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Geometric Interpretation of the Minkowski Metric

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Abstract

A novel geometric interpretation of the Minkowski metric and thus of phenomena in special relativity is provided. It is shown that a change of basis in Minkowski space is the equivalent of a change of basis in Euclidean space if a basis element is replaced by its dual element. The methodology of the proof includes infinitesimal changes of basis using the Lie-algebras of the involved groups. As a consequence, a direct mapping between Euclidean and Minkowski space is defined.

1 Introduction

The special relativity can be formulated within the four-dimensional Minkowski space $\mathbb{R}(3,1)$, which is defined by the following scalar product:

$$\mathbf{dv} \cdot \mathbf{dw} = \sum_{\mu\nu} g_{\mu\nu} dv^\mu dw^\nu \qquad \text{with} \qquad g_{\mu\nu} = [S1] \cdot \mathrm{diag}\,(-1,+1,+1,+1)\,. \quad (1)$$

The covariant tensor $g_{\mu\nu}$ is called Minkowski metric, $\mathbf{dv}$ and $\mathbf{dw}$ can be any four-vector elements of the Minkowski space. The algebraic sign $[S1] = \pm 1$ can be assigned freely depending on the convention [1].

Although this description is elegant and successful, it does not allow a direct geometrical interpretation of the underlying quantities. A direct geometric/ trigonometric interpretation in the Euclidean sense is not possible because the scalar product defined by $g_{\mu\nu}$ is not positive definite, which in turn implies that the norm induced by the scalar product $\|\mathbf{dv}\| = \sqrt{\mathbf{dv} \cdot \mathbf{dv}}$ can assume imaginary values (Figure 1). This means that imaginary distances between two points can occur in the vector space, and there is no intuitive trigonometric equivalent for this.

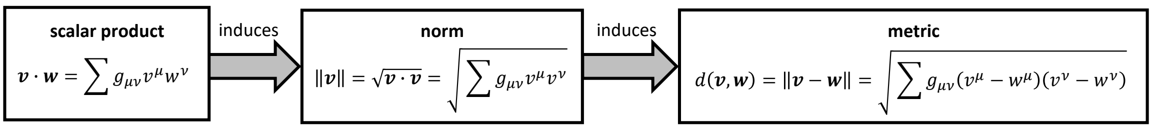

Figure 1: Relation between scalar product, induced norm and metric.

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This is seen as a significant shortcoming for authors who seek geometric interpretations of the underlying systems of equations (e.g. [2] and references included therein; [3], [4], [5]).

The purpose of this article is to offer the missing piece needed to acknowledge geometrical theories of space-time by considering the Minkowski metric as a measurement rule when observing the Euclidean space by means of a mixed basis consisting of elements of the vector- and the dual space.

2 Basics and notation

In the following, co- and contravariant vectors are written in bold, second order tensors are written in bold and have capital letters; greek indexes run from 0 to 3, latin indexes from 1 to 3, unless otherwise noted. To avoid confusion, elements of the dual space and their components are marked with a $()^*$.

Let $\mathbb{R}^4$ be a four-dimensional Euclidean vector space with states $\mathbf{v} \in \mathbb{R}^4$. These states can be expressed as a linear combination of a canonical basis $\{\mathbf{e}^*_\mu\} \in \mathbb{R}^{4*}$ with elements from the dual space (orig. [6], available at [7]; introductions e.g. in [8], [9]):

$$\mathbf{v} = \sum_\mu \mathbf{e}^*_\mu x^\mu \qquad \text{where} \qquad x^\mu = (\mathbf{v}^* \cdot \mathbf{e}^\mu) = \sum_\nu \left(v^*_\nu \cdot (e^\mu)^\nu\right) \qquad \mathbf{x} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \mathrel{\widehat{=}} \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}\,. \tag{2}$$

The scalar coefficients x^μ are called coordinates of the state $\mathbf{v}$ with respect to the basis $\{\mathbf{e}^*_\mu\}$. They can be summarized as a coordinate vector $\mathbf{x}$ (Eq. 2 on the right, visualized by four stacked boxes). The coordinates are generated by the dual basis $\{\mathbf{e}^\mu\} \in \mathbb{R}^4$, which is defined by $\left(e^*_\mu\right)_\nu = \sum_\xi g^{Eukl.}_{\nu\xi} \left(e^\mu\right)^\xi$ with the Euclidean metric tensor $g^{Eukl.}_{\nu\xi} = \delta_{\nu\xi}$.

Analogously, any dual state $\mathbf{w}^* \in \mathbb{R}^{4*}$ can be generated by a canonical dual basis:

$$\begin{aligned} \mathbf{w}^* = \sum_\mu y^*_\mu \mathbf{e}^\mu \qquad \text{where} \qquad y^*_\mu &= \left(\mathbf{e}^*_\mu \cdot \mathbf{w}\right) = \sum_\nu \left(\left(e^*_\mu\right)_\nu \cdot w^\nu\right) \\ \mathbf{y}^* &= \begin{pmatrix} y^*_0 & y^*_1 & y^*_2 & y^*_3 \end{pmatrix} \mathrel{\widehat{=}} \begin{array}{|c|c|c|c|}\hline & & & \\ \hline \end{array}\,. \end{aligned} \tag{3}$$

The coefficients y^*_μ are called dual coordinates of the dual state $\mathbf{w}$ with respect to the dual basis $\{\mathbf{e}^\mu\}$. They can be consolidated as a dual coordinate vector $\mathbf{y}^*$ (Eq. 3 last row, visualized by four boxes side by side).

Thus, scalar and tensor products can be directly defined and visualized (in coordinates):

$$\mathbf{y}^* \cdot \mathbf{x} = \sum_\mu y^*_\mu x^\mu \in \mathbb{R} \qquad\qquad \mathbf{x} \otimes \mathbf{y}^* = \mathbf{M} \in \mathbb{R}^{4\times4} \quad \text{with} \quad {M^\mu}_\nu = x^\mu y^*_\nu \tag{4}$$

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline \end{array} \cdot \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} = \boxed{} \qquad\qquad \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|}\hline & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}\,. \tag{5}$$

The state $\mathbf{v}$ does not vary when there's a change of basis. For this to be the case, the coordinates $\mathbf{x}$ must change inversely to the basis.

Changes of orthonormal bases in the four-dimensional Euclidean space are isomorphic to the Lie-group $SO(4)$, which can be represented by the special orthogonal matrices of fourth order $SO(4) = \{\mathbf{R} \in GL(4) \mid \mathbf{R^T R = 1\!\!1}, \det \mathbf{R} = 1\}$. Consider the change of basis from $\{\mathbf{e}^*_\nu\}$ to $\{\mathbf{e}'^*_\mu\}$. The elements of the new basis can be written as a linear combination of the elements of the old basis:

$$\mathbf{e}'^*_\mu = \sum_\nu \mathbf{e}^*_\nu \; R_\mu{}^\nu \,. \tag{6}$$

This translates into the following transformation rule for the coordinate vectors:

$$\mathbf{x' = R^{-1} \cdot x = R^T \cdot x} \qquad\qquad x'^\mu = \sum_\xi \left(R^{-1}\right)_\xi{}^\mu \; x^\xi = \sum_\xi R^\mu{}_\xi \; x^\xi \,. \tag{7}$$

The transpose of the matrix $\mathbf{R}$ is expressed by the index interchange. The inverse transformation of the coordinates is true for any change of basis; the fact that this corresponds to the transposed transformation matrix is a consequence of the orthonormality of the considered bases. The components of the transformation matrix and its inverse are:

$$R_\mu{}^\nu = \mathbf{e}'^*_\mu \cdot \mathbf{e}^\nu \qquad\qquad \left(R^{-1}\right)_\mu{}^\nu = R^\nu{}_\mu = \mathbf{e}^*_\mu \cdot \mathbf{e}'^\nu \,. \tag{8}$$

It is said that the basis $\{\mathbf{e}^*_\mu\}$ transforms covariantly, whereas the coordinates $\mathbf{x}$ transform contravariantly. As required, the state $\mathbf{v}$ remains unchanged:

$$\mathbf{v}' = \sum_\mu \mathbf{e}'^*_\mu x'^\mu = \sum_{\mu,\nu,\xi} \mathbf{e}^*_\nu \; R_\mu{}^\nu \; R^\mu{}_\xi \; x^\xi = \sum_{\nu,\xi} \mathbf{e}^*_\nu \; \delta^\nu_\xi \; x^\xi = \sum_\nu \mathbf{e}^*_\nu x^\nu = \mathbf{v} \,. \tag{9}$$

3 Definitions

Definition 1 (mixed basis)**.** *A mixed basis $\{\mathbf{e^0}, \mathbf{e}^*_1, \mathbf{e}^*_2, \mathbf{e}^*_3\}$ is defined as a basis where all elements belong to the canonical basis, except one which is replaced by its dual element, with* $(e^*_0)_\nu = \sum_\xi g^{Eukl.}_{\nu\xi} \; (e^0)^\xi$.

$$\mathbf{e^0} \mathrel{\widehat{=}} \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array} \qquad \mathbf{e}^*_1 \mathrel{\widehat{=}} \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array} \qquad \mathbf{e}^*_2 \mathrel{\widehat{=}} \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array} \qquad \mathbf{e}^*_3 \mathrel{\widehat{=}} \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \end{array} \,.$$

Definition 2 (mixed change of basis)**.** *Let a mixed change of basis be the change of basis between two mixed bases $\{\mathbf{e^0}, \mathbf{e}^*_1, \mathbf{e}^*_2, \mathbf{e}^*_3\}$ and $\{\mathbf{e'^0}, \mathbf{e}'^*_1, \mathbf{e}'^*_2, \mathbf{e}'^*_3\}$.*

Remark 1. *It is clear that mixed bases as in Definition 1 do not change like regular bases in Euclidean space; nor can the scalar or tensor product of the Euclidean space be applied to the mixed bases.*

The question now is whether it is possible to define a vector space in which the mixed bases can be described in a coherent mathematical manner. Hence the following proposition:

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4 Proposition and proof

Proposition 1. *A mixed basis* $\{\mathbf{e^0}, \mathbf{e_1^*}, \mathbf{e_2^*}, \mathbf{e_3^*}\}$ *in Euclidean space as the one in Definition 1 undergoing mixed changes of basis as the ones in Definition 2 behaves just like a regular basis* $\{\mathbf{f_\mu}\}$ *would in the Minkowski space, where the scalar product is defined as:*

$$\mathbf{y} \cdot \mathbf{x} = \sum_{\mu\nu} g_{\mu\nu} y^\mu x^\nu \qquad \text{with} \qquad g_{\mu\nu} = \operatorname{diag}(-1, +1, +1, +1) \ .$$

Proof. Let $\mathbb{R}^4$ be the Euclidean space with the canonical basis $\{\mathbf{e_\mu^*}\}$ and dual basis $\{\mathbf{e^\mu}\}$; $\mathbb{R}(3,1)$ the Minkowski space with the orthonormalized basis $\{\mathbf{f_\mu^*}\}$. It has to be shown that, when doing a change of basis, the element $\mathbf{e^0}$ of the dual Euclidean basis transforms like the element $\mathbf{f_0^*}$ of the Minkowski basis.

The regular changes of basis in Euclidean space $SO(4) = \{\mathbf{R} \in GL(4) \mid \mathbf{R^T R} = \mathbb{1}, \det \mathbf{R} = 1\}$ can be expressed as an exponential series:

$$\mathbf{R} = e^{t\mathbf{A}} \qquad\qquad \mathbf{R^{-1}} = \mathbf{R^T} = e^{-t\mathbf{A}} \ . \tag{10}$$

Where t is the parameter of the transformation that could be interpreted as a rotation angle and $\mathbf{A}$ is a skew symmetric matrix, where $\mathbf{A^T} = \mathbf{-A}$. The skew symmetry of these matrices is what finally leads to the orthogonality of the transformations and the coordinate vectors to change with the transposed transformation rule. More precisely, the matrices $\mathbf{A}$ build, together with the commutator $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$, the Lie-algebra $so(4)$ of the Lie-group $SO(4)$: the algebra that generates all infinitesimal orthogonal linear coordinate transformations, which through the exponential mapping span the whole of the $SO(4)$ Lie-group (for introductions see e.g. [10], [11]).

In the case of infinitesimal transformations (t very small) it is sufficient to only consider the first terms of the exponential series. The special role of the Lie-algebra is seen here:

$$\mathbf{R} = e^{t\mathbf{A}} \approx (\mathbf{1} + t\mathbf{A}) \qquad\qquad \mathbf{R^{-1}} = e^{-t\mathbf{A}} \approx (\mathbf{1} - t\mathbf{A}) \ . \tag{11}$$

The elements $\mathbf{A}$ of the Lie-algebra can once more be expressed as a linear combination of a basis, which in the case of $so(4)$ consists of six skew symmetrical matrices, e.g.:

$$\mathbf{L_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{L_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \mathbf{L_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

$$\mathbf{K_1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K_2} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{K_3} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

with

$$[\mathbf{L_i},\mathbf{L_j}] = \varepsilon_{ijk}\mathbf{L_k} \qquad [\mathbf{K_i},\mathbf{K_j}] = \varepsilon_{ijk}\mathbf{L_k} \qquad [\mathbf{L_i},\mathbf{K_j}] = \varepsilon_{ijk}\mathbf{K_k} \tag{14}$$

and therefore

$$t\mathbf{A} = \sum_{i=1,2,3} \left(t^i\mathbf{L_i} + s^i\mathbf{K_i}\right). \tag{15}$$

To apply a change of the mixed basis, a transformation rule has to be constructed which transforms the element $\mathbf{e_0}$ of the basis with the same coefficients that a coordinate vector $\mathbf{e^0}$ is transformed with. The infinitesimal change of a normalized coordinate vector $\mathbf{e^0} = (1,0,0,0)^\mathbf{T}$ can be written as follows:

$$\begin{aligned} \mathbf{e'^0} = (\mathbf{1} - t\mathbf{A})\,\mathbf{e^0} \qquad &= \left(\mathbf{1} - \sum_{i=1,2,3} s^i\mathbf{K_i}\right)\mathbf{e^0} \\ &= \left(\mathbf{1} + \sum_{i=1,2,3} s^i\mathbf{K_i^{-1}}\right)\mathbf{e^0} \end{aligned} \tag{16}$$

or in componentwise notation:

$$\begin{aligned} \left(e'^0\right)^\nu = \left({\delta_0}^\nu - t{A_0}^\nu\right)\left(e^0\right)^0 &= \left({\delta_0}^\nu - \sum_{i=1,2,3} s^i {(K_i)_0}^\nu\right)\left(e^0\right)^0 \\ &= \left({\delta_0}^\nu + \sum_{i=1,2,3} s^i {\left(K_i^{-1}\right)_0}^\nu\right)\left(e^0\right)^0. \end{aligned} \tag{17}$$

Where the (pseudo-)inverse $\mathbf{K_i^{-1}}$ is the inverse on the subspace spanned by $\mathbf{e_i^*}$ and $\mathbf{e_0^*}$. On the other hand, the transformation rule for $\mathbf{e_0^*}$ is:

$$\mathbf{e_0'^*} = \sum_\nu \mathbf{e_\nu^*}\left({\delta_0}^\nu + t{A_0}^\nu\right) = \sum_\nu \mathbf{e_\nu^*}\left({\delta_0}^\nu + \sum_{i=1,2,3} s^i\,{(K_i)_0}^\nu\right). \tag{18}$$

For the new mixed transformation to be useful, it must again function as a Lie algebra. Although it cannot be presumed that such a Lie algebra exists, if a corresponding algebra is found that fulfills the requirements, its existence is proven automatically.

Looking for a Lie algebra, it must satisfy by definition the transformation rule 18. By definition of the base transformations, the coordinates must also satisfy equation 17.

In order for $\mathbf{e_0^*}$ to transform additionally in the same way as $\mathbf{e^0}$, according to equations 17 (last term) and 18 (last term), the following must hold true: ${\left(K_i^{-1}\right)_0}^\nu = {(K_i)_0}^\nu$. Each $\mathbf{K_i}$ must therefore be its inverse in the subspace spanned by itself.

This condition cannot be fulfilled by regular Euclidean transformations, since in this case the $\mathbf{K_i}$ are skew symmetric, $\mathbf{K_i^{-1}} = -\mathbf{K_i}$.

To nonetheless find a change of basis which has the same effect on the element of the basis $\mathbf{e_0^*}$ as it has on the coordinate vector $\mathbf{e^0}$, one must depart from changes of basis in the ordinary Euclidean sense. Consider a new transformation rule (marked with a tilde):

$$t\widetilde{\mathbf{A}} = \sum_{i=1,2,3} \left(t^i \mathbf{L_i} + \tilde{s}^i \widetilde{\mathbf{K}}_\mathbf{i} \right) . \tag{19}$$

Different elements $\widetilde{\mathbf{K}}_\mathbf{i}$ of the basis are sought, which are their own inverses in their own spanned subspace. A possible choice is:

$$\widetilde{\mathbf{K}}_\mathbf{1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \widetilde{\mathbf{K}}_\mathbf{2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \widetilde{\mathbf{K}}_\mathbf{3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{20}$$

The new elements retain the form of the transformations for $\mathbf{e_0^*}$, yet without orthogonality, since the skew symmetry of the Lie-algebra's elements had to be abandoned.

Hence, a transformation rule was constructed which takes a basis of the Euclidean space $\{\mathbf{e_\mu^*}\}$ and changes it in the way a mixed basis $\{\mathbf{e^0, e_1^*, e_2^*, e_3^*}\}$ would change when exposed to regular Euclidean changes of basis. The contravariant transformation property of $\{\mathbf{e^0}\}$ was hereby transferred to $\{\mathbf{e_0}\}$.

The unchanged matrices $\{\mathbf{L_i}\}$ build, together with the new matrices $\{\widetilde{\mathbf{K}}_\mathbf{i}\}$ and the commutator as Lie bracket, the Lie-algebra $so(3,1)$, with its elements $\widetilde{\mathbf{A}} \in so(3,1)$. By means of the exponential mapping this Lie-Algebra translates into the Lorentz group $SO(3,1)$, where:

$$\mathbf{\Lambda} = e^{t\widetilde{\mathbf{A}}} \qquad\qquad \mathbf{\Lambda} \in SO(3,1) . \tag{21}$$

Yet the elements of the Lorentz group $\mathbf{\Lambda}$ are defined as those changes of basis taking place within Minkowski space $\mathbb{R}(3,1)$, having metric tensor $g_{\mu\nu} = \mathrm{diag}\,(-1,+1,+1,+1)$.

Thus, if one creates a transformation in Euclidean space, where one of the elements of the Euclidean basis transforms contravariantly, the defined basis will be equivalent to a regular basis of the Minkowski space and the discovered transformation will be equivalent to regular changes of bases in the Minkowski space. □

5 Results

A transformation which changes mixed bases into each other according to Definition 2, was created. For this, orthogonality had to be abandoned and Euclidean lengths are no longer preserved. It has been shown that the changes between mixed bases are transferable to the Minkowski space, where a coherent description, preserving the Minkowski length element, is possible.

This result corresponds to a mapping between Euclidean space and Minkowski space. Complex distances in the Minkowski space can thus be interpreted geometrically as effects resulting from the switch of a basis' element with its corresponding dual element.

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6 Discussion

The mapping between Euclidean and Minkowski space also implies a phenomenological mapping. As a consequence, one important realization is that all effects arising within the Minkowski space can be explained by purely metric quantities – in other words, quantities obtained through a measurement of distance. The cause of the effects observed can thus be directly transferred to the act of measurement within a mixed basis.

This may have far-reaching consequences for physics, especially when interpreting relativistic effects. However this topic is not pursued further here, as the purpose of this article is only to test the self-coherence of the mathematical foundation.

On my private homepage `http://elasticuniverse.org` an analogous concept with direct applications to physics can be found in unreviewed form. The proof developed here, in form of an earlier, less accurate version, is employed. This article concerns itself with its clarification and the generalization of its results.

However, some questions remain unanswered: How can the mapping be applied in real (finite) problems, how can it be used for calculations, and what influence does the topology of the two involved spaces have?

6.1 Next steps

Geometrically, changing the sign of an element of the metric tensor corresponds to switching the role of basis and coordinate in the affected component. This in principle defines a mapping between the Euclidean and Minkowski space. But this mapping is not a homomorphism, it might not be bijective or linear. In a next step, therefore, this mapping will be examined in more detail.

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