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From Black Holes to Neutrino Stars

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Abstract

Due to limitation of the binding energy of a self-gravitating matter, the radius of a body is at least twice larger than the Schwarzschild radius. The total energy is adsorbed at the body surface, giving rise of a size-dependent surface tension. Since the Hawking temperature appears to be the critical one, the black holes possess zero surface tension. Microscopic neutrino stars are also introduced.

Keywords: self-gravitation, black holes, neutrino stars.

A black hole is so condensed that even light cannot escape due to the gravitational attraction. Leaving apart any resistance due to particle repulsion and exclusion the black hole formation seems also energetically impossible. Imagine a self-gravitating star, emitting radiation in all. The kinetic energy of the matter particles is continuously decreasing and the star is shrinking towards a black hole. However, due to the mass-energy equivalence, the star is losing its mass as well, thus reducing the inner strength of the self-gravitational attraction. As the Schwarzschild radius decreases also with decreasing mass, this Zeno effect leads to the conclusion that only an empty black hole could form with zero size, when the mass of the star becomes zero.

To make the picture consistent, let us start from the Newton gravity. If the mass density ρ is radial-symmetrically distributed, the corresponding gravitational potential ϕ obeys the Poisson equation

$$\partial_r(r^2\partial_r\phi)/r^2 = 4\pi G\rho \qquad \partial_r\phi = Gm_r/r^2 \tag{1}$$

The second integral form involves the mass $m_r \equiv \int_0^r \rho 4\pi r^2 dr$, enclosed within a central sphere with radius r. Naturally, m_∞ contains the total mass M_0 of the self-gravitating matter. Using Eq. (1), one can calculate the overall potential energy

$$U = \int_0^\infty \phi\rho 2\pi r^2 dr = -(G/2)\int_0^\infty (m_r/r)^2 dr \tag{2}$$

Since no other interactions are considered and the excess energy is freely released as radiation in vacuum, the body contracts permanently due to the gravitational attraction. At the end, a mass point is formed with $m_r = M_0$ everywhere. According to Eq. (1), it corresponds to the classical Newton potential $\phi = -GM_0 / r$. As it is well known, U diverges for a mass point, which indicates a generic problem of the Newton gravity.

According to the Einstein special relativity theory, the total energy at rest $E = M_0c^2 + U$ is positively defined. The lowest value $E = 0$ limits the binding energy to the maximal 100 % mass defect. The integral and differential forms of $-U / c^2 = M_0$ reads

$$(G/2)\int_0^\infty (m_r / cr)^2 dr = \int_0^\infty (\partial_r m_r) dr \qquad \partial_r m_r = (G/2)(m_r / cr)^2 \tag{3}$$

The integration Eq. (3) leads straightforward to the radial mass distribution

$$m_r = M_0 / (1 + R_0 / r) \qquad R_0 \equiv GM_0 / 2c^2 \tag{4}$$

which is not constant anymore due to the binging energy limitation. Substituting Eq. (4) in Eq. (1) yields after integration the restricted gravitational potential

$$\phi = -2c^2 \ln(1 + R_0 / r) \tag{5}$$

Far from the center it tends to the Newton potential, while near the center the logarithmic potential $2c^2 \ln(r / R_0)$ is much weaker than $-GM_0 / r$. That is why the potential energy (2) is finite. One can derive the mass density distribution by differentiating directly m_r from Eq. (4)

$$\rho = M_0 R_0 / 4\pi r^2 (r + R_0)^2 \tag{6}$$

If the mass M_0 is very small, Eq. (4) approaches a mass point as expected. In the opposite case of large masses, Eqs. (4) and (6) tends to $m_r = M_0 (r / R_0)$ and $\rho = M_0 / 4\pi R_0 r^2$, which are independent of the mass M_0. They describe a compact body with radius R_0 and for $r > R_0$ the density $\rho = 0$ is zero and $m_r = M_0$. One can calculate the corresponding pressure p via the hydrostatic force balance

$$\partial_r p = -\rho \partial_r \phi = -\rho G m_r / r^2 \tag{7}$$

It is zero outside the body but remarkably $p = \rho c^2$ resembles inside an equation of state. As is seen, the limitation of the binding energy resolves the mass point singularity. However, since R_0 is one fourth of the Schwarzschild radius, a black hole singularity still holds. It points out that probably the maximal mass will prevent such a peculiarity as well.

It is interesting what the effect would be of the binding energy limitation on black holes. To answer this question, we are going to repeat the analysis above, using the Einstein general relativity theory. In the frames of the latter, the problem is described by the Tolman-Oppenheimer-Volkoff (TOV) equations[1]

$$\partial_r \phi = G(m_r + 4\pi r^3 p / c^2) / r^2 (1 - 2Gm_r / c^2 r) \qquad \partial_r p = -(\rho + p / c^2) \partial_r \phi \tag{8}$$

from which Eqs. (1) and (7) follow, respectively, in the non-relativistic limit. Note that the relativistic mass M is smaller than M_0 and the difference $M_0 - M = -U / c^2$ is the binding energy mass defect. For a mass point, Eq. (8) reduces to $\partial_r \phi = GM / r^2 (1 - 2GM / c^2 r)$, since $m_r = M$, $\rho = 0$ and $p = 0$. Integrating this equation results straightforward in the well-known relativistic potential, where $r_s \equiv 2GM / c^2$ is the Schwarzschild radius,[2]

$$2\phi / c^2 = \ln(1 - r_s / r) \tag{9}$$

Far from the center, ϕ tends to the Newton potential $-GM / r$. The singularity at r_s marks the event horizon,[3] where the surface of the black hole takes place.

Let us apply now the energy limitation to the TOV equations (8). In general, the pressure $p = X\varepsilon$ is proportional to the energy density $\varepsilon = \rho c^2$ and we got $X = 1$ in the semi-relativistic analysis above. Introducing $p = X\rho c^2$ in Eq. (8) yields

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$$\partial_r \phi = G(m_r + Xr\partial_r m_r) / r^2(1 - 2Gm_r / c^2 r) \qquad Xc^2\partial_r \rho = -(1+X)\rho\partial_r \phi \tag{10}$$

Searching for a black hole, we are looking for a compact body of self-gravitating matter. For radii larger than the body radius R the standard expressions $m_r = M$ and $\rho = 0$ hold. It follows immediately from Eq. (10) that Eq. (9) is the potential outside the body. It is well known that inside the body $m_r = M(r/R)$ and $\rho = M / 4\pi R r^2$ are solutions of Eq. (10),[4] which is also supported by our semi-relativistic analysis. Introducing them in Eq. (10) leads to

$$R / r_s = 1 + (1+X)^2 / 4X \qquad \phi = \phi_R + 2c^2 \ln(r/R) X / (1+X) \tag{11}$$

To determine the important value of the factor X, one can employ a general formula,[4] relating the relativistic mass M with the mass M_0 at the origin,

$$M_0 = \int_0^M \frac{dm_r}{\sqrt{1 - 2Gm_r / c^2 r}} = \sqrt{1 + 4X / (1+X)^2}\, M \tag{12}$$

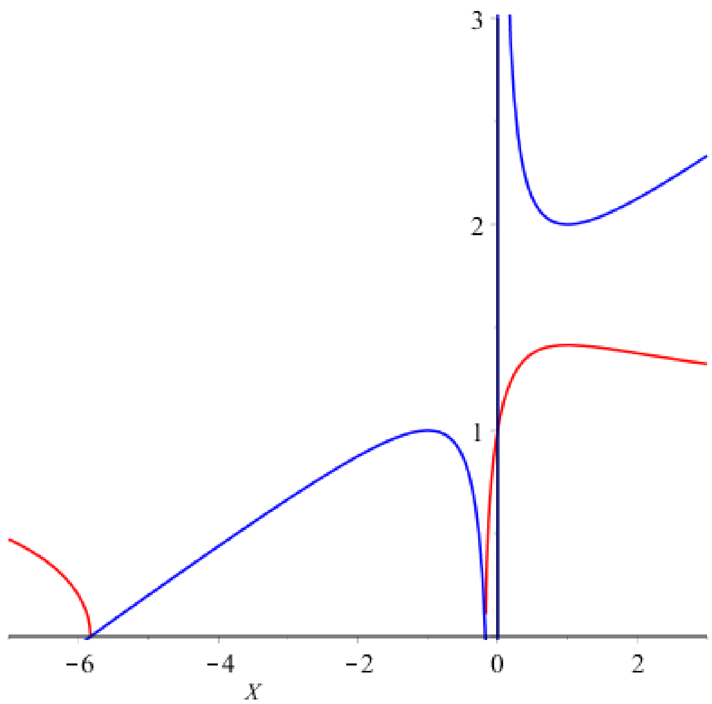

Fig. 1. Plot of R / r_s (blue) and M_0 / M (red) as a function of the factor X

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The positive ratios R/r_s from Eq. (11) and the real ratios M_0/M from Eq. (12) are plotted in Fig. 1. As is seen, there is no overlap between them for $X<0$. Moreover, the negative X is always related to a negative mass defect, which indicates lack of bounded body for dark matter. Looking for a positive binding energy, the necessary inequality $M_0>M$ imposes that $X>0$. If the matter is super-relativistic with $X=1/3$, the corresponding mass defect is 24.4 %. If the particles are moving much slower than the speed of light, the doubled non-relativistic factor $X=2/3$ results in 28.6 % mass defect. The maximal mass defect about 29.3 % at $X=1$ should correspond to the lowest body temperature in order to prohibit any further energy loss by quantum reasons. It is less than 30 % and the matter particles are probably at rest. Logically, $X=1$ corresponds to the minimal body radius $R=2r_s=4GM/c^2$ from Eq. (11). According to Fig. (1), the radius of a self-gravitating body is at least twice larger than the Schwarzschild radius.

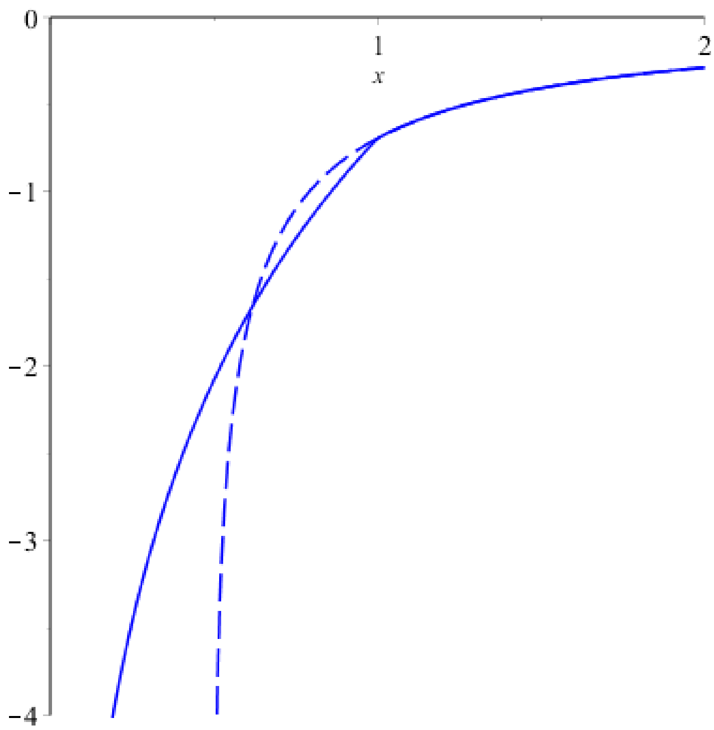

Fig. 2. The dimensionless potential $2\phi/c^2$ from Eq. (13) (solid) and Eq. (9) (dash) versus r/R

The finite value of the surface potential $\phi_R=-c^2\ln 2/2$ follows from Eq. (9) at $R=2r_s$. Hence, the overall potential (11) acquires the form, where H is the Heaviside step function,

$$2\phi/c^2 = H(R-r)\ln(r^2/2R^2)+H(r-R)\ln(1-R/2r) \tag{13}$$

Its plot in Fig. 2 shows lack of singularity in contrast to the Schwarzschild potential (9). The potential (13) possesses a kink at R, which reflects the effect of a pressure jump at the body surface. Hence, the body possesses a capillary pressure, $p_R = Mc^2 / 4\pi R^3$, which is related to the surface tension σ via the Laplace law $p_R = 2\sigma / R + \partial_R \sigma$. The latter can be integrated directly to obtain $\sigma = Mc^2 / A$, where $A = 4\pi R^2$ is the area of the body surface. It is a particular example of a size-dependent surface tension, described via the Tolman formula.[5] It vanishes at a flat surface and the bending elasticity $2R\sigma = c^4 / 8\pi G$ coincides with the universal Einstein expression. Remarkably, the full energy $Mc^2 = \sigma A$ of the body is at the surface, which supports our expectation that the body is energetically empty and that is why it cannot shrink anymore.

The problem now is what causes the pressure $p = \rho c^2$ if the particles are not moving. This equation looks very similar to the ideal gas equation of state $p = Nk_B T / V$ at constant temperature and we are going to explore such an identity. The characteristic potential of the body is the Helmholtz free energy $F(T,V,N)$ as a function of the natural parameters. Substituting the ideal gas pressure in the thermodynamic relation $p = -(\partial_V F)_{T,N}$ allows direct integration to obtain

$$F = Nk_B T[\ln(8l_P^3 N / V) - 1 + g(T)] \tag{14}$$

where $l_P \equiv (G\hbar / c^3)^{1/2}$ is the Planck length and g is an unspecified function. Using now the equivalence of the pressure and energy density at $X = 1$ determines the energy $E = pV = Nk_B T$. In the general case it reads $E = Nk_B T / X$. Substituting Eq. (14) into the thermodynamic relation $E = F - T(\partial_T F)_{V,N}$ yields the unspecified function $g = \ln(m_P c^2 / 4\pi k_B T)$, where $m_P \equiv (\hbar c / G)^{1/2}$ is the Planck mass. So, the free energy reads

$$F = Nk_B T[\ln(2G\hbar^2 N / \pi c^2 k_B TV) - 1] \tag{15}$$

Using this fundamental equation, one can calculate the chemical potential of the gas

$$\mu = (\partial_N F)_{T,V} = 2k_B T \ln(\hbar / 2\pi k_B T) + k_B T \ln(8\pi G p / c^2) \tag{16}$$

Considering the self-gravitating body with pressure $p = \rho c^2$, at constant temperature the gradients of the chemical and gravitational potentials cancel exactly in agreement with Eq. (10), $\partial_r \mu / k_B T + 2\partial_r \phi / c^2 = 0$. Integrating the latter equation yields the gravito-chemical potential

$$\tilde{\mu} = \mu + 2k_B T\phi / c^2 = 2k_B T \ln(\hbar c / 4\pi R k_B T) \tag{17}$$

which is constant everywhere in the body. Since the body is energetically hollow, the corresponding $\tilde{\mu}(T) = 0$ determines the temperature of the body

$$T = \hbar c / 4\pi R k_B \tag{18}$$

This is, in fact, the minimal energy-time Heisenberg relation. In black holes Eq. (18) reduces at $R = r_s$ to the Hawking temperature, $T_H = \hbar c / 4\pi r_s k_B$.[6] Therefore, due to $R = 2r_s$ the temperature of the body is half of the Hawking temperature. The relation $Mc^2 = \sigma A$ implies also that the body entropy compensates exactly the negative pressure-volume term $S = pV / T = Nk_B$ to cancel the internal energy completely. Substituting here the temperature (18) yields that the body entropy coincides with the Bekenstein-Hawking one[7]

$$S = Mc^2 / T = k_B A / 4l_P^2 \tag{19}$$

Replacing this expression in the thermodynamic Maxwell equation $\partial_T \sigma = -\partial_A S$ allows direct integration to obtain the temperature dependence of the surface tension $\sigma = k_B (T_c - T) / 4l_P^2$ with the critical temperature T_c. Substituting here $\sigma = Mc^2 / A$ and $T = T_H / 2$ yields straightforward that the critical temperature $T_c = T_H$ is the Hawking one. Therefore, a black hole possesses zero surface tension $\sigma = 0$, which means a zero mass M.

The mass of a gas particle $\bar{m} = M / N = k_B T / c^2$ is extremely small at moderate temperatures. Because any movement in the body is frozen, temperature causes solely some energy fluctuation $\bar{m}c^2$ with lifetime $\tau \geq \hbar / 2\bar{m}c^2 = 2\pi R / c$. Since the mass and energy are synonyms, $\bar{m}$ should be considered as the typical mass fluctuation as well. The temperature dependence of the free energy (15) follows directly from the following exponential Boltzmann distribution

$$f_m = (c^2 / k_B T)\exp(-mc^2 / k_B T) \tag{20}$$

Perhaps, the neutrino is the only known particle possible to transmit the energy fluctuations in a cold body. Moreover, $\bar{m} = m_P^2 / 16\pi M$ could be explained via the seesaw mechanism.[8] From this perspective, it is interesting to consider a pure neutrino star. Measurements of the neutrino mass are very difficult, but important progress is achieved in the measurement of the difference in the squares of masses $\Delta m_{12}^2 \equiv < |m_1^2 - m_2^2| >$ via KamLAND.[9] Assuming logically independent neutrinos in the ideal gas, it follows

$$\Delta m_{12}^2 = \int_0^\infty \int_0^\infty |m_1^2 - m_2^2| f_{m_1} f_{m_2} dm_1 dm_2 = 3\bar{m}^2 \tag{21}$$

Using the experimental value $\Delta m_{12}^2 = 79$ meV^2/c^4, the corresponding temperature of the neutrino star $T = 60$ K is low enough to neglect any motion. One can estimate further from Eq. (18) a very small radius $R = 3$ µm of the neutrino micro-star, while its mass $M = 10^{21}$ kg is 6000 times smaller than the mass of the Earth. The number of neutrinos in the star is $N = Mc^2 / k_B T = 10^{59}$ and from the volume per a particle $\bar{v} = 4\pi R^3 / 3N = 10^{-75}$ m^3 one can estimate the neutrino diameter as smaller than 10^{-25} m.

Obviously, the Fermi energy is huge due to the enormous density of the neutrino star and the tiny neutrino mass. This implies that the neutrinos form probably bosonic Cooper pairs (e.g. majorons) to avoid the fermionic repulsion due to the Pauli principle. Perhaps, this is the reason for the factor 2 in the gravito-chemical potential $\tilde{\mu} = \mu + 2\bar{m}\phi$. Thus, the neutrino star is a Bose-Einstein condensate, since T is below the BEC critical temperature and $\tilde{\mu} = 0$. That is why the body cannot emit energy anymore to collapse in a black hole. In a previous paper, we derived that a self-gravitating quantum matter should always form a central hollow cavity.[10] To examine this prediction, one can introduce now the exact relativistic density $\rho = M / 4\pi r^2 R$ into the Bohm quantum potential to obtain

$$Q = -(\hbar^2 / 2\bar{m})\partial_r (r^2 \partial_r \rho^{1/2}) / r^2 \rho^{1/2} = \hbar^2 \delta(r) / \bar{m} r = 16\pi GM\hbar\delta(r) / cr \tag{22}$$

where δ is the Dirac delta-function. Equation (22) describes an infinite repulsion in the body center. This singularity is much stronger than the gravitational one and it will definitely cause a central hollow cavity due to the Heisenberg constraint $r > \hbar / m_r c$ or $r > 2l_P$. It is also evident from the unphysical divergences of the gravitational potential ϕ and the density ρ in the body center.

The paper is dedicated to Stephen Hawking (1942-2018).

1. R.C. Tolman, *Phys. Rev.* **55** (1939) 364
   J.R. Oppenheimer, G.M. Volkoff, *Phys. Rev.* **55** (1939) 374
2. K. Schwarzschild, *Preuss. Akad. Wiss. (Berlin), Math.-Phys.-Tech.* (1916) 189
3. D. Finkelstein, *Phys. Rev.* **110** (1958) 965
4. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, New York (2013)
5. R.C. Tolman, *J. Chem. Phys.* **17** (1949) 333
6. S.W. Hawking, *Nature* **248** (1974) 30
7. J.D. Bekenstein, *Lett. Nuovo Cim.* **4** (1972) 737
   S.W. Hawking, *Phys. Rev. Lett.* **26** (1971) 1344
8. M. Gell-Mann, P. Ramond, R. Slansky, in *Supergravity*, P. van Nieuwenhuizen, D.Z. Freedman (eds.), North Holland, Amsterdam (1979), p. 315
9. T. Araki et al. (KamLAND Collaboration), *Phys. Rev. Lett.* **94** (2005) 081801
10. R. Tsekov, arXiv:1705.08249 (2017)