DEGENERATE DAEHEE NUMBERS OF THE THIRD KIND

SUNG-SOO PYO, TAEKYUN KIM, AND SEOG-HOON RIM

Abstract. In this paper, we define new kind of Daehee numbers, the degenerate Daehee numbers of the third kind, using degenerate log function as generating function. We obtain some identities for the degenerate Daehee numbers of the third kind associated with the Daehee, degenerate Daehee and degenerate Daehee numbers of the second kind. In addition, we derive a differential equation associated with the degenerate log function. We deduce some identities from the differential equation.

Keywords: degenerate log function; degenerate Daehee numbers of the third kind; nonlinear differential equation.

1. Introduction

After Carlitz [1, 2], many mathematicians have studied degenerate functions and numbers (see [3, 9, 15, 16, 17, 18, 19, 23, 24]). They mainly used \((1 + \lambda t)^{\frac{1}{\lambda}}\) instead of \(e^t\) to degenerate polynomials and numbers. In [17], T. Kim and D.S. Kim called \((1 + \lambda t)^{\frac{1}{\lambda}}\) the degenerate exponential function and express it as \(e^\lambda t\). They also presented the degenerate gamma function and degenerate Laplace transformation using \(e^\lambda t\).

In [25], authors introduced four kind degenerate version of Cauchy numbers. In the degenerate Cauchy numbers of the first and second kind, \((1 + t)^{\frac{1}{\lambda}}\) was used instead of \(e^t\). We call this degenerate based on the exponential sense.

In accordance with exponential sense, \(\log(1 + \lambda t)^{\frac{1}{\lambda}}\) can be used for \(t\) to study degenerate numbers and polynomials. In this sense, in [15], T. Kim introduced the degenerate Cauchy numbers by the generating function to be

\[
\log e^{\lambda t}(1 + \log(1 + \lambda t)^{\frac{1}{\lambda}}) = \log(1 + t)^{\frac{1}{\lambda}}.
\]

(1)

It is natural to think a degenerate log function as the inverse function of the degenerate exponential function. The degenerate log function, denoted by \(\log_\lambda(t)\), is defined by the generating function to be

\[
\log_\lambda(t) = \frac{1}{\lambda}(t^\lambda - 1) \quad (\text{see [19]}).
\]

(2)

The Cauchy numbers or the second kind Bernoulli numbers, denoted by \(C_n\), are defined by the generating function to be

\[
\frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!} \quad (\text{see [15, 18, 21, 25]}).
\]
Recently, in [18], T. Kim introduced the degenerate Cauchy numbers by the generating function to be
\[ \frac{e^{\log(1+t)} - 1}{\log(1+t)} = \lambda \left( e^{\frac{x}{\lambda(1+t) - 1}} - 1 \right) = \sum_{n=0}^{\infty} C_{n}(\lambda) \frac{t^{n}}{n!}. \] (3)

If \( \lambda \) goes to 0, then \( C_{n}(\lambda) \) converges to \( C_{n} \). In this case, the author used \( \log(1+t) \) instead of \( \log t \) for degenerating. We call this degenerate based on the log sense.

As is well known, the Bernoulli numbers, denoted by \( B_{n} \), are defined by the generating function:
\[ t = \sum_{n=0}^{\infty} B_{n} \frac{t^{n}}{n!}. \] (4)

The Bernoulli numbers, which started with a study on the sum of the power series, has many relationships with other special numbers [2, 4, 5, 3, 6, 7, 13, 21, 22, 27]. In [1], the degenerate Bernoulli numbers are presented as follows:
\[ \frac{t}{e^{t} - 1} = \sum_{n=0}^{\infty} B_{n} \frac{t^{n}}{n!}. \] (5)

The Daehee numbers, denoted by \( D_{n} \), are defined by the generating function:
\[ \log(1+t) = \sum_{n=0}^{\infty} D_{n} \frac{t^{n}}{n!}, \] (6)

Although the Daehee numbers are readily available as \( D_{n} = (-1)^{n} \frac{n!}{n+1} \), the Daehee numbers have many interesting relationships with other special numbers. For example, the following show relationships between the Daehee numbers and the Bernoulli numbers.
\[ D_{n} = \sum_{m=0}^{n} B_{m} S_{1}(n, m), \] (7)
\[ B_{m} = \sum_{n=0}^{m} D_{n} S_{2}(m, n), \] (8)

where \( S_{1}(n, m) \) and \( S_{2}(n, m) \) are the Stirling numbers of the first kind and the second kind respectively (see [11]).

The degenerate Daehee numbers, denoted by \( D_{\lambda}(n) \), are introduced as follows (see [9]):
\[ \frac{\lambda \log(1+t) + \frac{x}{1+\lambda \log(1+t)}}{\log(1+\lambda t)} = \sum_{n=0}^{\infty} D_{\lambda}(n) \frac{t^{n}}{n!}. \]

If \( \lambda \) goes to 0, then \( D_{\lambda}(n) \) converges to \( D_{n} \). This degenerate Daehee numbers \( D_{\lambda}(n) \) are degenerate based on the exponential sense. Recently, D. S. Kim and et al. presented the degenerate Daehee polynomials and numbers of the second kind as follows [12].
\[ \frac{\log(1+t)}{(1+\lambda \log(1+t))} = \sum_{n=0}^{\infty} D_{\lambda, 2}(n, x) \frac{t^{n}}{n!}. \] (9)

When \( x = 0 \), \( D_{\lambda, 2}(n) = D_{\lambda, 2}(n, 0) \) are called the degenerate Daehee numbers of the second kind. These degenerate numbers are also based on the exponential sense.
Numbers

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**Table 1.** Three kinds of degenerate Daehee numbers

It is natural to think about degenerate Daehee numbers based on the log sense. We define the degenerate Daehee numbers of the third kind, denoted by \( D_{\lambda,3}(n) \), as follows.

\[
\frac{\log(1+t)}{t} = \frac{1}{\lambda} \left( (1+t)^\lambda - 1 \right) = \sum_{n=0}^{\infty} D_{\lambda,3}(n) \frac{t^n}{n!},
\]  

(10)

We note that

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} \left( (1+t)^\lambda - 1 \right) = \frac{\log(1+t)}{t}.
\]  

(11)

From (11), we know that \( \lim_{\lambda \to 0} D_{\lambda,3}(n) = D_n \) for each \( n \).

The Table 1 summarizes the three types of degenerate Daehee numbers.

In this paper, we define the degenerate Daehee numbers based on the log sense. And we obtain some identities which are connected with the Daehee, the degenerate Daehee and the degenerate Daehee numbers of the second kind. Additionally, we deduce a differential equation using the degenerate log function, and we derive some identities related to the degenerate Daehee numbers from this differential equation.

### 2. Degenerate Daehee numbers of the third kind

From now on, for any real \( x \) and non negative integer \( n \), we denote \( (x)_n \) for falling factorial \( (x)_n = x(x-1)(x-2) \cdots (x-n+1) \) and \( (x)_0 = 1 \). We use \( S_1(m,n) \) and \( S_2(m,n) \) to denote the Stirling number of the first kind and the second kind respectively.

From the definition of the degenerate Daehee numbers of the third kind, we get

\[
\frac{1}{\lambda} \left( (1+t)^\lambda - 1 \right) = \frac{1}{\lambda} \sum_{n=0}^{\infty} (\lambda)_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} (\lambda)_{n+1} \frac{t^n}{n!}.
\]  

(12)

This (12) yields the following.

\[
D_{\lambda,3}(n) = \frac{(\lambda)_{n+1}}{\lambda(n+1)} = \frac{\lambda(\lambda-1)(\lambda-2) \cdots (\lambda-n)}{\lambda(n+1)}.
\]  

(13)

From (13), it is easy to show that \( \lim_{\lambda \to 0} D_{\lambda,3}(n) = D_n \).

Now let us investigate the relationship between the Daehee numbers and the degenerate Daehee numbers.
\[
\frac{1}{\lambda} ((1 + t)^\lambda - 1) = \frac{\log(e^{\frac{1}{\lambda}((1 + t)^\lambda - 1)} - 1 + 1)}{t} = \sum_{l=0}^{\infty} D_l \left( e^{\frac{1}{\lambda}((1 + t)^\lambda - 1)} - 1 \right)^l
\]

\[
= \sum_{l=0}^{\infty} D_l \sum_{m=l}^{\infty} S_2(m, l) \frac{1}{m!} \lambda^m ((1 + t)^\lambda - 1)^m
\]

\[
= \sum_{l=0}^{\infty} D_l \sum_{m=l}^{\infty} S_2(m, l) \frac{1}{m!} \lambda^m \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \sum_{n=0}^{\infty} (k\lambda)_n \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^{m} \binom{m}{k} (-1)^{m-k} D_l S_2(m, l)(k\lambda)_n \frac{t^n}{m!\lambda^m n!}
\]

(14)

The equation (14) yields a relationship between the Daehee numbers and the degenerate Daehee numbers.

**Theorem 1.** For any nonnegative integer \(n\),
\[
D_{\lambda,\beta}(n) = \sum_{m=0}^{\infty} \sum_{l=0}^{m} \sum_{k=0}^{l} \binom{m}{k} (-1)^{m-k} D_l S_2(m, l)(k\lambda)_n \frac{t^n}{m!\lambda^m n!}.
\]

From the definition of the Daehee numbers (6) and the degenerate Daehee numbers (10), we get the following.

\[
\frac{\log(1 + t)}{t} = \frac{1}{x} \left\{ \left( 1 + \left( \lambda \log(1 + t) + 1 \right)^\frac{1}{\lambda} - 1 \right)^\lambda - 1 \right\}
\]

\[
= \sum_{l=0}^{\infty} D_{\lambda,\beta}(l) \frac{\left( \lambda \log(1 + t) + 1 \right)^\frac{1}{\lambda} - 1}{l!}
\]

(15)

From the equation (15), we have a kind of inversion formula for Theorem 1.

**Theorem 2.** For any nonnegative integer \(n\),
\[
D_n = \sum_{l=0}^{\infty} \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{m}{k} \lambda^k D_{\lambda,\beta}(l) S_1(n,k) \frac{t^n}{l!}
\]

In [15], the degenerate Stirling numbers of the first kind, denoted by \(S_{1,\lambda}(n,k)\), are introduced by
\[
\frac{1}{k!} (\log\lambda(1 + t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}.
\]

(16)
The equation (7) notice that the Daehee numbers can be represented by the Bernoulli and Stirling numbers of the first kind. The next equation is a degenerate version of the equation (7).

\[
\log_\lambda(1 + t) = \frac{t}{\lambda} \frac{((1 + t)^\lambda - 1)}{((1 + \lambda(\frac{t}{\lambda}((1 + t)^\lambda - 1)))^{\frac{1}{\lambda}} - 1}
\]

\[
= \sum_{l=0}^{\infty} \beta_{l,\lambda} \left(\frac{t}{\lambda} \left(1 + \lambda(\frac{t}{\lambda}((1 + t)^\lambda - 1))\right)^l\right)
\]

\[
= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \beta_{l,\lambda} S_{1,\lambda}(n, l) \frac{t^n}{n!}
\]

By comparing the coefficients in the equation (17), we can obtain the following theorem.

**Theorem 3.** For any nonnegative integer \( n \),

\[
D_{\lambda,3}(n) = \sum_{l=0}^{n} \beta_{l,\lambda} S_{1,\lambda}(n, l).
\]

In [15], the degenerate Stirling numbers of the second kind were defined by the generating function

\[
\frac{1}{k!} (e^t - 1)^k = \left(1 + \lambda(\frac{t}{\lambda} - 1)\right)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!}.
\]

Using (18) and the definition of the degenerate Bernoulli numbers and the degenerate Daehee numbers, we get the following.

\[
\frac{t}{(1 + \lambda)^{\frac{1}{\lambda}} - 1} = \frac{1}{\lambda} \left(1 + \lambda(\frac{1}{\lambda} - 1)\right)\left((1 + \lambda(\frac{t}{\lambda} - 1))^{\frac{1}{\lambda}} - 1\right)
\]

\[
= \sum_{l=0}^{\infty} D_{\lambda,3}(l) \frac{1}{l!} \left((1 + \lambda(\frac{t}{\lambda} - 1))^{\frac{1}{\lambda}} - 1\right)^l
\]

\[
= \sum_{n=0}^{\infty} \sum_{l=0}^{n} D_{\lambda,3}(l) S_{2,\lambda}(n, l) \frac{t^n}{n!}
\]

The equation (19) give us an inversion formula of Theorem 3 which is a degenerate version of the equation (8).

**Theorem 4.** For any nonnegative integer \( n \),

\[
\beta_{n,\lambda} = \sum_{l=0}^{n} D_{\lambda,3}(l) S_{2,\lambda}(n, l).
\]
Now let us observe a relation between the degenerate Daehee numbers of the third kind and the Bernoulli numbers.

\[
\frac{\log(1 + t)}{t} = \frac{e^{\log(1 + t)} - 1}{t} = \sum_{l=0}^{\infty} B_l \frac{(\log(1 + t))^l}{l!} \sum_{m=1}^{\infty} \frac{1}{t^m} \frac{(\log(1 + t))^m}{m!}
\]

\[
= \sum_{l=0}^{\infty} B_l \sum_{k=l}^{\infty} \frac{S_{1,\lambda}(k, l)}{k!} \sum_{m=1}^{\infty} \frac{1}{t^m} \frac{S_{1,\lambda}(n, m)}{m!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{l=0}^{n-k} \binom{n}{k} B_l S_{1,\lambda}(n - k, l) \frac{S_{1,\lambda}(n + 1, m)}{n + 1} \right) \frac{t^n}{n!}. \tag{20}
\]

Obtain the following theorem from the definition of the degenerate Daehee numbers of the third kind (10) and from the previous (20).

**Theorem 5.** For any nonnegative integer \(n\),

\[D_{\lambda,3}(l) = \sum_{n=0}^{\infty} \sum_{l=0}^{n-k} \sum_{m=1}^{n} \frac{B_l S_{1,\lambda}(n - k, l) S_{1,\lambda}(n + 1, m)}{n + 1} \frac{t^n}{n!}.\]

Substituting \(e^{\log(1 + t)} - 1\) by \(t\) in the definition of the second kind (9), then the left side of (9) becomes

\[
\frac{\log(1 + \log(1 + t))}{t} = \frac{\log(1 + t)}{t}, \tag{21}
\]

and the right side becomes

\[
\sum_{k=0}^{\infty} D_{\lambda,2}(k) \frac{(\log(1 + t))^k}{k!} = \sum_{k=0}^{\infty} D_{\lambda,2}(k) \sum_{l=0}^{\infty} S_{2,\lambda}(l, k) \frac{(\log(1 + t))^l}{l!}
\]

\[
= \sum_{k=0}^{\infty} D_{\lambda,2}(k) \sum_{l=0}^{\infty} S_{2,\lambda}(l, k) \sum_{n=l}^{\infty} \frac{S_{1,\lambda}(n, l) t^n}{n!} \frac{t^n}{n!} \tag{22}
\]

Obtain the following theorem by comparing the coefficients of the equation (21) with the equation (22).

**Theorem 6.** For any nonnegative integer \(n\),

\[D_{\lambda,3}(n) = \sum_{l=0}^{n} \sum_{k=0}^{l} D_{\lambda,2}(k) S_{2,\lambda}(l, k) S_{1,\lambda}(n, l).\]

To represent \(D_{\lambda,3}(n)\) as \(D_{\lambda,2}(n)\), substituting \(e^{\lambda(\log(1 + t))} - 1 = (1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1\) by \(t\) in the definition of the degenerate Daehee numbers of the third kind (10), then the left side becomes
\[
\frac{\log(1 + ((1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1))}{(1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1} = \frac{\frac{1}{\lambda}((1 + ((1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1))^{\lambda} - 1)}{(1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1}
\]
and the right side becomes
\[
\sum_{k=0}^{\infty} D_{\lambda,3}(k) \left( e_{\lambda}(\log(1 + t)) - 1 \right)^k
= \sum_{k=0}^{\infty} D_{\lambda,3}(k) \sum_{l=k}^{\infty} S_{2,\lambda}(l, k) \frac{(\log(1 + t))^l}{l!}
= \sum_{k=0}^{\infty} D_{\lambda,3}(k) \sum_{l=k}^{\infty} S_{2,\lambda}(l, k) \sum_{n=l}^{\infty} S_{1}(n, l) \frac{t^n}{n!}.
\]

From (23) and (24), we obtain an inversion identity of Theorem 6.

**Theorem 7.** For any nonnegative integer \(n\),
\[
D_{\lambda,2}(n) = \sum_{l=0}^{n} \sum_{k=0}^{l} D_{\lambda,3}(k) S_{2,\lambda}(l, k) S_{1}(n, l).
\]

3. Differential equations arising from the generating function of degenerate Dahee numbers

From now on, we use \(F = F(t)\) to denote the degenerate log function:
\[
F(t) = \log_{\lambda}(1 + t) = \frac{1}{\lambda}((1 + t)^{\lambda} - 1),
\]
and for a natural number \(N\), \(F^{(N)}\) to denote the \(N\)-th derivative of \(F\), that is,
\[
F^{(0)} = F(t), \quad F^{(N)} = \frac{d}{dt} F^{(N-1)}.
\]
Differentiating the two sides of the equation (25) results in the following:
\[
\frac{d}{dt} F(t) = \frac{d}{dt} \log_{\lambda}(1 + t) = (1 + t)^{\lambda-1}
= \frac{1}{1 + t} \left\{ \lambda (1 + t)^{\lambda} - 1 + 1 \right\}
= \frac{\lambda}{1 + t} (1 + t)^{\lambda} - 1 + \frac{1}{1 + t}
= \frac{\lambda}{1 + t} F(t) + \frac{1}{1 + t}.
\]
Further differentiate of the equation (26) yields
\[
\frac{d^2}{dt^2} F(t) = -\frac{\lambda}{(1+t)^2} F(t) + \frac{\lambda}{1+t} F'(t) - \frac{1}{(1+t)^2} \\
= -\frac{\lambda}{(1+t)^2} F(t) + \frac{\lambda}{1+t} \left\{ \frac{\lambda}{1+t} F(t) + \frac{1}{1+t} \right\} - \frac{1}{(1+t)^2} \\
= -\frac{\lambda(\lambda - 1)}{(1+t)^2} F(t) + \frac{\lambda - 1}{(1+t)^2}.
\]

(27)

From the observation (26) and (27), we assume that
\[
F(N) = \frac{(\lambda)_N}{(1+t)^N} F + \frac{(\lambda)_N}{\lambda(1+t)^N}.
\]

(28)

Let us take differentiate both sides of the (28), then we have
\[
F^{(N+1)} = \frac{-N}{(1+t)^{N+1}} (\lambda)_N F + \frac{1}{(1+t)^N} (\lambda)_N F^{(1)} + \frac{-N}{\lambda(1+t)^{N+1}} (\lambda)_N \\
= \frac{(\lambda)_N}{(1+t)^{N+1}} (\lambda - N) F + \frac{(\lambda)_N}{\lambda(1+t)^{N+1}} (\lambda - N) \\
= \frac{(\lambda)_{N+1}}{(1+t)^{N+1}} F + \frac{(\lambda)_{N+1}}{\lambda(1+t)^{N+1}}.
\]

(29)

From (28) and (29), mathematical induction gives us the following theorem.

**Theorem 8.** For any positive integer \(N\), the differential equation
\[
F^{(N)} = \frac{(\lambda)_N}{(1+t)^N} F + \frac{(\lambda)_N}{\lambda(1+t)^N}
\]

(30)

has a solution
\[
F(t) = \log_{\lambda}(1+t) = \frac{1}{\lambda}((1+t)^{\lambda} - 1).
\]

We note that
\[
F^{(N)} = \frac{d^N}{dt^N} \log_{\lambda}(1+t) = \frac{d^N}{dt^N} \left( \frac{\log_{\lambda}(1+t)}{t} \cdot t \right) \\
= \frac{d^N}{dt^N} \left( \sum_{n=1}^{\infty} nD_{\lambda,3}(n-1) \frac{t^n}{n!} \right) \\
= \sum_{n=0}^{\infty} (n+N)D_{\lambda,3}(n+N-1) \frac{t^n}{m!}
\]

(31)

On the other hand
\[
\frac{1}{(1+t)^N} F = \sum_{l=0}^{\infty} \binom{N+l-1}{l} (-1)^l t^l (\log_{\lambda}(1+t)) \\
= \sum_{l=0}^{\infty} \binom{N+l-1}{l} (-1)^l t^l \sum_{m=1}^{\infty} \frac{1}{\lambda(\lambda)_m} \frac{t^m}{m!} \\
= \sum_{n=1}^{\infty} \sum_{m=1}^{n} \frac{1}{\lambda(\lambda)_m} \binom{N+n+m-1}{n} \frac{t^n}{m!}
\]

(32)

and
\[
\frac{(\lambda)^N}{\lambda (1+t)^N} = \frac{(\lambda)^N}{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \binom{N+n-1}{n} t^n
= \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda)^N}{n!} \frac{(N+n-1)}{n} t^n.
\] (33)

As a result (28), by comparing the coefficients of (29), (30) and (31), we get the following identity.

**Theorem 9.** For any nonnegative integer \( m \) and \( N \),
\[
(n + N)D_{\lambda,3}(n + N - 1) = \left( \sum_{m=1}^{n} \frac{1}{\lambda} (\lambda)_m \binom{N+n-m-1}{N-1} \binom{n}{m} (n-m)! + (-1)^n \frac{(\lambda)^N}{\lambda} \right) n! \quad \text{if } n \geq 1
= \begin{cases} (-1)^n (\lambda)^N n! & \text{if } n = 0. 
\end{cases}
\]

In the other viewpoint of the right side of (28), we get
\[
\frac{(\lambda)^N}{(1+t)^N} F = \frac{(\lambda)^N}{\lambda} \left( \frac{(\lambda-1)^{N-1}}{(1+t)^N} (\lambda F + 1) - \frac{(\lambda-1)^N}{(1+t)^{N-1}} \left( \frac{1}{\lambda^N} ((1+t)^\lambda - 1) + 1 \right) \right) = \frac{(\lambda-1)^{N-1}}{(1+t)^N} \left( \frac{1}{\lambda^N} ((1+t)^\lambda - 1) + 1 \right)
\]
(34)

The equation (29) and (34) yield the following.

**Theorem 10.** For nonnegative integer \( n \) and \( N \),
\[
(n + N)D_{\lambda,3}(n + N - 1) = (\lambda - 1)^{N-1} (\lambda - N)_n.
\]

4. Results and discussion

In this paper, we have studied the degenerate Dahee numbers of the third kind. To define degenerate function, we use degenerate log function \( \log_\lambda(t) = \frac{1}{\lambda} ((1+t)^\lambda - 1) \). After we defined the degenerate Dahee numbers of the third kind, we obtained two relations between the Dahee numbers and the degenerate Dahee numbers of the second kind with Stirling numbers, Theorem 1 and Theorem 2. After that, we have two relationships between the degenerate Dahee numbers of the third kind and the degenerate Bernoulli numbers, which are dual relations between the Dahee numbers and the Bernoulli numbers, Theorem 3 and Theorem 4. We obtained some relations between the degenerate Dahee numbers of the second kind and the third kind, Theorem 6 and Theorem 7.

In section 3, we derive a differential equation from the degenerate log function, Theorem 6. From the differential equation in the Theorem 6, we deduce some identities for the degenerate Dahee numbers of the second kind.
5. Conclusion

In [1, 2], L. Carlitz considered the degenerate exponential function. By using this degenerate exponential function, he studied the degenerate Bernoulli numbers and polynomials which are given by the generating function. In this view points, we consider the inverse function of Carlitz’s degenerate exponential function which is called the degenerate logarithmic function. From our degenerate logarithmic function, we derive several identities of special numbers. Through our results, we are able to see that degenerate log function is a useful tool for study of special number theory.

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