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Article

A Generalized Fractional Power Series for Solving a Class of Nonlinear Fractional Integro-Differential Equation

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Abstract: In this paper, we investigate an analytical solution of a class of nonlinear fractional integro-differential equation base on a generalized fractional power series expansion. The fractional derivatives are described in the conformable's type. The new approach is a modified form of the well-known Taylor series expansion. The illustrative examples are presented to demonstrate the accuracy and effectiveness of the proposed method.

Keywords: fractional power series; integro-differential equations; conformable derivative

1. Introduction

Fractional calculus and fractional differential equations are widely explored subjects thanks to the great importance of scientific and engineering problems. For example, fractional calculus is applied to model the fluid-dynamic traffics [1], signal processing [2], control theory [3], and economics [4]. For more details and applications about fractional derivative, we refer the reader to [5–8]. Many mathematical formulations contain nonlinear integro-differential equations with fractional order. However, the integro-differential equations are usually difficult to solve analytically, so it is required to obtain an efficient approximate solution. For instance we can mention the following papers. Rawashdeh [9] applied collocation method to study the integro-differential equations of fractional order, authors of [10] applied spectral collocation method to solve stochastic fractional integro-differential equations. Momani [11] applied the Adomian decomposition method (ADM) to approximate solutions for fourth-order integro-differential equations of fractional order. Nawaz [12] applied the variational iteration method and homotopy perturbation method for the fourth-order fractional integro-differential equations, authors of [13] presented a computational method based on the second kind Chebyshev wavelet to solve fractional nonlinear Fredholm integro-differential equations. In Ref. [14] approximated solution of fractional integro-differential equations by Taylor expansion method. Among these methods, the Taylor expansion method is more attractive. Hitherto several fractional power series expansions have been presented in the literature [15–21]. In Ref. [18] the authors presented a new algorithm for obtaining a series solution for a class of fractional differential equations. Syam [19] investigate a numerical solution of fractional Lienard's equation by using the residual power series method. In Ref. [20] the authors develop a new method to solve rational or irrational order fractional differential equations. This method is called the restricted fractional differential transform method (RFDTM). Recently, Jaradat [21] proposed a new method based on Taylor series expansion for solving the fractional (integro)-differential equations and compared numerical solution with exact solution. A new series expansion is proposed to obtain closed-form solutions of the fractional (integro-)differential equations in the Caputo's type. This expansion provide a more integrated representation of the fractional power series with a related convergence theorem called a generalized fractional power series (GFPS).

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In this paper, we adopt the conformable fractional derivative with GFPS and apply it to solve a class of nonlinear integro-differential equation

$$T^{\alpha}[y(t)] = h(t) + \int_0^1 k(t,\tau)[y(\tau)]^q d\tau, \;\; q \geq 1, \tag{1}$$

subject to the initial conditions

$$y^{(i)}(0) = a_i, \tag{2}$$

where a_i, $i = 0, 1, ..., r-1$, with $r-1 < \alpha \leq r$, $r \in \mathbb{N}$, are constants. $k(t,\tau)$ and $h(t)$ are smooth functions. The derivative which we use in this paper is the conformable fractional derivative. We organize our paper as follows. In Section 2, we present some preliminaries which we use in this paper. In Section 3, we present the proposed method which is the GFPS in conformable fractional derivative. Some analytical and numerical results are presented in Section 4. We end this paper by conclusions which presented in Section 5.

2. Preliminaries

In this section, we present some definitions and properties of the conformable fractional derivative and GFPS. The derivative in Equation (1) is in the conformable fractional derivative. The conformable fractional derivative is defined as follows; see [22]. Throughout the rest of this section, we assume $\alpha \in (0,1]$.

Definition 1. *Given a function $f : [0,\infty) \to \mathbb{R}$, where the conformable fractional derivative of f order α is defined by*

$$T^{\alpha}[f(t)] = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{3}$$

for all $t > 0$.

Theorem 1. *If f and g be α-differentiable at a point $t > 0$, then*

$$T^{\alpha}[af + bg] = aT^{\alpha}[f] + bT^{\alpha}[g],$$

for all a, $b \in \mathbb{R}$.

The power rule of the conformable fractional derivative is given as follows.

Theorem 2. *The conformable fractional derivative of the power function is given by*

$$T^{\alpha}[t^p] = pt^{p-\alpha},$$

for all $p \in \mathbb{R}$.

We implement the generalized fractional power series (GFPS) [21] to solve Equation (1). We start by the following definition and some related properties to the GFPS.

Definition 2. *A generalized fractional power series of the form*

$$\sum_{i+j=0}^{\infty} c_{ij} t^{i\alpha+j} = c_{00} + c_{01}t^1 + c_{10}t^{\alpha} + c_{02}t^2 + c_{11}t^{\alpha+1} + c_{20}t^{2\alpha} + \cdots \tag{4}$$

where $t > 0$, is called generalized fractional power series (GFPS) about $t = 0$. c_{ij} denote the coefficients of the series, where $i, j \in \mathbb{N}$.

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Moreover, the GFPS is naturally obtained as a Cauchy product of two power series, as following

$$\sum_{i+j=0}^{\infty} c_{ij} t^{i\alpha+j} = \left(\sum_{i=0}^{\infty} a_i t^{i\alpha}\right)\left(\sum_{j=0}^{\infty} b_j t^j\right), \quad (5)$$

where $c_{ij} = a_i b_j$.

Proposition 1. *If $\sum_{k=0}^{\infty} a_k t^{k\alpha}$ converges for some $t = a > 0$, then it converges absolutely for $t \in (0, a)$.*

Proof. See [21].

Corollary 1. *If $\sum_{k=0}^{\infty} b_k t^k$ converges for some $t = b > 0$, then it converges absolutely for $t \in (0, b)$.*

Proof. See [21].

Theorem 3. *Consider the two power series $A = \sum_{k=0}^{\infty} a_k t^{k\alpha}$ and $B = \sum_{k=0}^{\infty} b_k t^k$ such that A converges absolutely to a for $t = t_a > 0$ and B converges to b for $t = t_b > 0$. Then the Cauchy product of A and B converges to ab for $t = t_c > 0$ where $t_c = min\{t_a, t_b\}$.*

Proof. See [21].

3. The Generalized Conformable Fractional Power Series Method

In this section, we present a generalized conformable fractional power series method to solve problem (1) and (2). We assume that the solution $y(t)$ takes the form

$$y(t) = \sum_{i+j=0}^{\infty} c_{ij} t^{i\alpha+j}, \quad (6)$$

where $y(0) = y_0$ and c_{ij} are constants to be determined. Clearly, $c_{00} = y_0$.

Theorem 4. *The generalized conformable fractional power series (GCFPS) of order α is given by*

$$\begin{aligned} T^{\alpha}[y(t)] &= \sum_{i=1}^{\infty} c_{i0}(i\alpha) t^{(i-1)\alpha} + \sum_{j=1}^{\infty} c_{0j}(j) t^{j-\alpha} \\ &+ \sum_{i+j=0}^{\infty} c_{i+1,j+1}[(i+1)\alpha + j + 1] t^{i\alpha+j+1}, \end{aligned} \quad (7)$$

where t be a positive real number.

Proof. Assuming we can interchange the summation and the fractional derivative operator and using Theorems 1 and 2, by term-by-term differentiation within the interval of convergence of $t > 0$, if

$$y(t) = \sum_{i+j=0}^{\infty} c_{ij} t^{i\alpha+j}$$

then we obtain Equation (7).

The proposed GCFPS expansion (7) will be utilized to introduce a parallel scheme to the power series solution method. The illustrative examples are presented to demonstrate the accuracy and effectiveness of the proposed method in Section 4.

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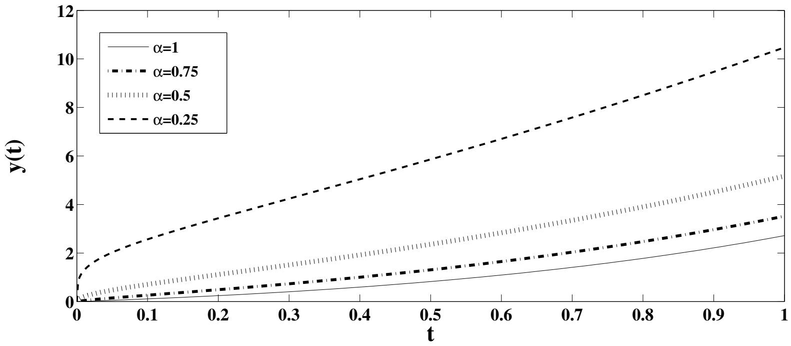

Figure 1. The approximate solution of Example 1 for $\alpha = 0.25, 0.5, 0.75, 1$.

4. Numerical Results

In this section, we have dealt with three examples of the nonlinear integro-differential equations to exhibit the usefulness of GCFPS expansion. It should be noted here that all the necessary calculations and graphical analysis are done by using MATLAB2017a.

Example 1. *Consider the nonlinear Fredholm fractional integro-differential equation*

$$T^{\alpha}[y(t)] = te^t + e^t - t + \int_0^1 ty(\tau)d\tau, \;\; 0 \leq t < 1, \;\; 0 < \alpha \leq 1 \tag{8}$$

subject to the initial condition $y(0) = 0$.

In the previous discussion and using the initial condition, the proposed generalized fractional power series solution to Equation (8) has the form

$$y(t) = \sum_{i+j=1}^{\infty} c_{ij} t^{i\alpha + j}. \tag{9}$$

By substituting Equation (9) into Equation (8), the coefficients c_{ij}, $i + j \geq 1$ are determined by equating the coefficients of like powers of t through determining a formal recurrence relation. We have obtained

$$c_{11} = \frac{\alpha+2}{\alpha^2+3\alpha+1}\left[\frac{\alpha(\alpha+1)+1}{\alpha(\alpha+1)} + \sum_{j=2}^{\infty}\frac{j+1}{(\alpha+j)(\alpha+j+1)j!}\right], \tag{10}$$

$$c_{1j} = \frac{j+1}{(\alpha+j)j!} \;\; \text{for} \;\; j = 0, 2, 3, 4, \ldots, \tag{11}$$

and $c_{ij} = 0$ otherwise. Therefore, the exact solution of Equation (8) is

$$y(t) = c_{11}t^{\alpha+1} + \frac{t^{\alpha}}{\alpha} + \sum_{j=2}^{\infty}\frac{j+1}{(\alpha+j)j!}t^{\alpha+j}, \tag{12}$$

with c_{11} as in Equation (10). Particularly with $\alpha = 1$, it can be obtained that the exact solution for the classical version of Equation (8) is

$$y(t) = t^2 + t + \sum_{j=2}^{\infty}\frac{t^{j+1}}{j!} = te^t. \tag{13}$$

Figure 1 illustrates the approximate solutions for $\alpha = 0.25, 0.5, 0.75, 1$ in $I \in [0, 1)$.

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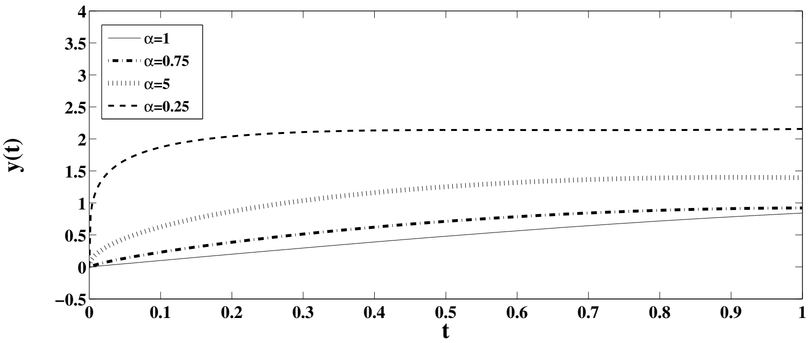

Figure 2. The approximate solution of Example 2 for $\alpha = 0.25, 0.5, 0.75, 1$.

Example 2. *Consider the Volterra integro-differential equation:*

$$T^{\alpha}[y(t)] = 1 - \int_0^t y(\tau)d\tau, \ \ 0 \leq t \leq 1, \ \ 0 \leq \alpha \leq 1, \ \ y(0) = 0. \tag{14}$$

Upon substituting all the relevant quantities into the Equation (14) and collecting powers of t , we have

$$c_{10} = \frac{1}{\alpha}, \tag{15}$$

$$c_{i+1,i} = \frac{1}{\alpha}\left[\frac{(-1)^i}{(\alpha+1)(2\alpha+1)\cdots(i\alpha+i)((i+1)\alpha+i)}\right] \text{ for } i = 1,2,3,\ldots, \tag{16}$$

where $c_{ij} = 0$ otherwise. Then, the exact solution is

$$y(t) = \frac{1}{\alpha}t^{\alpha} + \sum_{i=1}^{\infty} c_{i+1,i}t^{(i+1)\alpha+i}, \tag{17}$$

where $c_{i+1,i}$ satisfies Equation (16). Particularly, we can see the approximate solutions for $\alpha = 1$ which are derived for different values of t. Then, the exact solution in a closed form $y(t) = \sin t$. Figure 2 shows the effect of α on the solution for $\alpha = 0.25, 0.5, 0.75, 1$ in $I \in [0, 1)$.

Example 3. *Consider the nonlinear Fredholm fractional integro-differential equation*

$$T^{\frac{1}{2}}[y(t)] = 2t^{\frac{3}{2}} - t^{\frac{1}{2}} - \frac{t}{1260} + \int_0^1 t\tau[y(\tau)]^4 d\tau \tag{18}$$

subject to the initial condition $y(0) = 0$.

Since the definite integral in Equation (18) completely depends on the variable τ, the solution is spanned by the monomials $\{t, t^{\frac{3}{2}}, t^2\}$. That is

$$y(t) = c_{01}t + c_{11}t^{\frac{3}{2}} + c_{02}t^2 \tag{19}$$

with

$$T^{\frac{1}{2}}[y(t)] = c_{01}t^{\frac{1}{2}} + \frac{3}{2}c_{11}t + 2c_{02}t^{\frac{3}{2}} \tag{20}$$

Substituting all the relevant quantities into the Equation (18) and equating the coefficients of like powers of t from both sides, we obtain $c_{01} = -1, \ c_{02} = 1$, and c_{11} satisfies

$$c_{11}\left(c_{11}^3 - \frac{128}{255}c_{11}^2 + \frac{4}{21}c_{11} - \left(\frac{1024}{20995} + 12\right)\right) = 0. \tag{21}$$

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Subsequently, we have exact solutions in the form $y(t) = t^2 - t + c_{11}t^{\frac{3}{2}}$ where c_{11} satisfies Equation (21).

5. Conclusions

In this paper, we have investigated the analytical solution of a class of nonlinear integro-differential equation based on the GCFPS method. Three examples of our numerical results are presented. From Figures 1 and 2, we see that as α is increasing, the approximate solution is decreasing. The results reveal that the exact solutions are obtained in the form of a rapidly convergent series with an easily computable component. In conclusion, the proposed scheme could be used further in studying identical applications. It can be extended to solve a variety of fractional differential and integral equations in sciences and engineering.

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Abbreviations

The following abbreviations are used in this manuscript:

RFDTM	Differential transform method
GFPS	Generalized fractional power series
GCFPS	Generalized conformable fractional power series

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