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Effective null Raychaudhuri equation

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The effects on Raychaudhuri's equation of an intrinsically discrete or particle nature of spacetime are investigated, through consideration of null congruences emerging from, or converging to, a generic point of spacetime, i.e. in geometric circumstances somehow prototypical of singularity issues. This is done from an effective point of view, meaning through a (continuous) description of spacetime modified to embody the existence of an intrinsic discreteness on the small scale, adding to previous results for non-null congruences.

Various expressions for the effective rate of change of expansion are derived. They in particular provide finite values for the limiting effective expansion and its rate of variation when approaching the focal point; this on top of resulting non vanishing the limiting cross-sectional area itself of the congruence.

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Recently, an effective metric, or qmetric, bitensor q_{ab} has been introduced [1–3], capable of implementing the existence of an intrisic discreteness or particle nature of spacetime at the microscopic scale, while keeping the benefits of a continuous description for calculus [4]. q_{ab} acts like a metric in that it provides a (modified) squared distance between two generic spacelike or timelike separated events P and p (considered as base and field point, respectively), which approaches the squared distance as of ordinary g_{ab} metric when P and p are far away. Contrary to a metric however, the squared distance approaches ϵL^2 (with $\epsilon = 1(-1)$ for spacelike (timelike) separation) in the coincidence limit $p \to P$, with L being an invariant length characterizing the qmetric.

In [5] an extension of this qmetric approach to include the case of null separated events has been considered, and an expression of q_{ab} for them has been provided. This case could be directly relevant for the study of horizons. In case of null geodesics near a focal point, this might be exploited for example to study event horizons at their birth (described e.g. in [6] (in particular Figure 57), [7] Figure 34.7, and [8] Box 12.1). When these geodesics are meant as histories of ultrarelativistic or massless particles, what we are led to is singularity formation issues. In view of this, aim of this note is to investigate how the null Raychaudhuri equation gets modified by intrinsic discreteness of spacetime, as captured by the qmetric, near a focal point.

In [1–3], the qmetric is introduced as something which leads to replace the quadratic distance $\sigma^2(p,P)$ between spacelike/timelike separated events by an effective distance $[\sigma^2]_q = S_L(\sigma^2)$ dependent on the characterizing scale L, subject to the requirements $S_L \to \epsilon L^2$ when $\sigma^2 \to 0$ and $S_L \sim \sigma^2$ when σ^2/L^2 is large, and to an additional request on the form of the effective kernel $[G]_q$ of the d'Alembertian, namely that $[G]_q(\sigma^2) = G(S_L)$ in all maximally symmetric spacetimes. This fixes the expression of $q_{ab}(p,P)$ to the form

$$q_{ab} = A g_{ab} + \epsilon \left(\frac{1}{\alpha} - A\right) t_a t_b, \tag{1}$$

where t^a is the normalized tangent vector ($g_{ab}t^a t^b = \epsilon$; $t_a = g_{ab}t^b$) at p to the geodesics connecting P and p, g_{ab} is considered at p, and α and A are functions of σ^2, given by

$$\alpha = \frac{S_L}{\sigma^2\ {S'_L}^2}, \tag{2}$$

$$A = \frac{S_L}{\sigma^2} \left(\frac{\Delta}{\Delta_S}\right)^{\frac{2}{D-1}}. \tag{3}$$

Here the prime symbol indicates differentiation with respect to σ^2, Δ is van Vleck determinant ([9–12]; see [13–15])

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$$\Delta(p,P) = -\frac{1}{\sqrt{g(p)g(P)}}\det\Big[-\nabla_a^{(p)}\nabla_b^{(P)}\frac{1}{2}\sigma^2(p,P)\Big]$$

($g = \det g_{ab}$), and $\Delta_S(p,P) = \Delta(\tilde p, P)$ with $\tilde p$ being that point on the geodesic through P and p (on the same side of p) with $\sigma^2(\tilde p, P) = S_L(p,P)$.

The extension of this approach to include the null case [5] is done shifting the focus of attention from quadratic distance, which is identically vanishing in this case, to affine parameterization. Exploiting the fact that an affine parameter λ, assigned with a null geodesics γ, is a distance as measured along γ by suitable canonical observers parallel transported along it, the qmetric is introduced as something which leads to replace $\lambda(p,P)$ (having $\lambda(P,P)=0$) with an effective parameterization $[\lambda]_q = \tilde\lambda(\lambda)$, which depends on the characterizing scale L (we omit the explicit indication of this dependence), with the requirements $\tilde\lambda \to L$ when $\lambda \to 0$ and $\tilde\lambda \sim \lambda$ when λ/L is large, and the same additional request on the form of the effective kernel $[G]_q$ of the d'Alembertian as above, specialized to null geodesics, namely what springs for points on null geodesics from being $[G]_q(\sigma^2) = G(S_L)$ in all maximally symmetric spacetimes. This gives, for $q_{ab}(p,P)$ with P and p null separated, the expression

$$q_{ab} = A_\gamma g_{ab} - \Big(\frac{1}{\alpha_\gamma} - A_\gamma\Big) l_{(a}\, m_{b)},$$

with $l^a = \frac{dx^a}{d\lambda}$ and m^a null with $g_{ab}m^a l^b = -2$ considered at p (as well as g_{ab} is), $l_a = g_{ab}l^b$, $m_a = g_{ab}m^b$, and α_γ and A_γ functions of λ given by

$$\alpha_\gamma = \frac{1}{(d\tilde\lambda/d\lambda)^2}, \tag{4}$$

$$A_\gamma = \frac{\tilde\lambda^2}{\lambda^2}\Big(\frac{\Delta}{\Delta_S}\Big)^{\frac{2}{D-2}}\Big(\frac{d\tilde\lambda}{d\lambda}\Big)^{-\frac{2}{D-2}}.$$

Here $\Delta_S(p,P) = \Delta(\tilde p, P)$, where $\tilde p$ is that point on γ (on the same side of p) which has $\lambda(\tilde p, P) = \tilde\lambda$ with $(\partial^a\sigma^2)_{|\tilde p} = \partial^a S_L = 2\tilde\lambda l^a_{|\tilde p}$.

The functions α_γ and A_γ are defined for points on the null geodesic from P and then only on the submanifold Γ consisting of the null congruence of all null geodesics emerging from P (considered as base point). Crucial in the derivation of these expressions, is to consider the d'Alembertian at points of Γ in a form which has no derivations of the vectors tangent to the congruence taken along directions outside Γ [5]. This has been accomplished through the following expression for the d'Alembertian (meant as applied to a generic function $f(\sigma^2)$ in a maximally symmetric spacetime)

$$\Box f = \nabla_a\nabla^a f = \big(4 + 2\lambda\nabla_i l^i\big)\frac{df}{d\sigma^2}$$

($i = 1, ..., D-1$ are indices of components on Γ), i.e. in terms of a quantity, $\nabla_i l^i = \theta$, the expansion of Γ, in which all variations are in Γ. Expressions of $[\nabla_i l^i]_q$ have been then readily obtained as

$$[\nabla_i l^i]_q = \nabla_i\Big(\frac{d\lambda}{d\tilde\lambda}l^i\Big) + \frac{1}{2}\frac{d\lambda}{d\tilde\lambda}q^{bc}l^a\nabla_a q_{bc} \tag{5}$$

$$= \frac{d\lambda}{d\tilde\lambda}\nabla_i l^i - \frac{d\lambda}{d\tilde\lambda}\frac{d}{d\lambda}\ln\frac{d\lambda}{d\tilde\lambda} + \frac{1}{2}(D-2)\frac{d\lambda}{d\tilde\lambda}\frac{d}{d\lambda}\ln A_\gamma, \tag{6}$$

where q^{ab} is the inverse of q_{ab}. These expressions provide the expansion $[\theta]_q$ of the null congruence Γ according to the qmetric. Aim of this brief report, is to discuss what the associated effective null Raychaudhuri equation is and to explore both this and the effective expansion $[\theta]_q$ at coincidence limit $p \to P$. The results we obtain refer to a null congruence emerging from generic P, but can equivalently be read as referring to a null congruence converging to P upon substitution $\lambda \to -\lambda$, $\tilde\lambda \to -\tilde\lambda$ and $L \to -L$.

We begin by noting that, if we use of the expressions for α_γ and A_γ and introduce the quantity

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$$A^*_\gamma = A_\gamma \Big(\frac{d\tilde{\lambda}}{d\lambda}\Big)^{\frac{2}{D-2}} = \frac{\tilde{\lambda}^2}{\lambda^2}\Big(\frac{\Delta}{\Delta_S}\Big)^{\frac{2}{D-2}}, \tag{7}$$

we can recast equation (6) as

$$[\theta]_q = \sqrt{\alpha_\gamma}\Big[\theta + (D-2)\frac{d}{d\lambda}\ln\sqrt{A^*_\gamma}\Big]. \tag{8}$$

From this, considering the derivative of θ according to the qmetric

$$\begin{aligned}\Big[\frac{d\theta}{d\lambda}\Big]_q &= [l^a\nabla_a\theta]_q \\ &= [l^a]_q\ \partial_a[\theta]_q \\ &= \frac{d\lambda}{d\tilde{\lambda}}\ l^a\partial_a[\theta]_q \\ &= \frac{d\lambda}{d\tilde{\lambda}}\ \frac{d}{d\lambda}[\theta]_q \\ &= \frac{d}{d\tilde{\lambda}}[\theta]_q,\end{aligned}$$

we find

$$\begin{aligned}\Big[\frac{d\theta}{d\lambda}\Big]_q &= \alpha_\gamma\frac{d\theta}{d\lambda} + \frac{1}{2\sqrt{\alpha_\gamma}}\ [\theta]_q\ \frac{d\alpha_\gamma}{d\lambda} + (D-2)\ \alpha_\gamma\frac{d^2}{d\lambda^2}\ln\sqrt{A^*_\gamma} \\ &= \alpha_\gamma\frac{d\theta}{d\lambda} + \frac{1}{2}\Big[\theta + (D-2)\frac{d}{d\lambda}\ln\sqrt{A^*_\gamma}\Big]\frac{d\alpha_\gamma}{d\lambda} + (D-2)\ \alpha_\gamma\frac{d^2}{d\lambda^2}\ln\sqrt{A^*_\gamma}.\end{aligned} \tag{9}$$

In the 3rd equality above, use has been made of $[l^a]_q = dx^a/d\tilde{\lambda} = (d\lambda/d\tilde{\lambda})l^a$.

Equation (9) is supposed to be the qmetric rate of change of the expansion for the null congruence Γ. It exhibits quite a close resemblance to the qmetric rate of change of expansion found in [16] for congruences of unit-tangent spacelike/timelike integral curves emerging from P (eq. (22) in that paper), which, when the congruence is specialized to (spacelike/timelike) geodesics (which is the context to which the qmetric (1) refers to), reads

$$\Big[\frac{d\theta}{d\lambda}\Big]_q = \alpha\frac{d\theta}{d\lambda} + \frac{1}{2}\Big[\theta + (D-1)\frac{d}{d\lambda}\ln\sqrt{A}\Big]\frac{d\alpha}{d\lambda} + (D-1)\ \alpha\frac{d^2}{d\lambda^2}\ln\sqrt{A}, \tag{10}$$

where α and A are given in equations (2) and (3). We see that equations (9) and (10) are obtained one from the other through the replacements $(D-2), \alpha_\gamma, A^*_\gamma \leftrightarrow (D-1), \alpha, A$.

Making use of the explicit expressions for α_γ and A^*_γ (equations (4) and (7)), and of the convenient expression

$$\theta = \frac{D-2}{\lambda} - \frac{d}{d\lambda}\ln\Delta \tag{11}$$

relating the expansion and the van Vleck determinant in null congruences ([14]; see also [5]), expressions (8) and (9) of the expansion and of its rate of change can be given the form

$$[\theta]_q = \frac{D-2}{\tilde{\lambda}} - \frac{d}{d\tilde{\lambda}}\ln\Delta_S, \tag{12}$$

$$\Big[\frac{d\theta}{d\lambda}\Big]_q = -\frac{D-2}{\tilde{\lambda}^2} - \frac{d^2}{d\tilde{\lambda}^2}\ln\Delta_S. \tag{13}$$

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In these (exact) expressions, any dependence of $[\theta]_q$ and $[d\theta/d\lambda]_q$ on α_γ and A^*_γ has been translated into a dependence on $\tilde{\lambda}$ and Δ_S. Comparison with equation (11), and its derivative

$$\frac{d\theta}{d\lambda} = -\frac{D-2}{\lambda^2} - \frac{d^2}{d\lambda^2} \ln \Delta, \tag{14}$$

shows that the effective expansion and its effective rate of change at p with $\lambda = \lambda(p, P)$ turn out to be nothing but the expansion and its rate of change evaluated at point $\tilde{p}$ on the same null geodesic through P and p with $\lambda(\tilde{p}, P) = \tilde{\lambda}$. From

$$\frac{d\theta}{d\lambda} = -\frac{\theta^2}{D-2} - \frac{2}{\lambda}\frac{d}{d\lambda} \ln \Delta + \frac{1}{D-2}\Big(\frac{d}{d\lambda} \ln \Delta\Big)^2 - \frac{d^2}{d\lambda^2} \ln \Delta \tag{15}$$

(on using (11) in (14)), accordingly we also get

$$\Big[\frac{d\theta}{d\lambda}\Big]_q = -\frac{{[\theta]_q}^2}{D-2} - \frac{2}{\tilde{\lambda}}\frac{d}{d\tilde{\lambda}} \ln \Delta_S + \frac{1}{D-2}\Big(\frac{d}{d\tilde{\lambda}} \ln \Delta_S\Big)^2 - \frac{d^2}{d\tilde{\lambda}^2} \ln \Delta_S. \tag{16}$$

This fact makes equations (12) and (13), as well as (16), quite useful when going to evaluate $[\theta]_q$ and $[d\theta/d\lambda]_q$ at coincidence limit. We find

$$\begin{aligned} [\theta]_0 &\equiv \lim_{\lambda\to 0} [\theta]_q \\ &= \frac{D-2}{L} - \frac{d}{dL} \ln \Delta_L \\ &= \frac{D-2}{L} - \frac{1}{3}\, L\, (R_{ab} l^a l^b)_{|P} + o\big[L\, (R_{ab} l^a l^b)_{|P}\big] \\ &= \frac{D-2}{L}\Big[1 - \frac{1}{3(D-2)}\delta + o(\delta)\Big] \end{aligned} \tag{17}$$

and

$$\begin{aligned} \Big[\frac{d\theta}{d\lambda}\Big]_0 &\equiv \lim_{\lambda\to 0} \Big[\frac{d\theta}{d\lambda}\Big]_q \\ &= -\frac{D-2}{L^2} - \frac{d^2}{dL^2} \ln \Delta_L \\ &= \frac{d}{dL} \lim_{\lambda\to 0} [\theta]_q \\ &= -\frac{D-2}{L^2} - \frac{1}{3} (R_{ab} l^a l^b)_{|P} + o\big[(R_{ab} l^a l^b)_{|P}\big] \\ &= -\frac{D-2}{L^2}\Big[1 + \frac{1}{3(D-2)}\delta + o(\delta)\Big], \end{aligned} \tag{18}$$

as well as

$$\Big[\frac{d\theta}{d\lambda}\Big]_0 = -\frac{{[\theta]_0}^2}{D-2} - \frac{2}{L}\frac{d}{dL} \ln \Delta_L + \frac{1}{D-2}\Big(\frac{d}{dL} \ln \Delta_L\Big)^2 - \frac{d^2}{dL^2} \ln \Delta_L, \tag{19}$$

where Δ_L is defined as $\Delta_L = \Delta(\bar{p}, P)$ with $\bar{p}$ on γ such that $\lambda(\bar{p}, P) = L$, we used of the expansion ([11] and [13–15])

$$\Delta(p, P) = 1 + \frac{1}{6}\lambda^2 (R_{ab} l^a l^b)_{|P} + o\Big[\lambda^2 (R_{ab} l^a l^b)_{|P}\Big]$$

of the van Vleck determinant, and we put $\delta \equiv L^2\, (R_{ab} l^a l^b)_{|P}$ with the expansions useful when $\delta \ll 1$; this sets a maximum allowed value for $(R_{ab} l^a l^b)_{|P}$. We see that, whereas classically, i.e. according to g_{ab}, both θ and $d\theta/d\lambda$

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diverge when $p \to P$ (being $\theta \sim \frac{D-2}{\lambda}$ and $\frac{d\theta}{d\lambda} \sim -\frac{D-2}{\lambda^2}$ for $\lambda \to 0$), according to the qmetric they both remain finite, the limiting values of $[\theta]_q$ and $[d\theta/d\lambda]_q$ turning out to be the expressions for θ and $d\theta/d\lambda$ computed at $\lambda = L$.

This adds, and corresponds, to the non-vanishing of the effective cross-sectional $(D-2)$-dim area of Γ in the coincidence limit $p \to P$. Indeed, from

$$\begin{aligned} \left[d^{D-1}V\right]_q &= \Big(\frac{\tilde\lambda}{\lambda}\Big)^{D-2} \frac{\Delta}{\Delta_S}\, d^{D-2}\mathcal{A}\, d\lambda \\ &\equiv [d^{D-2}\mathcal{A}]_q\, d\lambda \end{aligned} \tag{20}$$

([5] equation (32), on using the explicit expression for A_γ), where $\left[d^{D-1}V\right]_q$ is the effective volume element and $[d^{D-2}\mathcal{A}]_q$ the effective cross-sectional area of the volume element $d^{D-1}V = d^{D-2}\mathcal{A}\, d\lambda$ of Γ, we get

$$\begin{aligned} [d^{D-2}\mathcal{A}]_0 &\equiv \lim_{\lambda\to 0}[d^{D-2}\mathcal{A}]_q \\ &= L^{D-2}\frac{1}{\Delta_L}(d\chi)^{D-2}, \end{aligned} \tag{21}$$

where we consider as cross-sectional area element a $(D-2)$-cube of edge $\lambda d\chi$. This completes what we were searching for.

If we start now from the classical Raychaudhuri equation as applied to our (affinely parameterized) null congruence Γ, written as

$$\frac{d\theta}{d\lambda} = -\frac{1}{D-2}\,\theta^2 - \sigma_{ab}\,\sigma^{ab} - R_{ab}\,l^a l^b$$

(σ_{ab} is shear; the twist is vanishing due to surface-orthogonality), and use of (15), we get

$$\sigma_{ab}\,\sigma^{ab} + R_{ab}\,l^a l^b = \frac{d^2}{d\lambda^2}\ln\Delta + \frac{2}{\lambda}\,\frac{d}{d\lambda}\ln\Delta - \frac{1}{D-2}\Big(\frac{d}{d\lambda}\ln\Delta\Big)^2, \tag{22}$$

and, from (16),

$$[\sigma_{ab}\sigma^{ab}]_q + [R_{ab}l^a l^b]_q = \frac{d^2}{d\tilde\lambda^2}\ln\Delta_S + \frac{2}{\tilde\lambda}\,\frac{d}{d\tilde\lambda}\ln\Delta_S - \frac{1}{D-2}\Big(\frac{d}{d\tilde\lambda}\ln\Delta_S\Big)^2, \tag{23}$$

with its coincidence limit

$$\lim_{\lambda\to 0}\Big([\sigma_{ab}\sigma^{ab}]_q + [R_{ab}l^a l^b]_q\Big) = \frac{d^2}{dL^2}\ln\Delta_L + \frac{2}{L}\,\frac{d}{dL}\ln\Delta_L - \frac{1}{D-2}\Big(\frac{d}{dL}\ln\Delta_L\Big)^2. \tag{24}$$

In particular, we can read here the expression for $[R_{ab}l^a l^b]_q$ and its coincidence limit in the shearless case.

To conclude, we briefly comment on a consequence it appears we can extract from the above regarding singularities. Let us consider the spacetime associated to a spherical layer of photons, assumed to be pointlike particles, undergoing spherically symmetric collapse towards a focal point P (we could consider massive particles as well, but we choose photons to stick to the results presented above). In our picture, we can look at this as a spherically symmetric congruence of null geodesics emerging from P and tracked backwards in time, with the further crucial assumption that these geodesics are actual histories of photons, which are then considered as source of spacetime curvature. For these circumstances, the classical description tells us that a singularity unavoidably develops (this is a sort of prototypical case of singularity formation in general relativity). Indeed, photons reach P in a finite variation $\Delta\lambda$ of affine parameter, with diverging energy densities $\rho = E/\mathcal{A}$ (energy per unit transverse area). This means that in a finite $\Delta\lambda$ photon histories do cease to exist while some components of the Riemann tensor w.r.t. a basis parallelly propagated along the geodesics grow without limit, i.e. we have incomplete geodesics corresponding to a parallelly propagated singularity curvature [6].

According to qmetric description, in that same $\Delta\lambda$ photon histories keep staying away from P (since the spatial distance from the actual location p of the photon and P according to any canonical observer at P remains not lower than L) and energy density reaches a maximum insurmountable value $[\rho]_0 = \lim_{\lambda\to 0}[\rho]_q = E/[\mathcal{A}]_0$. Then

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photon histories do not cease to exist after $\Delta\lambda$ and, using the density $[\rho]_0$ as source of Einstein's equations, no components of Riemann in a parallelly propagated basis are any longer diverging. In this sense we can say then that the microstructure of spacetime, as captured by qmetric, removes a classically-blatant curvature singularity.

Assuming L is as small as orders of Planck's length the density $[\rho]_0$ actually challenges the domain of validity of Einstein's equations and the notion of spacetime, as can be envisaged by computing (equation (21))

$$[\mathcal{A}]_0 = 4\pi L^2 \frac{1}{\Delta_L}, \tag{25}$$

where $\Delta_L = \Delta(\bar{p}, P)$ is finite in spite of being the classical metric singular at P when $\lambda = 0$ ($\Delta(\bar{p}, P)$ is indeed computed for the metric configuration associated with $\lambda(\bar{p}, P) = L$, that is, clearly, with $\lambda \neq 0$). The qmetric thus embodies that, after $\Delta\lambda$, photons' spacetime, instead of becoming singular, gets changing its nature from continuous to discrete and calls for new equations, different from Einstein's ones, to rule its evolution.

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