

DEGENERATE DAEHEE NUMBERS OF THE THIRD KIND

SUNG-SOO PYO, TAEKYUN KIM, AND SEOG-HOON RIM

ABSTRACT. In this paper, we define new kind of Daehee numbers, the degenerate Daehee numbers of the third kind, using degenerate log function as generating function. We obtain some identities for the degenerate Daehee numbers of the third kind associated with the Daehee, degenerate Daehee and degenerate Daehee numbers of the second kind. In addition, we derive a differential equation associated with the degenerate log function. We deduce some identities from the differential equation.

Keywords: degenerate log function; degenerate Daehee numbers of the third kind; nonlinear differential equation.

1. Introduction

After Carlitz [1, 2], many mathematicians have studied degenerate functions and numbers (see [3, 9, 15, 16, 17, 18, 19, 23, 24]). They mainly used $(1 + \lambda t)^{\frac{1}{\lambda}}$ instead of e^t to degenerate polynomials and numbers. In [17], T. Kim and D.S. Kim called $(1 + \lambda t)^{\frac{1}{\lambda}}$ the degenerate exponential function and express it as e_{λ}^t . They also presented the degenerate gamma function and degenerate Laplace transformation using e_{λ}^t . In [25], authors introduced four kind degenerate version of Cauchy numbers. In the degenerate Cauchy numbers of the first and second kind, $(1 + \lambda t)^{\frac{1}{\lambda}}$ was used instead of e^t . We call this *degenerate based on the exponential sense*.

In accordance with exponential sense, $\log(1 + \lambda t)^{\frac{1}{\lambda}}$ can be used for t to study degenerate numbers and polynomials. In this sense, in [15], T. Kim introduced the degenerate Cauchy numbers by the generating function to be

$$\frac{\log e_{\lambda}(t) - 1}{\log(1 + \log e_{\lambda}(t))} = \frac{\log(1 + \lambda t)^{\frac{1}{\lambda}} - 1}{\log(1 + \log(1 + \lambda t)^{\frac{1}{\lambda}})} = \sum_{n=0}^{\infty} C_{\lambda}^{*}(n) \frac{t^n}{n!}. \quad (1)$$

It is natural to think a degenerate log function as the inverse function of the degenerate exponential function. The degenerate log function, denoted by $\log_{\lambda}(t)$, is defined by the generating function to be

$$\log_{\lambda}(t) = \frac{1}{\lambda}(t^{\lambda} - 1) \text{ (see [19])}. \quad (2)$$

The Cauchy numbers or the second kind Bernoulli numbers, denoted by C_n , are defined by the generating function to be

$$\frac{t}{\log(1 + t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!} \text{ (see [15, 18, 21, 25])}.$$

Recently, in [18], T. Kim introduced the degenerate Cauchy numbers by the generating function to be

$$\frac{e^{\log_\lambda(1+t)} - 1}{\log_\lambda(1+t)} = \frac{\lambda \left(e^{\frac{1}{\lambda}((1+t)^\lambda - 1)} - 1 \right)}{(1+t)^\lambda - 1} = \sum_{n=0}^{\infty} C_\lambda(n) \frac{t^n}{n!}. \quad (3)$$

If λ goes to 0, then $C_n(\lambda)$ converges to C_n . In this case, author used $\log_\lambda(t)$ instead of $\log t$ for degenerating. We call this *degenerate based on the log sense*.

As is well known, the Bernoulli numbers, denoted by B_n , are defined by the generating function :

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \quad (4)$$

The Bernoulli numbers, which started with a study on the sum of the power series, has many relationships with other special numbers [2, 4, 5, 3, 6, 7, 13, 21, 22, 27]. In [1], the degenerate Bernoulli numbers are presented as follows :

$$\frac{t}{e_\lambda(t) - 1} = \frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} = \sum_{n=0}^{\infty} \beta_{n,\lambda} \frac{t^n}{n!}. \quad (5)$$

The Daehee numbers, denoted by D_n , are defined by the generating function

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \quad (\text{see [4, 5, 8, 9, 22, 23, 24]}). \quad (6)$$

Although the Daehee numbers are readily available as $D_n = (-1)^n \frac{n!}{n+1}$, the Daehee numbers have many interesting relationships with other special numbers. For example, the following show relationships between the Daehee numbers and the Bernoulli numbers.

$$D_n = \sum_{m=0}^n B_m S_1(n, m), \quad (7)$$

$$B_m = \sum_{n=0}^m D_n S_2(m, n), \quad (8)$$

where $S_1(n, m)$ and $S_2(n, m)$ are the the Stirling numbers of the first kind and the second kind respectively(see [11]).

The *degenerate Daehee numbers*, denoted by $D_\lambda(n)$, are introduced as follows(see [9]):

$$\frac{\lambda \log(1 + \frac{1}{\lambda} \log(1 + \lambda t))}{\log(1 + \lambda t)} = \sum_{n=0}^{\infty} D_\lambda(n) \frac{t^n}{n!}.$$

If λ goes to 0, then $D_\lambda(n)$ converges to D_n . This degenerate Daehee numbers $D_\lambda(n)$ are degenerate based on the exponential sense. Recently, D. S. Kim and et al. presented the degenerate Daehee polynomials and numbers of the second kind as follows [12].

$$\frac{\log(1+t)}{(1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1} (1 + \lambda \log(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} D_{\lambda,2}(n, x) \frac{t^n}{n!}. \quad (9)$$

When $x = 0$, $D_{\lambda,2}(n) = D_{\lambda,2}(n, 0)$ are called the degenerate Daehee numbers of the second kind. These degenerate numbers are also based on the exponential sense.

Numbers	Generating function
Daehee numbers	$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}$
degenerate Daehee numbers	$\frac{\log(1+\log e_{\lambda}(t))}{\log e_{\lambda}(t)} = \sum_{n=0}^{\infty} D_{n,\lambda} \frac{t^n}{n!}$
degenerate Daehee numbers of the 2nd kind	$\frac{\log(1+t)}{e_{\lambda}(\log(1+t))-1} = \sum_{n=0}^{\infty} D_{\lambda,2}(n) \frac{t^n}{n!}$
degenerate Daehee numbers of the 3rd kind	$\frac{\log_{\lambda}(1+t)}{t} = \sum_{n=0}^{\infty} D_{\lambda,3}(n) \frac{t^n}{n!}$

TABLE 1. Three kinds of degenerate Daehee numbers

It is natural to think about degenerate Daehee numbers based on the log sense. We define *the degenerate Daehee numbers of the third kind*, denoted by $D_{\lambda,3}(n)$, as follows.

$$\frac{\log_{\lambda}(1+t)}{t} = \frac{\frac{1}{\lambda}((1+t)^{\lambda} - 1)}{t} = \sum_{n=0}^{\infty} D_{\lambda,3}(n) \frac{t^n}{n!}. \quad (10)$$

We note that

$$\lim_{\lambda \rightarrow 0} \frac{\frac{1}{\lambda}((1+t)^{\lambda} - 1)}{t} = \frac{\log(1+t)}{t}. \quad (11)$$

From (11), we know that $\lim_{\lambda \rightarrow 0} D_{\lambda,3}(n) = D_n$ for each n .

The Table 1 summarizes the three types of degenerate Daehee numbers.

In this paper, we define the degenerate Daehee numbers based on the log sense. And we obtain some identities which are connected with the Daehee, the degenerate Daehee and the degenerate Daehee numbers of the second kind. Additionally, we deduce a differential equation using the degenerate log function, and we derive some identities related to the degenerate Daehee numbers from this differential equation.

2. Degenerate Daehee numbers of the third kind

From now on, for any real x and non negative integer n , we denote $(x)_n$ for falling factorial $(x)_n = x(x-1)(x-2) \cdots (x-n+1)$ and $(x)_0 = 1$. We use $S_1(m, n)$ and $S_2(m, n)$ to denote the Stirling number of the first kind and the second kind respectively.

From the definition of the degenerate Daehee numbers of the third kind, we get

$$\begin{aligned} \frac{\frac{1}{\lambda}((1+t)^{\lambda} - 1)}{t} &= \frac{1}{\lambda t} \sum_{n=1}^{\infty} (\lambda)_n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\lambda)_{n+1}}{\lambda(n+1)} \frac{t^n}{n!}. \end{aligned} \quad (12)$$

This (12) yields the following.

$$D_{\lambda,3}(n) = \frac{(\lambda)_{n+1}}{\lambda(n+1)} = \frac{\lambda(\lambda-1)(\lambda-2) \cdots (\lambda-n)}{\lambda(n+1)}. \quad (13)$$

From (13), it is easy to show that $\lim_{\lambda \rightarrow 0} D_{\lambda,3}(n) = D_n$.

Now let us investigate the relationship between the Daehee numbers and the degenerate Daehee numbers.

$$\begin{aligned}
\frac{\frac{1}{\lambda}((1+t)^\lambda - 1)}{t} &= \frac{\log\left(e^{\frac{1}{\lambda}((1+t)^\lambda - 1)} - 1 + 1\right)}{t} = \sum_{l=0}^{\infty} D_l \frac{\left(e^{\frac{1}{\lambda}((1+t)^\lambda - 1)} - 1\right)^l}{l!} \\
&= \sum_{l=0}^{\infty} D_l \sum_{m=l}^{\infty} S_2(m, l) \frac{1}{m!} \frac{1}{\lambda^m} ((1+t)^\lambda - 1)^m \\
&= \sum_{l=0}^{\infty} D_l \sum_{m=l}^{\infty} S_2(m, l) \frac{1}{m!} \frac{1}{\lambda^m} \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \sum_{n=0}^{\infty} (k\lambda)_n \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{l=0}^m \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \frac{D_l S_2(m, l) (k\lambda)_n}{m! \lambda^m} \frac{t^n}{n!}.
\end{aligned} \tag{14}$$

The equation (14) yields a relationship between the Daehee numbers and the degenerate Daehee numbers.

Theorem 1. For any nonnegative integer n ,

$$D_{\lambda,3}(n) = \sum_{m=0}^{\infty} \sum_{l=0}^m \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \frac{D_l S_2(m, l) (k\lambda)_n}{m! \lambda^m}.$$

From the definition of the Daehee numbers (6) and the degenerate Daehee numbers (10), we get the following.

$$\begin{aligned}
\frac{\log(1+t)}{t} &= \frac{\frac{1}{\lambda} \left\{ \left(1 + \left((\lambda \log(1+t) + 1)^{\frac{1}{\lambda}} - 1 \right) \right)^\lambda - 1 \right\}}{t} \\
&= \sum_{l=0}^{\infty} D_{\lambda,3}(l) \frac{\left((\lambda \log(1+t) + 1)^{\frac{1}{\lambda}} - 1 \right)^l}{l!} \\
&= \sum_{l=0}^{\infty} \frac{D_{\lambda,3}(l)}{l!} \sum_{m=0}^l (-1)^{l-m} \binom{l}{m} (\lambda \log(1+t) + 1)^{\frac{m}{\lambda}} \\
&= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=0}^l \sum_{k=0}^n (-1)^{l-m} \binom{l}{m} \left(\frac{m}{\lambda} \right)_k \lambda^k \frac{D_{\lambda,3}(l) S_1(n, k)}{l!} \frac{t^n}{n!}.
\end{aligned} \tag{15}$$

From the equation (15), we have a kind of inversion formula for Theorem 1.

Theorem 2. For any nonnegative integer n ,

$$D_n = \sum_{l=0}^{\infty} \sum_{m=0}^l \sum_{k=0}^n (-1)^{l-m} \binom{l}{m} \left(\frac{m}{\lambda} \right)_k \lambda^k \frac{D_{\lambda,3}(l) S_1(n, k)}{l!}.$$

In [15], the degenerate Stirling numbers of the first kind, denoted by $S_{1,\lambda}(n, k)$, are introduced by

$$\frac{1}{k!} (\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!}. \tag{16}$$

The equation (7) notice that the Daehee numbers can be represented by the Bernoulli and Stirling numbers of the first kind. The next equation is a degenerate version of the equation (7).

$$\begin{aligned}\frac{\log_{\lambda}(1+t)}{t} &= \frac{\frac{1}{\lambda}((1+t)^{\lambda}-1)}{t} \\ &= \frac{\frac{1}{\lambda}((1+t)^{\lambda}-1)}{\left(1+\lambda\left(\frac{1}{\lambda}((1+t)^{\lambda}-1)\right)\right)^{\frac{1}{\lambda}}-1} \\ &= \sum_{l=0}^{\infty} \beta_{l,\lambda} \frac{\left(\frac{1}{\lambda}((1+t)^{\lambda}-1)\right)^l}{l!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \beta_{l,\lambda} S_{1,\lambda}(n,l) \frac{t^n}{n!}\end{aligned}\quad (17)$$

By comparing the coefficients in the equation (17), we can obtain the following theorem.

Theorem 3. For any nonnegative integer n ,

$$D_{\lambda,3}(n) = \sum_{l=0}^n \beta_{l,\lambda} S_{1,\lambda}(n,l).$$

In [15], the degenerate Stirling numbers of the second kind were defined by the generating function

$$\frac{1}{k!} (e_{\lambda}(t) - 1)^k = \frac{\left((1+\lambda t)^{\frac{1}{\lambda}} - 1\right)^k}{k!} = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}. \quad (18)$$

Using (18) and the definition of the degenerate Bernoulli numbers and the degenerate Daehee numbers, we get the following.

$$\begin{aligned}\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} &= \frac{\frac{1}{\lambda} \left(\left(1 + ((1+\lambda t)^{\frac{1}{\lambda}} - 1)\right)^{\lambda} - 1 \right)}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} \\ &= \sum_{l=0}^{\infty} D_{\lambda,3}(l) \frac{1}{l!} \left((1+\lambda t)^{\frac{1}{\lambda}} - 1 \right)^l \\ &= \sum_{n=l}^{\infty} \sum_{l=0}^n D_{\lambda,3}(l) S_{2,\lambda}(n,l) \frac{t^n}{n!}.\end{aligned}\quad (19)$$

The equation (19) give us an inversion formula of Theorem 3 which is a degenerate version of the equation (8).

Theorem 4. For any nonnegative integer n ,

$$\beta_{n,\lambda} = \sum_{l=0}^n D_{\lambda,3}(l) S_{2,\lambda}(n,l).$$

Now let us observe a relation between the degenerate Daehee numbers of the third kind and the Bernoulli numbers.

$$\begin{aligned}
 \frac{\log_{\lambda}(1+t)}{t} &= \frac{\log_{\lambda}(1+t)}{e^{\log_{\lambda}(1+t)} - 1} \frac{e^{\log_{\lambda}(1+t)} - 1}{t} \\
 &= \sum_{l=0}^{\infty} B_l \frac{(\log_{\lambda}(1+t))^l}{l!} \sum_{m=1}^{\infty} \frac{1}{t} \frac{(\log_{\lambda}(1+t))^m}{m!} \\
 &= \sum_{l=0}^{\infty} B_l \sum_{k=l}^{\infty} S_{1,\lambda}(k, l) \frac{t^k}{k!} \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} S_{1,\lambda}(n, m) \frac{t^{n-1}}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \sum_{l=0}^{n-k} \sum_{m=1}^n \binom{n}{k} \frac{B_l S_{1,\lambda}(n-k, l) S_{1,\lambda}(n+1, m)}{n+1} \right) \frac{t^n}{n!}.
 \end{aligned} \tag{20}$$

Obtain the following theorem from the definition of the degenerate Daehee numbers of the third kind (10) and from the previous (20).

Theorem 5. For any nonnegative integer n ,

$$D_{\lambda,3}(l) = \sum_{k=0}^n \sum_{l=0}^{n-k} \sum_{m=1}^n \binom{n}{k} \frac{B_l S_{1,\lambda}(n-k, l) S_{1,\lambda}(n+1, m)}{n+1}.$$

Substituting $e^{\log_{\lambda}(1+t)} - 1$ by t in the definition of the second kind (9), then the left side of (9) becomes

$$\frac{\log(1 + (e^{\log_{\lambda}(1+t)} - 1))}{(1 + \lambda \log(1 + e^{\log_{\lambda}(1+t)} - 1))^{\frac{1}{\lambda}} - 1} = \frac{\log_{\lambda}(1+t)}{t}, \tag{21}$$

and the right side becomes

$$\begin{aligned}
 \sum_{k=0}^{\infty} D_{\lambda,2}(k) \frac{(e^{\log_{\lambda}(1+t)} - 1)^k}{k!} &= \sum_{k=0}^{\infty} D_{\lambda,2}(k) \sum_{l=k}^{\infty} S_{2,\lambda}(l, k) \frac{(\log_{\lambda}(1+t))^l}{l!} \\
 &= \sum_{k=0}^{\infty} D_{\lambda,2}(k) \sum_{l=k}^{\infty} S_{2,\lambda}(l, k) \sum_{n=l}^{\infty} S_{1,\lambda}(n, l) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \sum_{l=0}^n \sum_{k=0}^l D_{\lambda,2}(k) S_{2,\lambda}(l, k) S_{1,\lambda}(n, l) \frac{t^n}{n!}.
 \end{aligned} \tag{22}$$

The following theorem is obtained by comparing the coefficients of the equation (21) with the equation (22).

Theorem 6. For any nonnegative integer n ,

$$D_{\lambda,3}(n) = \sum_{l=0}^n \sum_{k=0}^l D_{\lambda,2}(k) S_{2,\lambda}(l, k) S_{1,\lambda}(n, l).$$

To represent $D_{\lambda,3}(n)$ as $D_{\lambda,2}(n)$, substituting $e_{\lambda}(\log(1+t)) - 1 = (1 + \lambda \log(1+t))^{\frac{1}{\lambda}} - 1$ by t in the definition of the degenerate Daehee numbers of the third kind (10), then the left side becomes

$$\begin{aligned} \frac{\log_{\lambda}(1 + ((1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1))}{(1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1} &= \frac{\frac{1}{\lambda}((1 + ((1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1))^{\lambda} - 1)}{(1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1} \\ &= \frac{\log(1 + t)}{(1 + \lambda \log(1 + t))^{\frac{1}{\lambda}} - 1} \end{aligned} \quad (23)$$

and the right side becomes

$$\begin{aligned} \sum_{k=0}^{\infty} D_{\lambda,3}(k) \frac{(e_{\lambda}(\log(1 + t)) - 1)^k}{k!} &= \sum_{k=0}^{\infty} D_{\lambda,3}(k) \sum_{l=k}^{\infty} S_{2,\lambda}(l, k) \frac{(\log(1 + t))^l}{l!} \\ &= \sum_{k=0}^{\infty} D_{\lambda,3}(k) \sum_{l=k}^{\infty} S_{2,\lambda}(l, k) \sum_{n=l}^{\infty} S_1(n, l) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^n \sum_{k=0}^l D_{\lambda,3}(k) S_{2,\lambda}(l, k) S_1(n, l) \frac{t^n}{n!}. \end{aligned} \quad (24)$$

From (23) and (24), we obtain an inversion identity of Theorem 6.

Theorem 7. For any nonnegative integer n ,

$$D_{\lambda,2}(n) = \sum_{l=0}^n \sum_{k=0}^l D_{\lambda,3}(k) S_{2,\lambda}(l, k) S_1(n, l).$$

3. Differential equations arising from the generating function of degenerate Daehee numbers

From now on, we use $F = F(t)$ to denote the degenerate log function :

$$F(t) = \log_{\lambda}(1 + t) = \frac{1}{\lambda}((1 + t)^{\lambda} - 1), \quad (25)$$

and for a natural number N , $F^{(N)}$ to denote the N -th derivative of F , that is,

$$F^{(0)} = F(t), \quad F^{(N)} = \frac{d}{dt} F^{(N-1)}.$$

Differentiating the two sides of the equation (25) results in the following:

$$\begin{aligned} \frac{d}{dt} F(t) &= \frac{d}{dt} \log_{\lambda}(1 + t) = (1 + t)^{\lambda-1} \\ &= \frac{1}{1 + t} \left\{ \frac{\lambda}{\lambda} ((1 + t)^{\lambda} - 1 + 1) \right\} \\ &= \frac{\lambda}{1 + t} \frac{1}{\lambda} ((1 + t)^{\lambda} - 1) + \frac{1}{1 + t} \\ &= \frac{\lambda}{1 + t} F(t) + \frac{1}{1 + t}. \end{aligned} \quad (26)$$

Further differentiate of the equation (26) yields

$$\begin{aligned}\frac{d^2}{dt^2}F(t) &= -\frac{\lambda}{(1+t)^2}F(t) + \frac{\lambda}{1+t}F'(t) - \frac{1}{(1+t)^2} \\ &= -\frac{\lambda}{(1+t)^2}F(t) + \frac{\lambda}{1+t}\left\{\frac{\lambda}{1+t}F(t) + \frac{1}{1+t}\right\} - \frac{1}{(1+t)^2} \\ &= -\frac{\lambda(\lambda-1)}{(1+t)^2}F(t) + \frac{\lambda-1}{(1+t)^2}.\end{aligned}\quad (27)$$

From the observation (26) and (27), we assume that

$$F^{(N)} = \frac{(\lambda)_N}{(1+t)^N}F + \frac{(\lambda)_N}{\lambda(1+t)^N}.\quad (28)$$

Let us take differentiate both sides of the (28), then we have

$$\begin{aligned}F^{(N+1)} &= \frac{-N}{(1+t)^{N+1}}(\lambda)_NF + \frac{1}{(1+t)^N}(\lambda)_NF^{(1)} + \frac{-N}{\lambda(1+t)^{N+1}}(\lambda)_N \\ &= \frac{(\lambda)_N}{(1+t)^{N+1}}(\lambda-N)F + \frac{(\lambda)_N}{\lambda(1+t)^{N+1}}(\lambda-N) \\ &= \frac{(\lambda)_{N+1}}{(1+t)^{N+1}}F + \frac{(\lambda)_{N+1}}{\lambda(1+t)^{N+1}}.\end{aligned}\quad (29)$$

From (28) and (29), mathematical induction gives us the following theorem.

Theorem 8. For any positive integer N , the differential equation

$$F^{(N)} = \frac{(\lambda)_N}{(1+t)^N}F + \frac{(\lambda)_N}{\lambda(1+t)^N}\quad (30)$$

has a solution

$$F(t) = \log_\lambda(1+t) = \frac{1}{\lambda}((1+t)^\lambda - 1).$$

We note that

$$\begin{aligned}F^{(N)} &= \frac{d^N}{dt^N} \log_\lambda(1+t) = \frac{d^N}{dt^N} \left(\frac{\log_\lambda(1+t)}{t} \cdot t \right) \\ &= \frac{d^N}{dt^N} \left(\sum_{n=1}^{\infty} n D_{\lambda,3}(n-1) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} (n+N) D_{\lambda,3}(n+N-1) \frac{t^m}{m!}.\end{aligned}\quad (31)$$

On the other hand

$$\begin{aligned}\frac{1}{(1+t)^N}F &= \sum_{l=0}^{\infty} \binom{N+l-1}{l} (-1)^l t^l (\log_\lambda(1+t)) \\ &= \sum_{l=0}^{\infty} \binom{N+l-1}{l} (-1)^l t^l \sum_{m=1}^{\infty} \frac{1}{\lambda} (\lambda)_m \frac{t^m}{m!} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^n \frac{1}{\lambda} (\lambda)_m \binom{N+n-m-1}{N-1} \binom{n}{m} (n-m)! \frac{t^n}{n!}\end{aligned}\quad (32)$$

and

$$\begin{aligned} \frac{(\lambda)_N}{\lambda} \frac{1}{(1+t)^N} &= \frac{(\lambda)_N}{\lambda} \sum_{n=0}^{\infty} (-1)^n \binom{N+n-1}{n} t^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda)_N}{\lambda} n! \binom{N+n-1}{n} \frac{t^n}{n!}. \end{aligned} \quad (33)$$

As a result (28), by comparing the coefficients of (29), (30) and (31), we get the following identity.

Theorem 9. For any nonnegative integer m and N ,

$$\begin{aligned} &(n+N)D_{\lambda,3}(n+N-1) \\ &= \begin{cases} \sum_{m=1}^n \frac{1}{\lambda} (\lambda)_m \binom{N+n-m-1}{N-1} \binom{n}{m} (n-m)! + (-1)^n \frac{(\lambda)_N}{\lambda} n! & \text{if } n \geq 1 \\ \frac{(\lambda)_N}{\lambda} & \text{if } n = 0 \end{cases}. \end{aligned}$$

In the other view point of the right side of (28), we get

$$\begin{aligned} \frac{(\lambda)_N}{(1+t)^N} F + \frac{(\lambda)_N}{\lambda(1+t)^N} &= \frac{(\lambda-1)_{N-1}}{(1+t)^N} (\lambda F + 1) \\ &= \frac{(\lambda-1)_{N-1}}{(1+t)^N} \left(\lambda \frac{1}{\lambda} ((1+t)^\lambda - 1) + 1 \right) \\ &= (\lambda-1)_{N-1} (1+t)^{\lambda-N} \\ &= \sum_{n=0}^{\infty} (\lambda-1)_{N-1} (\lambda-N)_n \frac{t^n}{n!}. \end{aligned} \quad (34)$$

The equation (29) and (34) yield the following.

Theorem 10. For nonnegative integer n and N ,

$$(n+N)D_{\lambda,3}(n+N-1) = (\lambda-1)_{N-1} (\lambda-N)_n.$$

4. RESULTS AND DISCUSSION

In this paper, we have studied the degenerate Daehee numbers of the third kind. To define degenerate function, we use degenerate log function $\log_\lambda(t) = \frac{1}{\lambda}((1+t)^\lambda - 1)$. After we defined the degenerate Daehee numbers of the third kind, we obtained two relations between the Daehee numbers and the degenerate Daehee numbers of the second kind with Stirling numbers, Theorem 1 and Theorem 2. After that, we have two relationships between the degenerate Daehee numbers of the third kind and the degenerate Bernoulli numbers, which are dual relations between the Daehee numbers and the Bernoulli numbers, Theorem 3 and Theorem 4. We obtained some relations between the degenerate Daehee numbers of the second kind and the third kind, Theorem 6 and Theorem 7.

In section 3, we drive a differential equation from the degenerate log function, Theorem 6. From the differential equation in the Theorem 6, we deduce some identities for the degenerate Daehee numbers of the second kind.

5. CONCLUSION

In [1, 2], L. Carlitz considered the degenerate exponential function. By using this degenerate exponential function, he studied the degenerate Bernoulli numbers and polynomials which are given by the generating function. In this view points, we consider the inverse function of Carlitz's degenerate exponential function which is called the degenerate logarithmic function. From our degenerate logarithmic function, we derive several identities of special numbers. Through our results, we are able to see that degenerate log function is a useful tool for study of special number theory.

ACKNOWLEDGEMENTS

This research was supported by Kyungpook National University Bokhyeon Research Fund, 2017.

REFERENCES

- [1] L. Carlitz, L : Extended Stirling and exponential numbers, *Duke Math. J.*, 32, 205–224 (1965).
- [2] L. Carlitz, L : Degenerate Stirling, Bernoulli and Eulerian numbers, *Utilitas Mathematica*, 15, 51–88(1979).
- [3] Dolgy, DV, Kim, T, Jang, LC, Kwon, HI, Seo, JJ : Some identities of symmetry for the degenerate q -Bernoulli polynomials under symmetry group of degree n , *Global J. of Pure and Applied Math.*, 12, 4385–4394(2016).
- [4] El-Desouky, BS, Mustafa, A : New results on higher-order Daehee and Bernoulli numbers and polynomials, *Adv. Difference Equ.*, 2016, doi:10.1186/s13662-016-0764-z(2016).
- [5] El-Desouky, BS, Mustafa, A, Abdel-Moneim, FM : Multiparameter Higher Order Daehee and Bernoulli Numbers and Polynomials, *Applied Math.*, **2017**, 775–785, DOI: 10.4236/am.2017.86060(2017).
- [6] He, Y : Symmetric identities for Carlitz's q -Bernoulli numbers and polynomials, *Advances in Difference Equations* 2013, 2013:246, doi:10.1186/1687-1847-2013-246(2013).
- [7] He, Y : Some results for Carlitz's q -Bernoulli numbers and polynomials, *Appl. Anal. Discrete Math.* 8, 304–319(2014).
- [8] Jang, GW, Kim, T : Revisit of identities of Daehee numbers arising from nonlinear differential equations, *Proc. Jangjeon Math. Soc.*, 20, 163–177(2017).
- [9] Jang, GW, Kwon, J, Lee, JG : Some identities of degenerate Daehee numbers arising from nonlinear differential equation, *Adv. Difference Equ.*, 2017, doi:10.1186/s13662-017-1265-4(2017).
- [10] Jang, GW, Kim, T : Some identities of ordered Bell numbers arising from differential equation, *Adv. Stud. Contemp. Math.(Kyungshang)*, 27, 385–397(2017).
- [11] Kim, DS, Kim, T : Daehee Numbers and Polynomials, *Appl. Math. Sci.* 7 (120) 5969–5976(2013).
- [12] Kim, DS, Kim, T, Kwon, HI, Jang, GW : Degenerate Daehee polynomials of the second kind, *Proc. Jangjeon Math. Soc.*, 21, 83–97(2018).
- [13] Kim, DS, Kim, T : Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation, *Bull. Korean Math. Soc.*, 52, 2001–2010(2015).
- [14] Kim, T : Degenerate Cauchy numbers and polynomials of the second kind. *Adv. Stud. Contemp. Math.(Kyungshang)* 27, 441–449(2017).
- [15] Kim, T : On the degenerate Cauchy numbers and polynomials, *Proc. Jangjeon Math. Soc.*, 18, 307–312(2015).
- [16] Kim, T : λ -analogue of Stirling numbers of the first kind, *Adv. Stud. Contemp. Math.(Kyungshang)* 27, 423–429(2017).
- [17] Kim, T, Kim, DS: Degenerate Laplace Transform and Degenerate Gamma Function, *Russ. J. Math. Phys.* 24, 241–248(2017).

- [18] Kim, T, Kim, DS, Jang, GW : Differential Equations Associated with Degenerate Cauchy numbers, Iran. J. Sci. Technol. Trans. Sci., doi:10.1007/s409950018-0531-y(2018).
- [19] Kim, T, Kim, DS, Kwon, HI : A note on degenerate Stirling numbers and their applications, Proc. Jangjeon Math. Soc., 21, 195–203(2018).
- [20] Kim, T, Kim, DS, Kwon, HI, Seo, JJ : Revisit nonlinear differential equations associated with Bernoulli numbers of the second kind, Global J. of Pure and Applied Math., 12, 1893–1901(2016).
- [21] Kim, T, Kim, DS, Dolgy, DV, Seo, JJ : Bernoulli polynomials of the second kind and their identities arising from umbral calculus, J. Nonlinear Sci. Appl., 9, 860–869(2016).
- [22] Ozden, H, Cangul, IN, Simsek, Y : Remarks on q-Bernoulli numbers associated with Daehee numbers. Adv. Stud. Contemp. Math. (Kyungshang) 18, 41–48(2009).
- [23] Park, JW, Kwon, J : A note on the Degenerate High order Daehee polynomials, Appl. Math. Sic., 9, 4635–4642(2015).
- [24] Pyo, SS, Kim, T, Rim, SH : Identities of the degenerate Daehee numbers with the Bernoulli numbers of the second kind arising from nonlinear differential equation, J. Nonlinear Sci. Appl., 10, 6219–6228(2017).
- [25] Pyo, SS, Kim T, Rim, SH : Degenerate Cauchy numbers of the third kind, J. Inequalities Appl., 10, 6219–6228(2017).
- [26] Simsek, Y : Identities on the Changhee numbers and apostol-type Daehee polynomials, Adv. Stud. Contemp. Math. (Kyungshang) 27, 199–212(2018).
- [27] Simsek, Y : Identities and relations related to combinatorial numbers and polynomials, Proc. Jangjeon Math. Soc., 20 No 1. 127–135 (2017).
- [28] Wang, NL, Li, H : Some identities on the Higher-order Daehee and Changhee Numbers, Pure and Appl. Math. J. 4, 33–37(2015).

DEPARTMENT OF MATHEMATICS EDUCATION, SILLA UNIVERSITY, BUSAN, KOREA
E-mail address: ssoopyo@gmail.com

DEPARTMENT OF MATHEMATICS, KWANGWOON UNIVERSITY, SEOUL, SOUTH KOREA
E-mail address: tkkim@kw.ac.kr

DEPARTMENT OF MATHEMATICS EDUCATION, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU, SOUTH KOREA(CORRESPONDING AUTHOR)
E-mail address: shrim@knu.ac.kr