0}(\mathbb{Z}_{>0})$ and $\mathbb{R}_{\geq 0}(\mathbb{Z}_{\geq 0})$ denote the set of all, positive and non-negative real numbers (integers), respectively. For $\mathbb{R}^n$, the vector space of real n-tuples, $\|\cdot\|$ denotes the Euclidean norm and m denotes the Lebesgue measure on $\mathbb{R}^n$. $\mathbf{0} \in \mathbb{R}^n$ denotes the zero vector. $B_\varepsilon = \{x \in \mathbb{R} \mid \|x\| < \varepsilon\}$ is the open $\varepsilon-$ball around $\mathbf{0}$ and B_ε^c is the complement of B_ε. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is integrable away from $\mathbf{0}$ meaning that it is Lebesgue-integrable on B_ε^c for all $\varepsilon > 0$. For functions $f, g : \mathbb{R}^n \to \mathbb{R}$, f is said to be of the same order as g meaning that $f(x) = O(g(x))$, i.e., $\left|\frac{f(x)}{g(x)}\right|$ is bounded as $\|x\| \to \infty$. $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be non-singular, if $m(f^{-1}(A)) = 0$ for every measurable set A with $m(A) = 0$. For a set $V \subset \mathbb{R}^n$, 1_V denotes the characteristic function of V. For a function $f : \mathbb{R}^n \to \mathbb{R}^n$, $\mathrm{D}f$ denotes the Jacobian of f and $\nabla \cdot f$ denotes the divergence of f. For $f : \mathbb{R}^n \to \mathbb{R}$, ∇f denotes the gradient of f. For a matrix A, A^{T} denotes the transpose of A. For symmetric matrices A and B, we use the notation $A < B$ ($A \leq B$) to mean that $B - A$ is positive (semi-)definite. Finally, we use the phrases 'almost all', 'almost every' and 'almost everywhere' in the sense of Lebesgue measure, namely, the set of points for which the argument fails is contained in a set of zero Lebesgue measure.

II. Almost Globaly Stability of Switched Systems in Continuous-Time

In this section, we present sufficient conditions for almost global stability of nonlinear switched systems with time-dependent switching. In relation to the obtained conditions, the provided examples illustrate the almost global stability of nonlinear switched systems and the lack thereof.

Initially, we state some results on the almost global stability of autonomous systems, not only for the sake of completeness but also for their use in showing the global existence of almost all solutions of switched systems. Consider the following ordinary differential equation

$$\dot{x} = f(x), \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $f(\mathbf{0}) = \mathbf{0}$. The following theorem can be seen as a modified version of Rantzer's theorem for autonomous systems for which almost all solutions are known to exist for all positive times.

Theorem 1. *(Adapted from Theorem 4.2 in [11]) Suppose that, for almost every $x_0 \in \mathbb{R}^n$, a forward-complete solution*

$x:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ of (1) *with $x(0)=x_0$ exists. Assume that there exists a non-negative, continuously differentiable function $\rho:\mathbb{R}^n\setminus\{\mathbf{0}\}\to\mathbb{R}$ satisfying the following conditions*

- *$\rho(x)$ is integrable away from $\mathbf{0}$, and*
- *$\nabla\cdot(\rho f)(x)>0$ for almost all $x\in\mathbb{R}^n\setminus\{\mathbf{0}\}$.*

Then, almost all solutions of (1) *converge to $\mathbf{0}$ as $t\to\infty$.*

The following corollary of this theorem is used to ensure the global existence of almost all solutions.

Corollary 1. *Assume that there exists a non-negative, continuously differentiable function $\rho:\mathbb{R}^n\setminus\{\mathbf{0}\}\to\mathbb{R}$ satisfying*

- *$(1+\|f(x)\|)\rho(x)$ is integrable away from $\mathbf{0}$, and*
- *$\nabla\cdot(\rho f)(x)>0$ for almost all $x\in\mathbb{R}^n\setminus\{\mathbf{0}\}$.*

Then, for almost every initial state $x_0\in\mathbb{R}^n$, a forward-complete solution $x:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ of (1) *with $x(0)=x_0$ exists and converges to $\mathbf{0}$ as $t\to\infty$.*

Proof: Consider the time scaling $t_{\text{new}}=\int_0^t[1+\|f(x(s))\|]\,ds$, under which the scaled solutions $x(t_{\text{new}})$ satisfy the scaled system $dx/dt_{\text{new}}=f_{\text{new}}(x)$, where $f_{\text{new}}(x):=f(x)/(1+\|f(x)\|)$. Solutions of the scaled vector field f_{new} exist globally and they produce the same trajectories as $x(t)$ with the direction of time preserved (see [26, page 184]). Therefore, it is enough to show the convergence of almost all solutions of f_{new} to $\mathbf{0}$, since the convergence of trajectories $x(t)$ to a bounded set implies their existence for all $t\in\mathbb{R}_{\geq 0}$. This can be done by applying Theorem 1 with $\rho_{\text{new}}(x):=(1+\|f(x)\|)\rho(x)$, noting that $\nabla\cdot(\rho_{\text{new}}f_{\text{new}})(x)=\nabla\cdot(\rho f)(x)>0$. ■

Remark 1. *Corollary 1 differs from Rantzer's original theorem in that it assumes the integrability of $(1+\|f(x)\|)\rho(x)$ away from $\mathbf{0}$ instead of the integrability of $\|f(x)\|\rho(x)/\|x\|$ away from $\mathbf{0}$. We prefer the former condition as it implies the integrability of $\rho(x)$ away from $\mathbf{0}$, which is used in the proofs of the main theorems for switched systems below.*

Let us consider a nonlinear switched system given by

$$\dot{x}(t)=f_{\sigma(t)}(x(t)),\quad \sigma\in\mathcal{S}_\tau,\quad t\in[0,\infty). \tag{2}$$

Here, $\sigma:[0,\infty)\to\{1,\dots,N\}$ is called a switching signal which is a right-continuous, piecewise constant function with finitely many discontinuities on any finite interval. $\mathcal{S}_\tau$ denotes the set of all switching signals satisfying $t_k-t_{k-1}\geq\tau$, $k\in\mathbb{Z}_{>0}$, where t_k denote the kth discontinuity point of σ ($t_0=0$ is assumed) and τ is called a minimum dwell time. We call each system given by $\dot{x}=f_p(x)$, for $p\in\{1,2,\dots,N\}$ a subsystem of (2). We assume that each subsystem $f_p:\mathbb{R}^n\to\mathbb{R}^n$, $p\in\{1,2,\dots,N\}$ is continuously differentiable and share a common equilibrium at $\mathbf{0}$, namely $f_p(\mathbf{0})=\mathbf{0}$.

Let us denote the value of $\sigma(t)$ for $t\in[t_{k-1},t_k)$ by p_k. A switching signal can then be identified using these values as

$$\sigma(t)=\Big((p_1,\Delta t_1),(p_2,\Delta t_2),\dots\Big), \tag{3}$$

where $\Delta t_k=t_k-t_{k-1}>\tau$ is the operation time for the subsystem f_{p_k} on the kth constant operation of the switched system. In examples, we will mostly use periodic switching signals, which we identify by a finite sequence (showing the shortest repeating pattern) as

$$\begin{aligned}\sigma(t)&=\Big((p_1,\Delta t_1),\dots,(p_n,\Delta t_n)\Big)\\&=\Big((p_1,\Delta t_1),\dots,(p_n,\Delta t_n),(p_1,\Delta t_1),\dots\Big),\end{aligned} \tag{4}$$

which has a minimum period of $\Delta t_1+\cdots+\Delta t_n$.

Definition 1. *The nonlinear switched system* (2) *is said to be* almost globally stable for a $\sigma\in\mathcal{S}_\tau$ *if the following condition holds:*

> *For almost every $x_0\in\mathbb{R}^n$, a forward-complete solution $x:\mathbb{R}_{\geq 0}\to\mathbb{R}^n$ of* (2) *for the switching signal σ and the initial state $x(0)=x_0$ exists and converges to $\mathbf{0}$ as $t\to\infty$.*

The system (2) *is said to be* almost globally stable *if it is almost globally stable for every $\sigma\in\mathcal{S}_\tau$.*

Note that if forward-complete solutions exist for almost all initial states for each subsystem, then forward-complete solutions of the switched system (2) exist for almost all initial states when $\sigma\in\mathcal{S}_\tau$ for some $\tau>0$. However, if $\sigma\notin\mathcal{S}_\tau$ for all τ, for example when the system undergoes infinitely many switching in finite time, i.e., chattering (the Zeno behaviour) takes place, then the complete solutions of the switched systems may not exist even if all subsystems have complete solutions [27]. Thus, the dwell time condition plays a role to ensure the existence of complete solutions of switched systems with time dependent switching.

A. Common Lyapunov Density in Continuous-Time

The following theorem shows that the existence of a common Lyapunov density implies almost global stability of a nonlinear switched system.

Theorem 2. *Consider the switched system* (2)*. Assume that there exist a constant $\kappa>0$ and a non-negative, continuously differentiable function $\rho:\mathbb{R}^n\setminus\{\mathbf{0}\}\to\mathbb{R}$ such that the following conditions are satisfied for all $p\in\{1,2,\dots,N\}$:*

CLD0: *$(1+\|f_p(x)\|)\rho(x)$ is integrable away from $\mathbf{0}$,*
CLD1: *$\nabla\cdot(\rho f_p)(x)\geq\kappa\rho(x)$ for all $x\in\mathbb{R}^n$.*

Then, the system (2) *is almost globally stable for any $\tau>0$.*

Proof: See Section IV. ■

The following example shows that a nonlinear switched system with a common Lyapunov density is almost global stable but may not exhibit global stability.

Example 1. *Consider the switched system* (2) *with $N=3$ and the subsystems given as*

$$f_1(x_1,x_2)=\begin{pmatrix}x_2-x_1+3x_1x_2\\-x_2-x_1+x_2^2-2x_1^2\end{pmatrix},$$

$$f_2(x_1,x_2)=\begin{pmatrix}-x_2-x_1+x_1^2-2x_2^2\\-x_2+x_1+3x_1x_2\end{pmatrix},$$

and

$$f_3(x_1,x_2)=\begin{pmatrix}x_2-x_1-x_1^2+2x_2^2\\-x_2-x_1-3x_1x_2\end{pmatrix}.$$

Let us consider $\rho(x) = \left(x_1^2 + x_2^2\right)^{-5/2}$. CLD0 is satisfied because $(1 + \|f_p\|)\,\rho$ is of the same order as $\|x\|^{-3}$ for $p = 1, 2, 3$. Moreover, it can be shown that $\nabla \cdot (\rho f_p) = 3\rho$ for $p = 1, 2, 3$. Therefore, CLD1 in Theorem 2 is satisfied for $\kappa = 3$. As a result, $\mathbf{0}$ is almost globally stable.

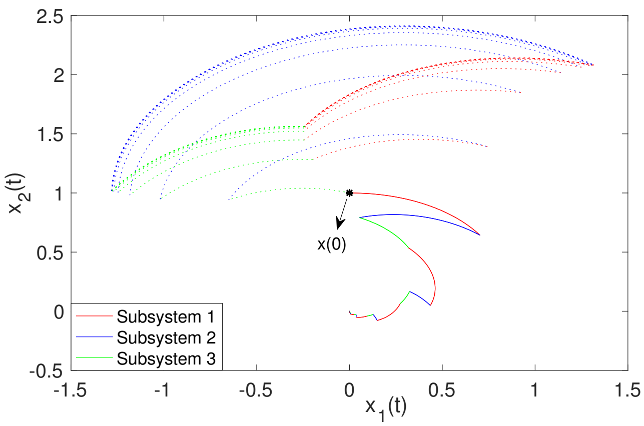

Fig. 2: A solution of Example 1. The dotted line is for the backward solution of the system for the initial state $x(0)$ as shown, which approaches to a limit cycle as $t \to -\infty$. The solid line is for the forward solution, which approaches to $\mathbf{0}$ as $t \to \infty$.

Fig. 2 exhibits a solution of the system for the following periodic switching signal

$$\sigma(t) = \big((1, 0.5), (2, 0.3), (3, 0.2)\big), \tag{5}$$

with period 1. It is seen in Fig. 2 that the backward solution (obtained by extending the switched system and the switching signal backward in time in a trivial way) approaches to a limit cycle as $t \to -\infty$, whereas the forward solution approaches to $\mathbf{0}$ as $t \to \infty$. The existence of the unstable limit cycle implies the lack of global stability for the switched system, i.e. not all initial states lead to convergence of solutions to the origin.

We have already discussed the role of the dwell time condition in ensuring the global existence of solutions of the switched system. The following example shows that, for a switched system with a common Lyapunov density, one can construct a state-dependent switching rule leading to divergence of solutions to infinity (in finite time), but in this case the resulting switching signal has no dwell time.

Example 2. *Consider the switched system (2) with $N = 2$ and subsystems f_1 and f_2 as defined in Example 1 above. Since the subsystems share a common Lyapunov density as discussed in Example 1, almost all solutions converge to the origin for any switching signal $\sigma \in \mathcal{S}_\tau$. Nevertheless, one can construct a switching rule, which leads to divergence of solutions but a chattering switching signal. Specifically, consider the following switching rule:*

$$\sigma(t+) = \begin{cases} 1, & \text{if } x_2(t) = 0.3x_1(t) \\ 2, & \text{if } x_2(t) = 4x_1(t). \end{cases} \tag{6}$$

Starting with the subsystem f_1 and with an initial state in the region $0.3x_1 < x_2 < 4x_1$, solutions stay in this region but have finite escape time. Such a solution for the initial condition $x_0 = (2, 3)$ is shown in Fig. 3. The resulting switching signal then becomes

$$\sigma(t) = \big((1, 0.0448), (2, 0.1596), (1, 0.1063), (2, 0.0734), (1, 0.0413), (2, 0.0253), (1, 0.0129), (2, 0.0075), \dots\big), \tag{7}$$

where the operation times of subsystems decrease to zero so fast that the switching times t_k approach to a constant value $t_\infty \cong 0.4791$. In other words, the switching signal undergoes chattering as $t \to t_\infty$ and is not defined for $t \geq t_\infty$.

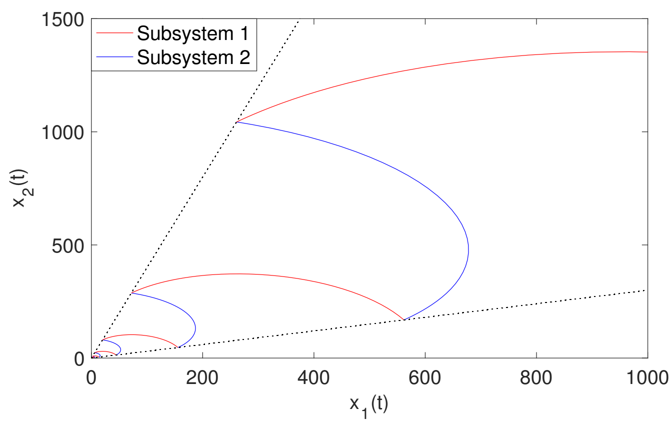

Fig. 3: A solution of Example 2 for the initial state $x(0) = (2, 3)$. The dotted lines show the switching boundaries given in (6). The solution escapes to infinity in finite time.

B. Multiple Lyapunov Densities and Dwell Time Computation in Continuous-Time

Inspired by the method of multiple Lyapunov functions for the global stability of switched nonlinear systems [28], we employ multiple Lyapunov densities for almost global stability as follows:

Theorem 3. *Consider the switched system (2). Assume that there exist constants $\tau_{\min} > 0$, $\kappa_p > 0$, $p \in \{1, 2, \dots, N\}$ and non-negative, continuously differentiable functions $\rho_p : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$, $p \in \{1, 2, \dots, N\}$ such that the following conditions are satisfied for all $p, m \in \{1, 2, \dots, N\}$:*

MLD0: $(1 + \|f_p(x)\|)\rho_p(x)$ *is integrable away from* $\mathbf{0}$,
MLD1: $\nabla \cdot (\rho_p f_p)(x) \geq \kappa_p \rho_p(x)$ *for all* $x \in \mathbb{R}^n$,
MLD2': $e^{-\kappa_p \tau_{\min}} \rho_p(x) \leq \rho_m(x)$ *for all* $x \in \mathbb{R}^n$.

Then, the system (2) is almost globally stable for any $\tau > \tau_{\min}$.

Proof: See Section IV. ■

Remark 2. *An improved version of Theorem 3 (Theorem 5) where the condition MLD2' is replaced by a less conservative condition (MLD2) will be stated later in Section IV.*

Note that the condition MLD2' is satisfied for some $\tau_{\min}$ if and only if the Lyapunov densities ρ_p, $p \in \{1, 2, \dots, N\}$ are comparable functions, i.e., there exist positive numbers c_{pm}, $p, m \in \{1, 2, \dots, N\}$ satisfying $\rho_p(x) \leq c_{pm}\rho_m(x)$ for

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all $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $p, m \in \{1, 2, \dots, N\}$. In this case MLD2' follows from $\tau_{\min} \geq \frac{\ln c_{pm}}{\kappa_p}$, $p, m \in \{1, 2, \dots, N\}$. On the other hand, if ρ_p, $p \in \{1, 2, \dots, N\}$ satisfy the condition MLD1, $\beta_p \rho_p$, $p \in \{1, 2, \dots, N\}$ with arbitrary positive constants β_p's also satisfy the condition MLD1 with the same κ_p's due to the linearity of the divergence. Therefore for a given set of comparable Lyapunov densities satisfying the conditions of Theorem 3, the condition MLD2' holds for any value of $\tau_{\min}$ that satisfies the following minimum dwell time condition:

$$\tau_{\min} \geq \min_{\beta_1, \dots, \beta_N \in \mathbb{R}_{>0}} \max_{p,m \in \{1,2,\dots,N\}} \frac{\ln\left(\frac{\beta_p}{\beta_m} c_{pm}\right)}{\kappa_p}. \tag{8}$$

In particular, for bimodal systems, the dwell time condition (8) can be written as

$$\tau_{\min} \geq \frac{\ln\left(\sqrt{c_{12} c_{21}}\right)}{\min\{\kappa_1, \kappa_2\}}. \tag{9}$$

This can be seen by observing that the min-max operation in the nominator of (8) for $N = 2$ is achieved by the geometric mean of $\frac{\beta_2}{\beta_1} c_{12}$ and $\frac{\beta_1}{\beta_2} c_{21}$ which is constant for all $\beta_1, \beta_2 \in \mathbb{R}_{>0}$. To summarize, we state the following corollary:

Corollary 2. *Consider the switched system* (2)*. Suppose that the assumptions of Theorem 3 are satisfied except for MLD2', with comparable multiple Lyapunov densities ρ_p, $p \in \{1, \dots, N\}$, i.e., there exist positive constants c_{pm}, $p, m \in \{1, \dots, N\}$ satisfying $\rho_p(x) \leq c_{pm} \rho_m(x)$ for all $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $p, m \in \{1, 2, \dots, N\}$. Then, the system* (2) *is almost globally stable for all $\tau_{\min}$ satisfying* (9) *for $N = 2$ and* (8) *for $N > 2$.*

The following example illustrate an application of Corollary 2

Example 3. *Consider the switched system* (2) *with $N = 2$ and with subsystems*

$$f_1(x_1, x_2) = \begin{pmatrix} -0.1x_1 + x_2 + 3x_1 x_2 \\ -x_1 - 0.1x_2 - 2x_1^2 + x_2^2 \end{pmatrix}$$

and

$$f_2(x_1, x_2) = \begin{pmatrix} -0.1x_1 - 2x_2 + 0.5x_1^2 - 4x_2^2 \\ 0.5x_1 - 0.1x_2 + 1.5x_1 x_2 \end{pmatrix}.$$

Let us consider $\rho_1(x_1, x_2) = \left(x_1^2 + x_2^2\right)^{-5/2}$ and $\rho_2(x_1, x_2) = \left((0.5x_1)^2 + x_2^2\right)^{-5/2}$. MLD0 is satisfied because $(1 + \|f_p\|)\, \rho_p$ is of the same order as $\|x\|^{-3}$ for $p = 1, 2$. Moreover, it can be shown that $\nabla \cdot (\rho f_p) = 0.3\rho$ for $p = 1, 2$. Therefore, MLD1 in Theorem 3 is satisfied for $\kappa_1 = \kappa_2 = 0.3$. $\rho_1(x)$ and $\rho_2(x)$ are comparable as $\rho_1(x) < c_{12} \rho_2(x)$, for $c_{12} = 1$ and $\rho_2(x) < c_{21} \rho_1(x)$, for $c_{21} = 2^5$. Applying (9)*, we obtain $\tau_{\min} \geq \frac{\ln(\sqrt{2^5})}{0.3} = 5.7762$. Hence, Corollary 2 implies that the origin is almost globally stable for all switching signals with dwell time $\tau > 5.7762$. Fig. 4 depicts solutions of the switched system for two different periodic switching signals, one that satisfies the dwell time condition (Fig. 4a) and another that does not satify the dwell time condition (Fig. 4b).*

III. Almost Global Stability of Discrete-Time Non-Autonomous Systems

In this section, we give sufficient conditions for the almost global stability of discrete-time non-autonomous systems. These results will be used in Section IV for the proofs of the main results for continuous-time switched systems.

A. Preliminaries for Transfer Operators

Let $\mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ denote the linear vector space of equivalence classes of measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$, where two functions are assumed to be equal if they agree on a set of full Lebesgue measure. Therefore, all equalities and inequalities for the functions in $\mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ should be understood to hold for almost all points in $\mathbb{R}^n$. In the sequel, we will omit the phrase 'almost all' for such relations. In particular, we say that a function $\rho \in \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ is positive meaning that $\rho(x) > 0$ for almost all $x \in \mathbb{R}^n$.

For a nonsingular map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(\mathbf{0}) = \mathbf{0}$, let us denote the Frobenius-Perron operator and the Koopman operator for f restricted to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ as $\mathbb{P}$ and $\mathbb{U}$, respectively. Recall that $\mathbb{P} : \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\}) \to \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ provides information about the evolution of densities (possibly with infinite mass) and is defined via

$$\int_V \mathbb{P}\rho dx = \int_{f^{-1}(V)} \rho m(dx). \tag{10}$$

This defines $\mathbb{P}$ uniquely due to the non-singularity of f and the Radon-Nikodym theorem for signed σ-finite measures [29, p.238]. When f is a differentiable and invertible map, $\mathbb{P}$ can be written explicitly (see [20, Remark 3.2.4.]) as

$$\mathbb{P}\rho(x) = \rho(f^{-1}(x)) \det\left(Df^{-1}\right). \tag{11}$$

On the other hand, Koopman operator $\mathbb{U} : \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\}) \to \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ provides the evolution of observables (possibly essentially unbounded) and is defined as

$$\mathbb{U}g(x) = g\left(f(x)\right).$$

The operators $\mathbb{P}$ and $\mathbb{U}$ are positive operators, i.e., $\rho > 0 \implies \mathbb{P}\rho > 0$ and $g > 0 \implies \mathbb{U}g > 0$. We use the following notation

$$\langle g, \rho \rangle := \int_{\mathbb{R}^n} g \mu_\rho(dx),$$

for the Lebesgue-integral of g with respect to the signed measure μ_ρ with density ρ, where $\mu_\rho(V) := \int_V \rho m(dx)$. Duality between $\mathbb{P}$ and $\mathbb{U}$ is expressed by

$$\langle \mathbb{U}g, \rho \rangle = \langle g, \mathbb{P}\rho \rangle,$$

and can easily be seen when g is a characteristic function:

$$\begin{aligned} \langle \mathbb{U}1_V, \rho \rangle &= \int_{\mathbb{R}^n} 1_V(f(x)) \mu_\rho(dx) \\ &= \int_{f^{-1}(V)} \mu_\rho(dx) = \int_{f^{-1}(V)} \rho m(dx) \\ &= \int_V \mathbb{P}\rho m(dx) = \langle 1_V, \mathbb{P}\rho \rangle \end{aligned}$$

Note that we allow both sides of the duality equation to be infinite.

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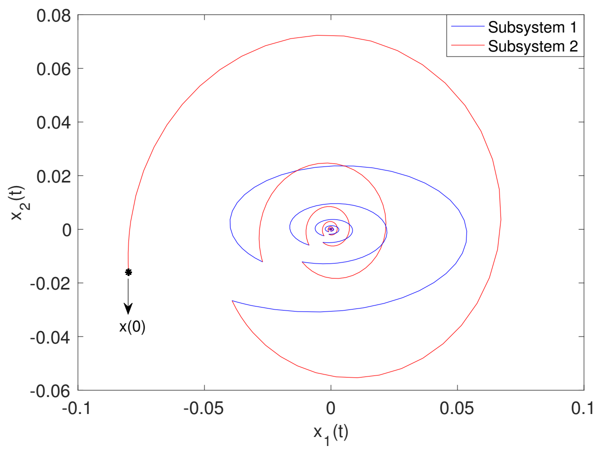
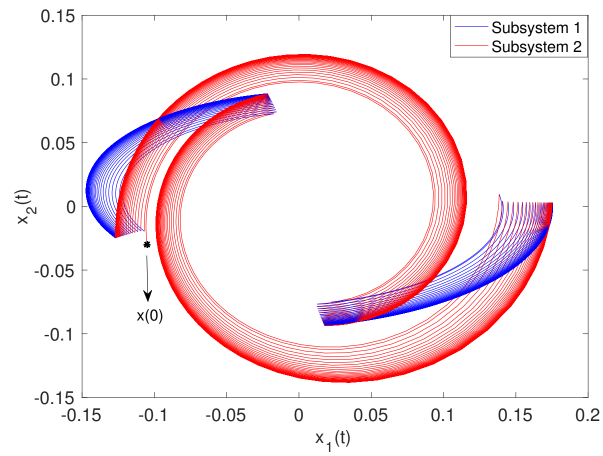


(a) A solution of Example 3 for the switching signal $\sigma(t) = ((1, 5.9), (2, 6.1))$ that satisfies the dwell time condition $\tau > 5.7762$. The solution approaches to $\mathbf{0}$.



(b) A solution of Example 3 for the switching signal $\sigma(t) = ((1, 4.7), (2, 1.7))$ that does not satisfy the dwell time condition $\tau > 5.7762$. The solution approaches to a limit cycle.

Fig. 4: Simulation results for Example 3.

B. Sufficient Conditions for the Almost Global Stability of Discrete-Time Non-Autonomous Systems

Consider the following discrete-time non-autonomous system:

$$x(k+1) = f_k\left(x(k)\right), \quad k \in \mathbb{Z}_{\geq 0}, \tag{12}$$

where $f_k : \mathbb{R}^n \to \mathbb{R}^n, k \in \mathbb{Z}_{\geq 0}$ are non-singular maps. We assume that $\mathbf{0}$ is a common fixed point for all maps; namely, $f_k(\mathbf{0}) = \mathbf{0}, k \in \mathbb{Z}_{\geq 0}$. Denote the solution of (12) for an initial state $x(0) = x_0 \in \mathbb{R}^n$ by $\phi_k(x_0) = f_{k-1} \circ \cdots \circ f_0(x_0)$ for $k \geq 0$. We say that the system (12) is *almost globally stable* if $\lim_{k\to\infty} \phi_k(x) = \mathbf{0}$ for almost every $x \in \mathbb{R}^n$.

For maps f_k for $k \in \mathbb{Z}_{\geq 0}$, let us denote the Frobenius-Perron operator and the Koopman operator for f_k restricted to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ as $\mathbb{P}_k$ and $\mathbb{U}_k$, respectively. Similarly, Frobenius-Perron operators and Koopman operators for the solution maps $\phi_k = f_{k-1} \circ \cdots \circ f_0$, $k \in \mathbb{Z}_{>0}$ restricted to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ can be written as

$$\begin{aligned} \mathbb{P}_{\to k} &:= \mathbb{P}_{k-1} \circ \cdots \circ \mathbb{P}_0, \\ \mathbb{U}_{\to k} &:= \mathbb{U}_0 \circ \cdots \circ \mathbb{U}_{k-1}, \end{aligned}$$

which are dual to each other, namely $\langle g, \mathbb{P}_{\to k}\rho\rangle = \langle \mathbb{U}_{\to k} g, \rho\rangle$. We set $\mathbb{P}_{\to 0}$ and $\mathbb{U}_{\to 0}$ to be the identity operators.

We can now state the following result, which is a direct consequence of the Borel-Cantelli lemma:

Lemma 1. *Solutions of* (12) *converge to* $\mathbf{0}$ *for almost all initial states if there exists a* $\rho \in \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ *such that* $\rho(x) > 0$ *and* $\bar{\rho} := \sum_{k=0}^{\infty} \mathbb{P}_{\to k}\rho$ *is integrable away from* $\mathbf{0}$ [1].

Note that if the conditions in Lemma 1 are satisfied, then ρ is also integrable away from $\mathbf{0}$. This is because $\rho \leq \bar{\rho}$ due to the positivity of $\mathbb{P}_k$, $k \in \mathbb{Z}_{\geq 0}$.

[1]Note that $\bar{\rho}$ is well-defined as a function from $\mathbb{R}^n$ to $\mathbb{R} \cup \{\infty\}$.

Proof: For an $\varepsilon > 0$, consider the events $E_k^{(\varepsilon)} = \{x \in \mathbb{R}^n \mid \mathbb{U}_{\to k} 1_{B_\varepsilon^c}(x) = 1\} = f_0^{-1} \cdots f_{n-1}^{-1}(B_\varepsilon^c)$, where B_ε^c is the complement of the $\varepsilon-$ball of $\mathbf{0}$. Define $E^{(\varepsilon)} := \limsup_{k\to\infty} E_k^{(\varepsilon)} = \cap_{k=0}^{\infty} \cup_{k'=k}^{\infty} E_{k'}^{(\varepsilon)}$, which is the set of all initial states for which the solution of (12) visits B_ε^c infinitely often. It suffices to show that $m(E^{(\varepsilon)}) = 0$ for any $\varepsilon > 0$. This is because the set $\cup_k E^{(\varepsilon_k)}$ for a sequence $\varepsilon_k \to 0$ is a set of Lebesque measure zero and its complement contains the set of all initial state that converge to $\mathbf{0}$. Consider the measure $\mu_\rho(V) := \int_V \rho m(dx)$, with respect to which m is absolutely continuous[2] (since $\rho(x) > 0$), i.e., $m(W) = 0$ whenever $\mu_\rho(W) = 0$. Therefore, we only need to show that $\mu_\rho(E^{(\varepsilon)}) = 0$. Note that, $\mu_\rho(E_k^{(\varepsilon)}) = \langle \mathbb{U}_{\to k} 1_{B_\varepsilon^c}, \rho\rangle = \langle 1_{B_\varepsilon^c}, \mathbb{P}_{\to k}\rho\rangle$. Since $\bar{\rho}$ is integrable away from $\mathbf{0}$, we have $\langle 1_{B_\varepsilon^c}, \bar{\rho}\rangle < \infty$, and

$$\sum_{k=0}^{\infty} \mu_\rho(E_k^{(\varepsilon)}) = \sum_{k=0}^{\infty} \langle 1_{B_\varepsilon^c}, \mathbb{P}_{\to k}\rho\rangle = \langle 1_{B_\varepsilon^c}, \bar{\rho}\rangle < \infty.$$

By Borel-Cantelli lemma, this implies $\mu_\rho(E^{(\varepsilon)}) = 0$ and therefore $m(E^{(\varepsilon)}) = 0$ for all $\varepsilon > 0$. ■

We now leverage multiple Lyapunov densities to ensure almost global stability of (12).

Theorem 4. *Consider the discrete-time non-autonomous system* (12)*. Assume that there exist a positive constant* $\alpha < 1$ *and a sequence of positive functions* $\rho_k \in \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\}), k \in \mathbb{Z}_{\geq 0}$ *dominated by a function* $\rho_{\max} \in \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$*, i.e.* $\rho_k \leq \rho_{\max}, k \in \mathbb{Z}_{\geq 0}$*, such that*

- $\rho_{\max}$ *is integrable away from* $\mathbf{0}$*, and*
- $\mathbb{P}_k\rho_k \leq \alpha\rho_{k+1}$ *for all* $k \in \mathbb{Z}_{\geq 0}$*,*

where $\mathbb{P}_k$ *denotes the Frobenius-Perron operator of* f_k *restricted to* $\mathbb{R}^n \setminus \{\mathbf{0}\}$ *for each* $k \in \mathbb{Z}_{\geq 0}$*. Then, the system* (12) *is almost globally stable, i.e.,* $\lim_{k\to\infty} \phi_k(x) = \mathbf{0}$ *for almost all* $x \in \mathbb{R}^n$.

[2]In fact, μ_ρ and m are equivalent, i.e, they have the same measure zero set. This is because μ_ρ is absolute continuous with respect to m by its definition.

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Proof: Define $\bar{\rho} := \sum_{k=0}^{\infty} \mathbb{P}_{\to k}\rho_0$. Note that $\mathbb{P}_k\rho_k < \alpha\rho_{k+1}$ implies $\mathbb{P}_{k+1}\mathbb{P}_k\rho_k \leq \mathbb{P}_{k+1}\alpha\rho_{k+1} \leq \alpha^2\rho_{k+2}$ due to the positivity of the Frobenius-Perron operator. Iterative application of this gives

$$\begin{aligned}\bar{\rho} &\leq \rho_0 + \alpha\rho_1 + \alpha^2\rho_2 + \cdots \\ &\leq \rho_{\max}\frac{1}{1-\alpha},\end{aligned}$$

which implies that $\bar{\rho}$ is integrable away from $\mathbf{0}$. Therefore, Lemma 1 for $\rho = \rho_0$ implies the result. ■

The following corollary states that the existence of a common Lyapunov density for the maps in (12) implies the convergence of almost all solutions to $\mathbf{0}$.

Corollary 3. *Consider the discrete time non-autonomous system* (12)*. Assume that there exist a positive constant $\alpha < 1$ and a positive, measurable function $\rho \in \mathcal{M}(\mathbb{R}^n \setminus \{\mathbf{0}\})$ such that*

- *ρ is integrable away from $\mathbf{0}$, and*
- *$\mathbb{P}_k\rho \leq \alpha\rho$ for all $k \in \mathbb{Z}_{\geq 0}$,*

where $\mathbb{P}_k$ denotes the Frobenius-Perron operator of f_k restricted to $\mathbb{R}^n \setminus \{\mathbf{0}\}$ for each $k \in \mathbb{Z}_{\geq 0}$. Then, the system (12) *is almost globally stable, i.e., $\lim_{k\to\infty} \phi_k(x) = \mathbf{0}$ for almost all $x \in \mathbb{R}^n$.*

Proof: The result follows from Theorem 4 by setting $\rho_k = \rho_{\max} = \rho$. ■

IV. Proofs of Theorem 2 and Theorem 3

In this section, proofs of Theorem 2 and Theorem 3 are given based on the results of the almost global stability of discrete-time non-autonomous systems. (Theorem 4 and Corollary 3). The proof of Theorem 3 relies on a less-conservative theorem stated below as Theorem 5, which uses Frobenius-Perron operator of solution maps.

The following lemmas provide the required links concerning almost global stability between the continuous-time and discrete-time cases.

Lemma 2. *For a continuously differentiable vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, assume that there exists a non-negative, continuously differentiable function $\rho : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ such that $(1 + \|f(x)\|)\rho(x)$ is integrable away from $\mathbf{0}$ and there exists $\kappa > 0$ such that $\nabla\cdot(\rho f) \geq \kappa\rho$. Then almost all solutions of $\dot{x} = f(x)$ exist, $\rho(x) > 0$ almost everywhere and for a fixed $t > 0$, $\mathbb{P}^{(t)}\rho \leq \mathrm{e}^{-\kappa t}\rho$, where $\mathbb{P}^{(t)}$ is the Frobenius-Perron operator of the time-t solution map Φ_t of $\dot{x} = f(x)$.*

Proof: See Appendix. ■

Given a nonlinear switched system (2) for a fixed $\sigma \in \mathcal{S}_\tau$, we consider the time-$\Delta T$ maps $\Phi_i^{(\Delta T)}$, $i \in \mathbb{N}$ which maps the states at time $i \cdot \Delta T$ to the states at time $(i+1)\cdot\Delta T$ under the dynamics of (2). This gives rise to a discrete-time non-autonomous system

$$x(k+1) = \Phi_k^{(\Delta T)}(x(k)), \quad k \in \mathbb{Z}_{\geq 0}, \tag{13}$$

which produce solutions that are discretizations of the continuous-time trajectories of (2). Note that the systems (13) for a $\sigma \in \mathcal{S}_\tau$ is a discrete-time non-autonomous system in the form of (12).

Lemma 3. *The switched system* (2) *is almost globally stable for $\sigma \in \mathcal{S}_\tau$ if and only if its discretization* (13) *for σ is almost globally stable for all sufficiently small $\Delta T > 0$.*

Proof: See Appendix. ■

Although Theorem 2 follows directly from Theorem 3, we state the proof of Theorem 2 seperately, for the sake of easy reading.

Proof: (Proof of Theorem 2) The existence of almost all solutions for subsystems follows from Corollary 1 and therefore the existence of almost all solutions of the switched system follows from $\sigma \in \mathcal{S}_\tau$. It remains to show that almost all solutions converge to $\mathbf{0}$. We first show that almost all solutions converge to $\mathbf{0}$ for the above-mentioned discretized system for a sufficiently small ΔT[3]. This can be done by proving that the assumptions in Corollary 3 are satisfied for (13) for the same density function ρ as follows: CLD0 implies that ρ is integrable. $\rho(x) > 0$ for almost all x follows from Lemma 2. It remains to show that there exists an $\alpha < 1$ such that, for all k, $\mathbb{P}_k\rho < \alpha\rho$, where $\mathbb{P}_k$ is the Frobenius-Perron operator of the map $\Phi_k^{(\Delta T)}$. Since $\sigma \in \mathcal{S}_\tau$, for a sufficiently small ΔT, $\Phi_k^{(\Delta T)}$ either consists of a time-ΔT map of a subsystem f_p or is a composition of two maps, a ΔT_1-map of a subsystem f_p and a ΔT_2-map of the next subsystem f_m, where $\Delta T_1 + \Delta T_2 = \Delta T$. In both cases, one can see that, CLD1 together with Lemma 2 imply $\mathbb{P}_k\rho(x) \leq \alpha\rho(x)$ for $\alpha := e^{-\kappa\Delta T}$. The first case is straightforward, and the second case follows from $\mathbb{P}_k\rho = \mathbb{P}_{k,2}\mathbb{P}_{k,1}\rho \leq \mathbb{P}_{k,2}e^{-\kappa\Delta T_1}\rho \leq e^{-\kappa(\Delta T_1+\Delta T_2)}\rho = \alpha\rho$, where $\mathbb{P}_{k,1}$ and $\mathbb{P}_{k,2}$ are the Frobenius-Perron operators of the two maps in consideration. Hence, Corollary 3 can be applied and almost everywhere convergence to $\mathbf{0}$ can be proven for all discretization with sufficiently small sampling time ΔT. Finally, Lemma 3 implies the result. ■

Proof of Theorem 3 will be given based on the following theorem where the condition MLD2' is replaced by a less conservative condition.

Theorem 5. *Consider the switched system* (2)*. Assume that there exist constants $\tau_{\min} > 0$, $\kappa_p > 0$, $p \in \{1, 2, \ldots, N\}$ and non-negative, continuously differentiable functions $\rho_p : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$, $p \in \{1, 2, \ldots, N\}$ such that the following conditions are satisfied for all $p, m \in \{1, 2, \ldots, N\}$*

MLD0: *$(1 + \|f_p(x)\|)\rho_p(x)$ is integrable away from $\mathbf{0}$,*
MLD1: *$\nabla\cdot(\rho_p f_p)(x) \geq \kappa_p\rho_p(x)$ for all $x \in \mathbb{R}^n$,*
MLD2: *$\mathbb{P}_p^{(\tau_{\min})}\rho_p(x) \leq \rho_m(x)$ for all $x \in \mathbb{R}^n$,*

where $\mathbb{P}_p^{(t)}$ is the Frobenius-Perron operator of the time-t map for the subsystem $\dot{x} = f_p(x)$. Then, the system (2) *is almost globally stable for any $\tau > \tau_{\min}$.*

Proof: The existence of almost all solutions of the switched system follows as in the proof of Theorem 2. To show that almost all solutions converge to $\mathbf{0}$, we show that almost all solutions converge to $\mathbf{0}$ for the discretization (13) for all sufficiently small ΔT. This is done by proving that the

[3] Smallness of ΔT is assumed only for the simplicity of the arguments in the proof.

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assumptions in Theorem 4 are satisfied for (13) for a sequence of density function labeled as

$$(\nu_k)_{k\in\mathbb{Z}_{>0}} = \left(\nu_1^{(1)},\dots,\nu_1^{(K_1)},\nu_2^{(1)},\dots,\nu_2^{(K_2)},\nu_3^{(1)},\dots\right).$$

Here, K_i is the number of sampling instants in the interval $[t_{i-1},t_i)$ (see Fig. 5).

Without loss of generality, we assume that the switching instants satisfy $t_i - t_{i-1} < 2\tau$ for all $i \in \mathbb{Z}_{\geq 0}$, since otherwise we could split the subsystem operation interval $[t_{i-1},t_i)$ into pieces of length greater than or equal to τ by adding dummy switching instants in this interval that represent switchings from the subsystem f_{p_i} to the same subsystem f_{p_i}. This result in an upper bound on K_i's as $K_i \leq K_{\max} := 2\tau/\Delta T$. Note that the first sampling instant in the interval $[t_{i-1},t_i)$ is $\bar{K}_{i-1}\Delta T$, where $\bar{K}_i = K_1 + \cdots + K_i$ for all $i \in \mathbb{Z}_{>0}$ and $\bar{K}_0 = 0$ (see Fig. 5). We define $\nu_i^{(j)}$ recursively as follows:

- $\nu_i^{(1)} = \mathbb{P}_{p_i}^{(\bar{K}_{i-1}\Delta T - t_{i-1})}\rho_{p_i}$
- $\nu_i^{(j+1)} = \mathbb{P}_{p_i}^{(\Delta T - \Delta\tau/K_i)}\nu_i^{(j)}$, for $j = 1,\dots,K_i - 1$,

where $\Delta\tau := \tau - \tau_{\min}$. Note that, recursive application of the above yields

$$\nu_i^{(K_i)} = \mathbb{P}_{p_i}^{(\Delta t_i - \Delta\tau + \Delta\tau/K_i - (t_i - (\bar{K}_i - 1)\Delta T))}\rho_{p_i}. \tag{14}$$

MLD0 implies that each ν_k is integrable away from $\mathbf{0}$ and Lemma 2 implies $\nu_k > 0$. It remains to show that there exists an $\alpha < 1$ such that, for all k, $\mathbb{P}_k\nu_k \leq \alpha\nu_{k+1}$ almost everywhere, where $\mathbb{P}_k$ is the Frobenius-Perron operator of the map $\Phi_k^{(\Delta T)}$. As done in the proof of Theorem 2, we assume that ΔT is sufficiently small such that $\Phi_k^{(\Delta T)}$ either consists of a time-ΔT map of a subsystem f_p or is a composition of two maps, a ΔT_1-map of a subsystem f_p and a ΔT_2-map of the next subsystem f_m, where $\Delta T_1 + \Delta T_2 = \Delta T$. In particular, for k not equal to any $\bar{K}_m$, $\Phi_k^{(\Delta T)}$ consists of one time-ΔT map (see Fig. 5) and from Lemma 2 we have

$$\begin{aligned}\mathbb{P}_k\nu_k &= \mathbb{P}_{p_i}^{(\Delta T)}\nu_i^{(j)} = \mathbb{P}_{p_i}^{(\Delta\tau/K_i)}\nu_i^{(j+1)} \leq e^{-\kappa_{p_i}\Delta\tau/K_i}\nu_i^{(j+1)}\\ &\leq \alpha\nu_{k+1},\end{aligned}$$

where i is such that $[(k-1)\Delta T, k\Delta T) \subset [t_{i-1},t_i)$, j is such that $k = \bar{K}_{i-1} + j$ and $\alpha := e^{-(\min_p \kappa_p)\Delta\tau/K_{\max}}$. For $k = \bar{K}_i$ for some i, $\Phi_k^{(\Delta T)}$ consists of two maps in general, as explained above, in particular with $\Delta T_1 = t_i - (\bar{K}_i - 1)\Delta T$ and $\Delta T_2 = \bar{K}_i\Delta T - t_i$. Using (14) we have,

$$\begin{aligned}\mathbb{P}_k\nu_k &= \mathbb{P}_{p_{i+1}}^{(\Delta T_2)}\mathbb{P}_{p_i}^{(\Delta T_1)}\nu_i^{(K_i)}\\ &= \mathbb{P}_{p_{i+1}}^{(\bar{K}_i\Delta T - t_i)}\mathbb{P}_{p_i}^{(\Delta t_i - \Delta\tau)}\mathbb{P}_{p_i}^{(\Delta\tau/K_i)}\rho_{p_i}.\end{aligned}$$

Using $\Delta t_i - \Delta\tau \geq \tau_{\min}$ and invoking Lemma 2 for the last Frobenius-Perron operator above, we get

$$\begin{aligned}\mathbb{P}_k\nu_k &\leq \mathbb{P}_{p_{i+1}}^{(\bar{K}_i\Delta T - t_i)}\mathbb{P}_{p_i}^{(\tau_{\min})}e^{-\kappa_{p_i}(\Delta\tau/K_i)}\rho_{p_i}\\ &\leq \mathbb{P}_{p_{i+1}}^{(\bar{K}_i\Delta T - t_i)}\mathbb{P}_{p_i}^{(\tau_{\min})}\alpha\rho_{p_i}\end{aligned}$$

Finally, MLD2 implies $\mathbb{P}_k\nu_k \leq \mathbb{P}_{p_{i+1}}^{(\bar{K}_i\Delta T - t_i)}\alpha\rho_{p_{i+1}} = \alpha\nu_{i+1}^{(1)} = \alpha\nu_{k+1}$. ■

Proof: (Proof of Theorem 3) Conditions MLD1 and MLD2' imply MLD2 by Lemma 2. Hence, the proof follows from Theorem 5. ■

Remark 3. *Values of multiple Lyapunov functions decrease with time (along solutions) monotonically on each operating interval and from one switching instant to the next, allowing increases at switching instants. As an analogue to this, integrals of multiple Lyapunov densities increase with time (over a set of states) on each operating interval and from one switching instant to the next, allowing decreases at switching instants (see Fig. 6). To be precise, assume that the switched system operates initially with the subsystem f_p on the interval $[t_{k-1} - t_k)$ and then switches to the subsystem f_m at the switching instant t_k. Recall that $\mu_\rho(V) := \int_V \rho m(dx)$. MLD1 and Lemma 2 imply that $\mathbb{P}_p^{(t)}\rho_p < \rho_p$ for $t > 0$. Integrating both sides over $\phi_t(V)$, where V is a measurable set of states and ϕ_t is the time-t solution map of the subsystem f_p, we get $\int_{\phi_t(V)} \mathbb{P}_p^{(t)}\rho_p m(dx) < \int_{\phi_t(V)} \rho_p m(dx)$, which implies that $\int_V \rho_p m(dx) < \int_{\phi_t(V)} \rho_p m(dx)$. As a result, $\mu_{\rho_p}(V)$, namely the integral of the density ρ_p over V increases on the interval $[t_{k-1} - t_k)$. On the other hand, since $t_k - t_{k-1} > \tau_{\min}$, we have*

$$\begin{aligned}\mathbb{P}_p^{(t_k - t_{k-1})}\rho_p &= \mathbb{P}_p^{(\tau_{\min})}\mathbb{P}_p^{(t_k - t_{k-1} - \tau_{\min})}\rho_p\\ &< \mathbb{P}_p^{(\tau_{\min})}\rho_p\\ &\leq \rho_m,\end{aligned}$$

where the first inequality follows from the positivity of the Frobenius-Perron operator and the second inequality follows from MLD2. Integrating both sides of the inequality $\mathbb{P}_p^{(t_k - t_{k-1})}\rho_p < \rho_m$ over $V_{t_k} := \phi_{t_k - t_{k-1}}(V_{t_{k-1}})$ for a measurable set $V_{t_{k-1}}$, we have $\int_{V_{t_k}} \mathbb{P}_p^{(t_k - t_{k-1})}\rho_p m(dx) < \int_{V_{t_k}} \rho_m m(dx)$ which implies $\int_{V_{t_{k-1}}} \rho_p m(dx) < \int_{V_{t_k}} \rho_m m(dx)$. Therefore, we obtain $\mu_{\rho_p}(V_{t_{k-1}}) < \mu_{\rho_m}(V_{t_k})$ meaning that integrals of densities $(\mu_{\rho_{\sigma(t)}})$ increase with time from one switching instant to the next. This change of integrals of densities over a given set of states is depicted in Fig. 6.

V. A Reduction to Linear Switched Systems

In this section, we show that, for linear switched systems with stable subsystems, Theorem 3 generalizes an already-known LMI condition based on multiple quadratic Lyapunov functions [24].

Let us consider the linear version of the switched system (2), namely

$$\dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma \in \mathcal{S}_\tau, \quad t \in [0,\infty), \tag{15}$$

where A_p's are Hurwitz stable matrices. The sufficient condition obtained in [24] for the exponential stability of (2) for a given dwell time τ is that there exist positive definite, symmetric matrices $P_1,\dots,P_N$ such that

$$A_p^{\mathrm{T}}P_p + P_pA_p < 0, \quad p \in \{1,\dots,N\} \tag{16}$$

$$e^{A_p^{\mathrm{T}}\tau}P_me^{A_p\tau} < P_p, \quad p,m \in \{1,\dots,N\}. \tag{17}$$

Let us consider densities for each subsystem as

$$\rho_p = (x^{\mathrm{T}}P_px)^{-\gamma}, \tag{18}$$

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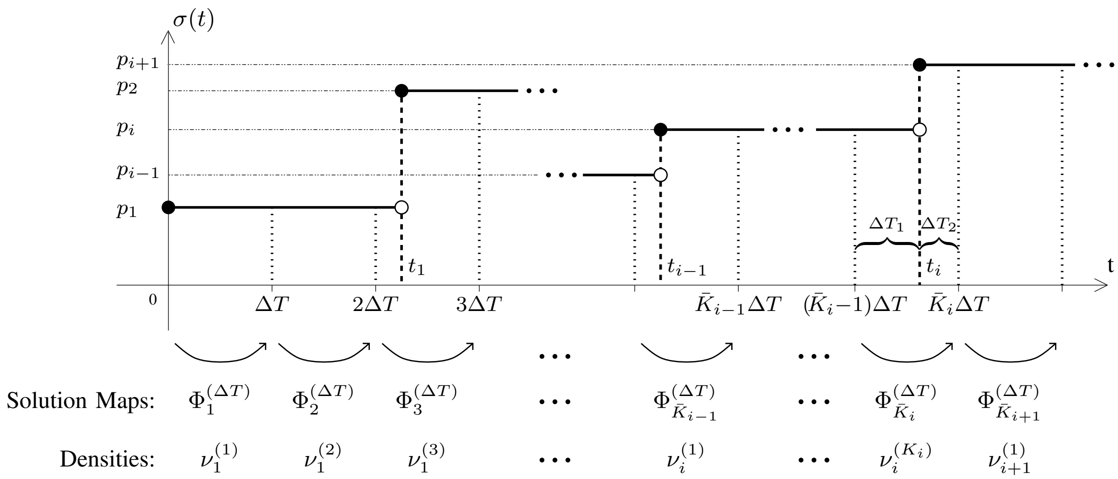

Fig. 5: An illustration for the proof of Theorem 3.

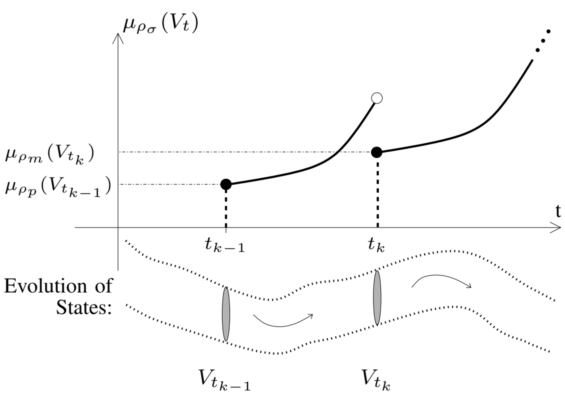

Fig. 6: The change of integrals of densities with time.

where P_p's are $n \times n$ positive definite symmetric matrices and $\gamma \geq 2$ is sufficiently large. We want to show that, for this particular choice of densities, Theorem 5 generalizes conditions (16) and (17): The integrability condition MLD0 in Theorem 5 is satisfied since $\|f_p(x)\|\rho_p(x) = \|A_p x\|/(x^{\mathrm{T}}P_p x)^{-\gamma}$ is of the same order as $\|x\|^{-(2\gamma-1)} \leq \|x\|^{-3}$. Let us show that conditions (16) and (17) can be obtained from MLD1 and MLD2, respectively.

Recall the condition MLD1, namely

$$\nabla \cdot (\rho_p f_p)(x) \geq \kappa_p \rho_p(x). \tag{19}$$

Substituting $\nabla \cdot (\rho_p f_p) = \nabla \rho_p f_p + \rho_p \nabla \cdot f_p$ and (18) in (19), we get

$$-\gamma(x^{\mathrm{T}}P_p x)^{-\gamma-1}x^{\mathrm{T}}\left(A^{\mathrm{T}}P_p + P_p A\right)x + (x^{\mathrm{T}}P_p x)^{-\gamma}\nabla \cdot f_p \geq \kappa_p (x^{\mathrm{T}}P_p x)^{-\gamma}.$$

Multiplying both sides by $-(x^{\mathrm{T}}P_p x)^{\gamma}/\gamma$, we obtain

$$A_p^{\mathrm{T}}P_p + P_p A \leq \frac{\nabla \cdot f_p - \kappa_p}{\gamma}. \tag{20}$$

Since $\nabla \cdot f_p = \mathrm{trace}(A_p) < 0$ by the stability of subsystems and γ can be made arbitrarily large, the right hand side of (20) can be made arbitrarily close to 0. As a result, MLD1 is equivalent to (16).

Consider now the condition MLD2 in Theorem 5, namely

$$\mathbb{P}_p^{(\tau_{\min})}\rho_p(x) \leq \rho_m(x). \tag{21}$$

Note that $\mathbb{P}_p^{(\tau_{\min})}$ is the Frobenius operator for the map $e^{A_p \tau_{\min}}$, which is an invertible differentiable map. By (11),

$$\mathbb{P}_p^{(\tau_{\min})}\rho(x) = \rho(e^{-A_p\tau_{\min}}(x))\det(e^{-A_p\tau_{\min}}). \tag{22}$$

Using (18) and (22), (21) can be written as

$$\left(x^{\mathrm{T}}e^{-A_p^{\mathrm{T}}\tau_{\min}}P_p e^{-A_p\tau_{\min}}x\right)^{-\gamma}\det(e^{-A_p\tau_{\min}}) \leq \left(x^{\mathrm{T}}P_m x\right)^{-\gamma},$$

which implies that

$$e^{-A_p^{\mathrm{T}}\tau_{\min}}P_m e^{-A_p\tau_{\min}} \leq e^{\frac{\mathrm{trace}(A_p)\tau_{\min}}{\gamma}}P_p \tag{23}$$

Since $\mathrm{trace}(A_p) < 0$ choosing γ large enough the right hand side of (23) can be made arbitrary close to P_p. Hence, MLD2 reduces to (16).

VI. Conclusion

We have derived sufficient conditions for the almost global stability of nonlinear switched systems with time-dependent switching. Our method is based on multiple Lyapunov densities and can be seen as the analogue of the multiple Lyapunov function technique, for the framework of almost global stability.

After this work, new directions in the field of almost global stability may open up. Firstly, the use of Lyapunov densities for the verification of temporal properties such as safety, reachability, eventuality and avoidance, studied in [30], can be considered for switched nonlinear systems. Secondly, following the ideas presented in [31] and [32] for the graph-based estimations of the minimum and average dwell time, the techniques in this paper can be used to obtain graph-based estimations of the minimum and average dwell time for the almost global stability of nonlinear switched systems.

 doi:10.20944/preprints201809.0514.v1

Appendix
Proofs of Lemma 2 and Lemma 3

The following two lemmas will be used for the proof of Lemma 2:

Lemma 4. *[1] Let $f \in \mathbb{C}^1(D, \mathbb{R}^n)$, where $D \subset \mathbb{R}^n$ is open and let $\rho \in C^1(D, R)$ be integrable. Let ϕ_t denote the solution map of $\dot{x} = f(x)$. For a measurable set Z, assume that $\phi_s(Z) = \{\phi_s(x) | x \in Z\}$ is a subset of D for all s between 0 and t. Then, we have*

$$\int_{\phi_t(Z)} \rho(x)dx - \int_Z \rho(x)dx = \int_0^t \int_{\phi_s(Z)} [\nabla \cdot (f\rho)](x)dxds.$$

Lemma 5. *[11] Let $\rho : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be a non-negative, continuously differentiable function such that ρ is integrable away from $\mathbf{0}$ and $\nabla \cdot (\rho f)(x) > 0$ for almost all $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Then for any fixed $t > 0$, ρ satisfies $\mathbb{P}^{(t)}\rho < \rho$, where $\mathbb{P}^{(t)}$ is the Frobenius-Perron operator for the time-t solution map ϕ_t of $\dot{x} = f(x)$ for a fixed $t > 0$.*

Proof: (Proof of Lemma 2)

Almost everywhere existence of solutions for $t > 0$, follows from Corollary 1 in Section II. $\rho(x) > 0$ almost everywhere follows from Theorem 1 in [33].[4]

Let us denote $\phi(t, x) := \phi_t(x)$, where $\phi_t(x)$ is the solution of $\dot{x} = f(x)$. For fixed $T \in \mathbb{R}_{>0}$, the set $\phi([-T, 0], Z)$ is compact due to the continuity of the map $\phi(t, x)$ in both time and space variables. Because $\mathbf{0}$ is invariant, any flow map $\phi(t, x)$, with $x \in Z$ will stay in $\mathbb{R}^n \setminus \mathbf{0}$ for all $t \geq 0$. Thus, $\mathbf{0}$ and $\phi([-T, 0], Z)$ are disjoint sets. Due to the normality of $\mathbb{R}^n$, there is a neighbourhood of $\mathbf{0}$ which is disjoint from $\phi([-T, 0], Z)$. Therefore, there exists an $\varepsilon_1 > 0$, such that $\phi([-T, 0], Z) \subset B^c_{\varepsilon_1}$.

Applying Lemma 4 with $t = T$ and $D \subset B^c_{\varepsilon_1}$, we obtain that

$$\int_{\phi_T(Z)} \rho(z)dz - \int_Z \rho(z)dz = \int_0^T \int_{\phi_s(Z)} [\nabla \cdot (f\rho)](z)dzds,$$

where $\phi_s(Z) = \{\phi_s(z) | z \in Z\}$. Then, by applying variable change $z = \phi_s(x)$, we have

$$\begin{aligned} &\int_Z \rho(\phi_T(x)) \det(\mathrm{D}\phi_T(x))\, dx - \int_Z \rho(z)dz \\ &= \int_0^T \int_Z [\nabla \cdot (f\rho)](\phi_s(x)) \det(\mathrm{D}\phi_s(x))\, dxds. \end{aligned} \tag{24}$$

Applying condition $\nabla \cdot (f\rho)(x) \geq \kappa\rho(x)$ to (24), we obtain that

$$\begin{aligned} &\int_Z \rho(\phi_T(x)) \det(\mathrm{D}\phi_T(x))\, dx - \int_Z \rho(x)dx \geq \\ &\int_0^T \int_Z \kappa\rho(\phi_s(x)) \det(\mathrm{D}\phi_s(x))\, dxds. \end{aligned}$$

Let us define $\mathbb{P}^{(-t)} := \left(\mathbb{P}^{(t)}\right)^{-1}$. Using the equality

$$\int_Z \rho(\phi_T(x)) \det(\mathrm{D}\phi_T(x))\, dx = \int_Z \mathbb{P}^{(-T)}\rho(x)dx,$$

which can be obtained from (10) for $f(x) := (\phi_t)^{-1}(x)$ and applying the above variable change, we get

$$\int_Z \mathbb{P}^{(-T)}\rho(x)dx - \int_Z \rho(x)dx \geq \int_0^T \int_Z \kappa\mathbb{P}^{(-s)}\rho(x)dxds. \tag{25}$$

From inequality $\nabla \cdot (f\rho)(x) \geq \kappa\rho(x) > 0$, utilising Lemma 5, one observes that $\mathbb{P}^{(s)}\rho < \rho$. Using the positivity of the Frobenius-Perron operator $\mathbb{P}^{(s)}$, we obtain $\mathbb{P}^{(-s)}\rho > \rho$, applying this to (25), we get

$$\begin{aligned} \int_Z \mathbb{P}^{(-T)}\rho(x)dx - \int_Z \rho(x)dx &> \int_0^T \int_Z \kappa\rho(x)dxds \\ &= \int_Z \kappa T\rho(x)dx. \end{aligned}$$

We have $\mathbb{P}^{(-T)}\rho(x) > (1+\kappa T)\rho(x)$ on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, since Z is an arbitrary compact subset of B^c_ε for an arbitrary $\varepsilon > 0$. By using the positivity of the Perron-Frobenious operator, we obtain that $\mathbb{P}^{(T)}\rho(x) < \frac{1}{(1+\kappa T)}\rho(x)$, for almost all $x \in B^c_{\varepsilon_1}$. We conclude that $\mathbb{P}^{(T)}\rho(x) < \frac{1}{(1+\kappa T)}\rho(x)$ for almost all $x \in B^c_\varepsilon$, since ε_1 is also arbitrary. Dividing the interval $[0, T]$, into equal pieces, $\Delta t = \frac{T}{n}$, we obtain $\mathbb{P}^{(\Delta t)}\rho(x) < \frac{1}{(1+\frac{\kappa T}{n})}\rho(x)$. Then, for all $n \in \mathbb{Z}_{>0}$,

$$\mathbb{P}^{(T)}\rho(x) = \left(\mathbb{P}^{(\Delta t)}\right)^n \rho(x) < \frac{1}{(1+\frac{\kappa T}{n})^n}\rho(x).$$

Taking the limit of the right hand side as $n \to \infty$, we get $\mathbb{P}^{(T)}\rho(x) \leq e^{-\kappa T}\rho(x)$.

■

Proof: (Proof of Lemma 3) The necessity part of the proof is trivial. To show the sufficiency, let us choose a sequence of sufficiently small numbers $\{\Delta T_i\}_{i \in \mathbb{Z}_{>0}} \to 0$ such that the discretization (13) with sampling time ΔT_i is almost globally stable for all $i \in \mathbb{Z}_{>0}$. In other words, for each $i \in \mathbb{Z}_{>0}$, there exists a set N_i of zero Lebesgue measure such that all initial points in $\mathbb{R}^n \setminus N_i$ converge to $\mathbf{0}$ for the discretization (13) with sampling time ΔT_i. Set $N := \cup_i N_i$, which has zero Lebesgue measure. It is enough to show that a solution of (2), say $x(t)$, for a given $\sigma \in \mathcal{S}_\tau$ and an initial state $x(0) = x_0 \in \mathbb{R}^n \setminus N$ converges to $\mathbf{0}$ if its discretization $x(k\Delta T_i)$, namely the solution of (13) with sampling time ΔT_i for $x(0) = x_0$, converges to $\mathbf{0}$ for all $i \in \mathbb{Z}_{>0}$. We show this by contradiction as follows: Let us assume that $\lim_{k\to\infty} x(k\Delta T_i) = \mathbf{0}$ for all $i \in \mathbb{Z}_{>0}$ and $\lim_{t\to\infty} x(t) \neq \mathbf{0}$. The second assumption implies that there exists an $\varepsilon > 0$ such that for each time T, there exists a larger time $T'(T)$ such that $x(T'(T)) \in B^c_\varepsilon$, whereas the first assumption implies that there exists a $k_1 \in \mathbb{Z}_{\geq 0}$ such that the sequence $x(k\Delta T_1)$ is contained in $B^c_{\varepsilon/2}$ for all $k \geq k_1$. Hence, by the continuity of $x(t)$ with respect to t, we can choose an increasing sequence of time instants $\{t_k\}_{k \in \mathbb{Z}_{\geq 0}}$ such that $\|x(t_0)\| = \varepsilon/2$ where $t_0 > k_1\Delta T_1$, $\|x(t_k)\| = \varepsilon$ for all odd k's and $\|x(t_k)\| = \varepsilon/2$ for all even k's. This can be done as follows: Set $k = k_1$, consider $x(k\Delta T_1) \in B_{\varepsilon/2}$ and $x(T'(k\Delta T_1)) \in B^c_\varepsilon$, and by continuity choose $t_0, t_1 \in (k\Delta T_1, T'(k\Delta T_1)]$ such that $\|x(t_0)\| = \varepsilon/2$ and $\|x(t_1)\| = \varepsilon$ (See Fig. 7). Repeat this process for $k = k_2$ satisfying $k_2\Delta T_1 > t_1$ to obtain t_2 and t_3, and so on. By the continuous differentiability of the subsystem vector fields f_p,

[4] The requirement of the existence of all solutions in the proof of this Theorem 1 in [33] can be replaced by the weaker requirement that almost all solutions exists for $t > 0$.

 doi:10.20944/preprints201809.0514.v1

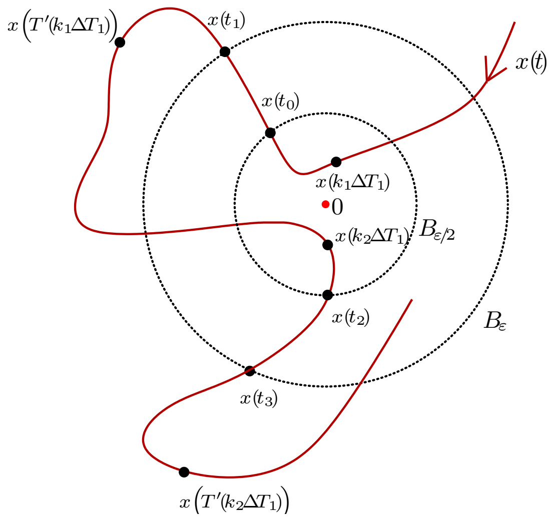

Fig. 7: An illustration for the proof of Lemma 3.

there exists a common local Lipschitz constant for B_ε valid for all vector fields. Therefore, $\liminf_k(t_{k+1}-t_k) > 0$, and hence one can choose a sufficiently small ΔT_j such that $x(k\Delta T_j)$ visits $B_\varepsilon \setminus B_{\varepsilon/2}$ infinitely often, which contradicts our second assumption above.

■

Acknowledgment

This work has been supported by the Independent Research Fund Denmark in the projects DeBaTe and CodeMe.

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 doi:10.20944/preprints201809.0514.v1

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