EXTENSION OF EIGENVALUE PROBLEMS ON GAUSS MAP OF RULED SURFACES

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Abstract. A finite-type immersion or smooth map is a nice tool to classify submanifolds of Euclidean space, which comes from eigenvalue problem of immersion. The notion of generalized 1-type is a natural generalization of those of 1-type in the usual sense and pointwise 1-type. We classify ruled surfaces with generalized 1-type Gauss map as part of a plane, a circular cylinder, a cylinder over a base curve of an infinite type, a helicoid, a right cone and a conical surface of $G$-type.

1. Introduction

Nash’s embedding theorem enables us to study Riemannian manifolds extensively by regarding a Riemannian manifold as a submanifold of Euclidean space with sufficiently high codimension. By means of such a setting, we can have rich geometric information from the intrinsic and extrinsic properties of submanifolds of Euclidean space. Inspired by the degree of algebraic varieties, B.-Y. Chen introduced the notion of order and type of submanifolds of Euclidean space. Furthermore, he developed the theory of finite-type submanifolds and estimated the total mean curvature of compact submanifolds of Euclidean space in the late 1970s ([3]).

In particular, the notion of finite-type immersion is a direct generalization of eigenvalue problem relative to the immersion of a Riemannian manifold into a Euclidean space: Let $x: M \to \mathbb{E}^m$ be an isometric immersion of a submanifold $M$ into the Euclidean $m$-space $\mathbb{E}^m$ and $\Delta$ the Laplace operator of $M$ in $\mathbb{E}^m$. The submanifold $M$ is said to be of finite-type if $x$ has a spectral decomposition by $x = x_0 + x_1 + \ldots + x_k$, where $x_0$ is a constant vector and $x_i$ are the vector fields satisfying $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$ ($i = 1, 2, \ldots, k$). If the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different, it is called $k$-type. Since this notion was introduced, many works have been made in this area (see [3, 5]). This notion of finite-type immersion was naturally extended to that of pseudo-Riemannian manifolds in pseudo-Euclidean space and it was also applied to

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smooth maps, particularly the Gauss map defined on submanifolds of Euclidean space or pseudo-Euclidean space ([1, 2, 3, 10, 11]).

Regarding the Gauss map of finite-type, B.-Y. Chen and P. Piccini ([6]) initiated to study submanifolds of Euclidean space with finite-type Gauss map and classified compact surfaces with 1-type Gauss map, that is, \( \Delta G = \lambda (G + C) \), where \( C \) is a constant vector and \( \lambda \in \mathbb{R} \). Since then, quite a few works on ruled surfaces and ruled submanifolds with finite-type Gauss map in Euclidean space or pseudo-Euclidean space have been established ([1, 2, 3, 4, 7, 8, 9, 12, 13, 14, 15]).

However, some surfaces including a helicoid and a right cone in Euclidean 3-space have an interesting property concerning the Gauss map: The helicoid in \( \mathbb{E}^3 \) parameterized by

\[
x(u, v) = (u \cos v, u \sin v, av), \quad a \neq 0
\]

has the Gauss map and its Laplacian respectively given by

\[
G = \frac{1}{\sqrt{a^2 + u^2}}(a \sin v, -a \cos v, u)
\]

and

\[
\Delta G = \frac{2a^2}{(a^2 + u^2)^2}G.
\]

The right (or circular) cone \( C_a \) with parametrization

\[
x(u, v) = (u \cos v, u \sin v, au), \quad a \geq 0
\]

has the Gauss map

\[
G = \frac{1}{\sqrt{1 + a^2}}(a \cos v, a \sin v, -1)
\]

which satisfies

\[
\Delta G = \frac{1}{u^2}(G + (0, 0, \frac{1}{\sqrt{1 + a^2}}))
\]

(cf. [4, 8]). The Gauss maps above are similar to 1-type, but it is not of 1-type Gauss map in the usual sense. Based upon such cases, B.-Y. Chen and the present authors defined the notion of pointwise 1-type Gauss map ([4]).

**Definition 1.1.** A submanifold \( M \) in \( \mathbb{E}^m \) is said to have pointwise 1-type Gauss map if the Gauss map \( G \) of \( M \) satisfies

\[
\Delta G = f(G + C)
\]

for some non-zero smooth function \( f \) and a constant vector \( C \). In particular, if \( C \) is zero, then the Gauss map is said to be of pointwise 1-type of the first kind. Otherwise, it is said to be of pointwise 1-type of the second kind.

Let \( p \) be a point of \( \mathbb{E}^3 \) and \( \beta = \beta(s) \) a unit speed curve such that \( p \) does not lie on \( \beta \). A surface parametrized by

\[
x(s, t) = p + t\beta(s)
\]
is called a conical surface. A typical conical surface is a right cone and a plane.

Let us consider a following example of a conical surface.

**Example 1.2.** ([15]) Let $M$ be a surface in $\mathbb{E}^3$ parameterized by

$$x(s, t) = (s \cos^2 t, s \sin t \cos t, s \sin t).$$

Then, the Gauss map $G$ can be obtained by

$$G = \frac{1}{\sqrt{1 + \cos^2 t}}(-\sin^3 t, (2 - \cos^2 t) \cos t, -\cos^2 t).$$

After a considerably long computation, its Laplacian turns out to be

$$\Delta G = fG + gC$$

for some non-zero smooth functions $f$, $g$ and a constant vector $C$. The surface $M$ is a kind of conical surfaces generated by a spherical curve $\beta(t) = (\cos^2 t, \sin t \cos t, \sin t)$ on the unit sphere $S^2(1)$ centered at the origin.

Inspired by such an example, we would like to generalize the notion of pointwise 1-type Gauss map as follows:

**Definition 1.3.** ([15]) The Gauss map $G$ of a submanifold $M$ in $\mathbb{E}^m$ is of generalized 1-type if the Gauss map $G$ of $M$ satisfies

$$\Delta G = fG + gC$$

(1.1)

for some non-zero smooth functions $f$, $g$ and a constant vector $C$.

Especially we define a conical surface with generalized 1-type Gauss map.

**Definition 1.4.** A conical surface with generalized 1-type Gauss map is called a conical surface of $G$-type.

**Remark 1.5.** ([15]) A conical surface of $G$-type is constructed by the functions $f$, $g$ and the constant vector $C$ by solving the differential equations generated by (1.1).

In the present paper, we classify a ruled surface with generalized 1-type Gauss map in $\mathbb{E}^3$ as a plane, a circular cylinder, a cylinder over a base curve of an infinite type generated by the given function $f$, $g$ and the constant vector $C$, a helicoid, a right cone and a conical surface of $G$-type.

#### 2. Preliminaries

Let $M$ be a surface of the 3-dimensional Euclidean space $\mathbb{E}^3$. The map $G : M \to S^2(1) \subset \mathbb{E}^3$ which maps each point $p$ of $M$ to a point $G_p$ of $S^2(1)$ by identifying the unit normal vector $N_p$ to $M$ at the point with $G_p$ is called the Gauss map of the surface $M$, where $S^2(1)$ is the unit sphere in $\mathbb{E}^3$ centered at the origin.
For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the metric on $M$, we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. $\tilde{G}$ ) the inverse matrix (resp. the determinant) of the matrix $(\tilde{g}_{ij})$. Then the Laplacian $\Delta$ on $M$ is in turn given by

$$\Delta = -\frac{1}{\sqrt{\tilde{G}}} \sum_{i,j} \frac{\partial}{\partial x^i} \left( \sqrt{\tilde{G}} \tilde{g}^{ij} \frac{\partial}{\partial x^j} \right). \quad (2.1)$$

Now, we define a ruled surface $M$ in the 3-dimensional Euclidean space $\mathbb{E}^3$. Let $\alpha = \alpha(s)$ be a regular curve in $\mathbb{E}^3$ defined on an open interval $I$ and $\beta = \beta(s)$ a transversal vector field to $\alpha'(s)$ along $\alpha$. Then the ruled surface $M$ can be parameterized by

$$x(s,t) = \alpha(s) + t\beta(s), \quad s \in I, \quad t \in \mathbb{R}$$

satisfying $\langle \alpha', \beta \rangle = 0$ and $\langle \beta, \beta \rangle = 1$, where $'$ denotes $d/ds$. The curve $\alpha$ is called the base curve and $\beta$ the director vector field or ruling. In particular, $M$ is said to be cylindrical if $\beta$ is constant, or, non-cylindrical otherwise.

Throughout this paper, we assume that all the geometric objects are smooth and all surfaces are connected unless otherwise stated.

3. Cylindrical ruled surfaces in $\mathbb{E}^3$ with generalized 1-type Gauss map

In this section, we study the cylindrical ruled surfaces with generalized 1-type Gauss map in $\mathbb{E}^3$.

Let $M$ be a cylindrical ruled surface in $\mathbb{E}^3$. Without loss of generality, we assume that $M$ is parameterized by

$$x(s,t) = \alpha(s) + t\beta,$$

where $\alpha(s) = (\alpha_1(s), \alpha_2(s), 0)$ is a plane curve parameterized by the arc-length $s$ and $\beta$ a constant vector, namely $\beta = (0, 0, 1)$. In this case, the Gauss map $G$ of $M$ is given by

$$G = \alpha' \times \beta = (\alpha_2', -\alpha_1', 0) \quad (3.1)$$

and the Laplacian $\Delta G$ of the Gauss map $G$ using (2.1) is obtained by

$$\Delta G = (-\alpha_2'', \alpha_1'', 0), \quad (3.2)$$

where $'$ stands for $d/ds$.

From now on, $'$ denotes the differentiation with respect to the parameter $s$ relative to the base curve.

Suppose that the Gauss map $G$ of $M$ is of generalized 1-type, i.e., $G$ satisfies equation (1.1). We now consider two cases for equation (1.1).

Case 1. $f = g$.

In this case, the Gauss map $G$ is of pointwise 1-type described in Definition 1.1. According to Classification Theorem in [8] and [9], we have the ruled surface $M$ is
part of a plane, a circular cylinder or a cylinder over a base curve of an infinite-type satisfying
\[
\sin^{-1} \left( \frac{c^2 f^{-\frac{1}{3}} - 1}{\sqrt{c_1^2 + c_2^2}} \right) - \sqrt{c_1^2 + c_2^2 - \left( c^2 f^{-\frac{1}{3}} - 1 \right)^2} = \pm c^3 (s + k),
\]  
(3.3)
where \( C = (c_1, c_2, 0) \), \( c \neq 0 \) and \( k \) are constants.

Case 2. \( f \neq g \).

By a direct computation using (3.1) and (3.2), we see that the third component \( c_3 \) of the constant vector \( C \) is zero. We put \( C = (c_1, c_2, 0) \). Then, we have the following system of ordinary differential equations
\[
-\alpha''''_2 = f \alpha'_2 + gc_1, \\
\alpha''''_1 = -f \alpha'_1 + gc_2.
\]  
(3.4)
Since \( \alpha \) is a unit speed curve, that is, \((\alpha'_1)^2 + (\alpha'_2)^2 = 1\), we may put
\[
\alpha'_1(s) = \cos \theta(s) \quad \text{and} \quad \alpha'_2(s) = \sin \theta(s)
\]
for a smooth function \( \theta = \theta(s) \) of \( s \). It enables equation (3.4) to be rewritten in the form
\[
(\theta')^2 \sin \theta - \theta'' \cos \theta = f \sin \theta + gc_1, \\
(\theta')^2 \cos \theta + \theta'' \sin \theta = f \cos \theta - gc_2,
\]
which give
\[
(\theta')^2 = f + g(c_1 \sin \theta - c_2 \cos \theta), \\
-\theta'' = g(c_1 \cos \theta + c_2 \sin \theta).
\]  
(3.5)
(3.6)
Taking the derivative of (3.5), we have
\[
2\theta' \theta'' = f' + g'(c_1 \sin \theta - c_2 \cos \theta) + g(c_1 \cos \theta + c_2 \sin \theta) \theta'.
\]
With the help of (3.5) and (3.6) it implies that
\[
\frac{3}{2} (\theta^2)' = f' + \frac{g'}{g}((\theta')^2 - f).
\]
Solving the above differential equation, we get
\[
\theta'(s)^2 = kg^\frac{2}{3}(s) + \frac{2}{3}g^\frac{2}{3}(s) \int g^{-\frac{2}{3}}(s) f(s) \left( \frac{f'}{f} - \frac{g'}{g} \right) ds, \quad k(\neq 0) \in \mathbb{R}.
\]
If we put
\[
\theta'(s) = \pm \sqrt{p(s)},
\]  
(3.7)
where \( p(s) = |kg^\frac{2}{3}(s) + \frac{2}{3}g^\frac{2}{3}(s) \int g^{-\frac{2}{3}}(s) f(s) \left( \frac{f'}{f} - \frac{g'}{g} \right) ds| \) for some non-zero constant \( k \), we get a base curve \( \alpha \) of \( M \) as follows:
\[
\alpha(s) = \left( \int \cos \theta(s) ds, \int \sin \theta(s) ds, 0 \right),
\]  
(3.8)
where \( \theta(s) = \pm \int \sqrt{p(s)} \, ds \). In fact, \( \theta' \) is the signed curvature of the base curve \( \alpha \) which is precisely determined by the given functions \( f, \ g \) and the constant vector \( C \).

Note that if \( f \) and \( g \) are constant, the Gauss map \( G \) is of 1-type in the usual sense. In this case, the signed curvature of the base curve \( \alpha \) is non-zero constant. So, the cylindrical ruled surface \( M \) is part of a circular cylinder.

Suppose that one of the functions \( f \) and \( g \) is not constant. Since a plane curve in \( \mathbb{E}^3 \) is of finite-type if and only if it is part of a straight line or a circle, the base curve defined by (3.8) is of an infinite-type ([5]). Thus, by putting together Cases 1 and 2, we have a classification theorem of cylindrical ruled surface with generalized 1-type Gauss map in \( \mathbb{E}^3 \).

**Theorem 3.1.** Let \( M \) be a cylindrical ruled surface in \( \mathbb{E}^3 \). Suppose that \( M \) has generalized 1-type Gauss map. Then it is an open part of a plane, a circular cylinder or a cylinder over a base curve of an infinite-type satisfying (3.3), (3.7) and (3.8).

### 4. Classification Theorem

In this section, we examine non-cylindrical ruled surfaces with generalized 1-type Gauss map in \( \mathbb{E}^3 \) and obtain a classification theorem.

Let \( M \) be a non-cylindrical ruled surface in \( \mathbb{E}^3 \) parameterized by a base curve \( \alpha \) and a director vector field \( \beta \). Up to a rigid motion, its parametrization is given by

\[
x(s, t) = \alpha(s) + t\beta(s)
\]

such that \( \langle \alpha', \beta \rangle = 0 \), \( \langle \beta, \beta \rangle = 1 \) and \( \langle \beta', \beta' \rangle = 1 \). Then, we have the natural frame \( \{x_s, x_t\} \) given by \( x_s = \alpha'(s) + t\beta'(s) \) and \( x_t = \beta(s) \).

From this setting, we have an orthonormal frame \( \{\beta, \beta', \beta \times \beta'\} \). For later use, we define the smooth functions \( q, u, Q \) and \( R \) as follows:

\[
q = \langle x_s, x_s \rangle, \quad u = \langle \alpha', \beta' \rangle, \quad Q = \langle \alpha', \beta \times \beta' \rangle, \quad R = \langle \beta'', \beta \times \beta' \rangle.
\]

In terms of the orthonormal frame \( \{\beta, \beta', \beta \times \beta'\} \), we obtain

\[
\begin{align*}
\alpha' &= u\beta' + Q\beta \times \beta', \\
\beta'' &= -\beta + R\beta \times \beta', \\
\alpha' \times \beta &= Q\beta' - u\beta \times \beta', \\
\beta \times \beta'' &= -R\beta',
\end{align*}
\]

(4.1)

from which, the smooth function \( q \) is given by

\[
q = t^2 + 2ut + u^2 + Q^2
\]

and the Gauss map \( G \) of \( M \) is obtained by

\[
G = \frac{x_s \times x_t}{||x_s \times x_t||} = q^{-1/2} (Q\beta' - (u + t)\beta \times \beta').
\]

(4.2)
Let $H$ and $K$ be the mean curvature and the Gaussian curvature of $M$ respectively. By straightforward computation in using the first and second fundamental forms, they are given as follows:

$$H = \frac{1}{2}q^{-3/2}(-Rt^2 - (2uR + Q')t + u'Q - Q^2 R - u^2 R - uQ'),$$
$$K = -\frac{Q^2}{q^2}. \quad (4.3)$$

**Remark 4.1.** If $R \equiv 0$, then the director vector field $\beta$ is a plane curve.

The following formula is well known with respect to the Laplacian of the Gauss map of $M$ in $\mathbb{E}^3$, which are easily obtained by applying the Gauss formula and the Weingarten formula:

$$\Delta G = 2\text{grad}H + (\text{tr}A^2)G,$$

where $A$ denotes the shape operator of the surface $M$.

From (4.3), we get

$$2\text{grad}H = 2e_1(H)e_1 + 2e_2(H)e_2 = q^{-3}B_1e_1 + q^{-5/2}A_1e_2 = q^{-7/2}(qA_1\beta + (u + t)B_1\beta' + QB_1\beta \times \beta'),$$

where $e_1 = \frac{x_1}{||x||}$, $e_2 = \frac{x_2}{||x||}$,

$$A_1 = Rt^3 + (3uR + 2Q')t^2 + (Q^2 R - 3u'Q + 3u^2 R + 4uQ')t + (uQ^2 R - 3uu'Q + u^3 R + 2u^2 Q' - Q^2 Q'),$$

$$B_1 = 3(u' + uu' + QQ')\{Rt^2 + (2uR + Q')t - u'Q + Q^2 R + u^2 R + uQ'\} + (t^2 + 2ut + u^2 + Q^2\{R' - (2u'R + 2uR' + Q'')t + u''Q - 2QQ'R - Q^2 R' - 2uu'R - u^2 R' - uQ''\}. $$

We also have

$$\text{tr}A^2 = q^{-3}D_1,$$

where

$$D_1 = \{-Rt^2 - (2uR + Q')t - u(uR + Q') + Q(u' - QR)\}^2 + 2Q^2(t^2 + 2ut + u^2 + Q^2).$$

Thus we obtain the Laplacian $\Delta G$ of the Gauss map $G$ of $M$ given by

$$\Delta G = q^{-7/2}[qA_1\beta + ((u + t)B_1 + D_1Q)\beta' + (QB_1 - D_1(u + t))\beta \times \beta']. \quad (4.4)$$

Suppose that $M$ has generalized 1-type Gauss map $G$. Then, with the help of (1.1), (4.2) and (4.4), we obtain

$$q^{-7/2}[qA_1\beta + ((u + t)B_1 + D_1Q)\beta' + (QB_1 - D_1(u + t))\beta \times \beta'] = f q^{-1/2}\{Q\beta' - (u + t)\beta \times \beta'\} + gC \quad (4.5)$$
for some non-zero smooth functions \( f, g \) and a constant vector \( \mathbb{C} \).

If we take the inner product to equation (4.5) with \( \beta, \beta' \) and \( \beta \times \beta' \) respectively, then we get the following:

\[
q^{-5/2}A_1 = g \langle \mathbb{C}, \beta \rangle, \quad (4.6)
\]

\[
q^{-7/2}\{(u + t)B_1 + D_1Q\} =fq^{-1/2}Q + g \langle \mathbb{C}, \beta' \rangle, \quad (4.7)
\]

\[
q^{-7/2}\{QB_1 - (u + t)D_1\} = -fq^{-1/2}(u + t) + g \langle \mathbb{C}, \beta \times \beta' \rangle. \quad (4.8)
\]

Combining equations (4.6), (4.7) and (4.8), we have

\[
qA_1\omega_2 - \{(u + t)B_1 + D_1Q\}\omega_1 +fq^3Q\omega_1 = 0, \quad (4.9)
\]

\[
qA_1\omega_3 - \{QB_1 - (u + t)D_1\}\omega_1 -fq^3(u + t)\omega_1 = 0, \quad (4.10)
\]

\[
\{(u + t)B_1 + D_1Q\}\omega_3 - \{QB_1 - (u + t)D_1\}\omega_2 -fq^3\{Q\omega_3 + (u + t)\omega_2\} = 0, \quad (4.11)
\]

where we have put \( \omega_1 = \langle \mathbb{C}, \beta \rangle, \omega_2 = \langle \mathbb{C}, \beta' \rangle \) and \( \omega_3 = \langle \mathbb{C}, \beta \times \beta' \rangle \).

On the other hand, differentiating a constant vector \( \mathbb{C} = \omega_1 \beta + \omega_2 \beta' + \omega_3 \beta \times \beta' \) with respect to the parameter \( s \) and using (4.1), we get

\[
\omega_1' - \omega_2 = 0, \quad (4.12)
\]

\[
\omega_2' + \omega_2 R = 0, \quad (4.12)
\]

\[
\omega_1 + \omega_2' - \omega_3 R = 0.
\]

Combining equations (4.9) and (4.10), we obtain

\[
A_1\{\omega_2(u + t) + \omega_3Q\} - B_1\omega_1 = 0. \quad (4.13)
\]

First of all, we consider the case of \( R \equiv 0 \).

**Theorem 4.2.** Let \( M \) be a non-cylindrical ruled surface in \( \mathbb{E}^3 \) with generalized 1-type Gauss map. If \( R \equiv 0 \), then \( M \) is part of a plane or a helicoid.

**Proof.** If the constant vector \( \mathbb{C} \) is zero in the definition given by (1.1), then the Gauss map \( G \) is nothing but of pointwise 1-type Gauss map of the first kind. By Characterization Theorem of a ruled surface with pointwise 1-type Gauss map of the first kind, \( M \) is part of a helicoid ([8]).

We now assume that the constant vector \( \mathbb{C} \) is non-zero. In this case, we will show \( Q \equiv 0 \) on \( M \) and thus \( M \) is part of a plane due to (4.3).

Suppose that the open subset \( U = \{ s \in \text{dom}(\alpha)|Q(s) \neq 0 \} \) of \( \mathbb{R} \) is not empty. Then, on a component \( U_C \) of \( U \), we have from (4.12) that \( \omega_3 \) is a constant and \( \omega_1'' = -\omega_1 \).

Observing equation (4.13), the left side is a polynomial in \( t \) with functions of \( s \) as the coefficients. Hence the leading coefficient must vanish and \( \omega^2_1Q' \) is a constant on \( U_C \) with the help of (4.12).

Next, by examining the coefficient of the term involving \( t^2 \) in (4.13), we obtain

\[
3\omega_2u'Q' - 2\omega_3QQ' + 3\omega_1u'Q' + \omega_1u''Q = 0. \quad (4.14)
\]
Similarly as above, from the coefficient of the linear term in $t$ of (4.13) with the help of (4.14), we get
\[ \omega_2 QQ' + \omega_3 u'Q - \omega_1 (u')^2 + \omega_1 (Q')^2 = 0. \]  
(4.15)
Also, the constant term in (4.13) with respect to the parameter $t$ is automatically zero. If we make use of (4.14), we obtain
\[ Q[\omega_1 \{3u(u')^2 + 3u'QQ' - 3u(Q')^2 - u''Q^2\} - 3\omega_2 uQQ' - \omega_3 (3uu'Q + Q^2Q')] = 0. \]
Hence, on $U_C$, we have
\[ \omega_1 \{3u(u')^2 + 3u'QQ' - 3u(Q')^2 - u''Q^2\} - 3\omega_2 uQQ' - \omega_3 (3uu'Q + Q^2Q') = 0. \]  
(4.16)
Using (4.14) and (4.15), equation (4.16) can be reduced to
\[ 2\omega_1 u'Q' + \omega_2 u'Q - \omega_3 QQ' = 0. \]  
(4.17)
Suppose that there is a point $s_0 \in U_C$ such that $u'(s_0) \neq 0$. Then, $u'(s) \neq 0$ everywhere on an open interval $I$ containing $s_0$. So, (4.15) yields
\[ \omega_3 Q = \frac{1}{u'} \{\omega_1 (u')^2 - \omega_1 (Q')^2 - \omega_2 QQ'\}. \]  
(4.18)
Putting (4.18) into (4.17), $(u^2 + Q^2)(\omega_2 Q + \omega_1 Q') = 0$, which implies $\omega_2 Q + \omega_1 Q' = 0$. Since $\omega_2 = \omega_1'$, we see that $\omega_1 Q$ is constant on $I$.

If $\omega_1 \equiv 0$ on some subinterval $J$ in $I$, $\omega_2 = 0$ on $J$. (4.15) gives $\omega_3 = 0$ on $J$. Since $C$ is a constant vector, $\mathbb{C}$ is zero vector, which is a contradiction. Thus, without loss of generality we may assume that $\omega_1 \neq 0$ everywhere on $I$ and it is of the form $\omega_1 = k_1 \cos(s + s_1)$ for some non-zero constant $k_1$ and $s_1 \in \mathbb{R}$. Since $\omega_2 Q$ is constant and $\omega_1 Q$ is constant on $I$, $\omega_1$ must be zero on $I$, which contradicts $\omega_1 = k_1 \cos(s + s_1)$ for some non-zero constant $k_1$. Therefore, the open interval $I$ is empty and thus $u' = 0$ on $U_C$. If we take it into account of (4.15) and (4.17), we get $Q'(\omega_2 Q + \omega_1 Q') = 0$ and $\omega_3 Q' = 0$, respectively.

Suppose that $Q'(s_2) \neq 0$ at some point $s_2 \in U_C$. Then $\omega_3 = 0$ and $\omega_1 Q$ is a constant on an open interval $J_1$ containing $s_2$. Similarly as above, since $\omega_2 Q'$ and $\omega_1 Q$ are constant on $J_1$, it follows that $\omega_1 = 0$ By (4.12), $\omega_2$ is zero. Hence the constant vector $C$ is zero, a contradiction. Thus $J_1$ is empty. Therefore, $Q$ is constant on $U_C$. By continuity, $Q$ is either a non-zero constant or zero on $M$. Because of (4.3), $M$ is minimal and it is an open part of a helicoid, which means that the Gauss map is of pointwise 1-type of the first kind. Therefore, the open subset $U$ is empty. Consequently, $Q$ is zero on $M$. Hence, $M$ is an open part of a plane. 

Now, without loss of generality we may assume that the function $R$ is not vanishing everywhere.
If \( f = g \), the non-cylindrical ruled surface \( M \) has pointwise 1-type Gauss map which is characterized as an open part of a right cone including the case that \( M \) is a plane or a helicoid depending upon whether the constant vector \( C \) is non-zero or zero ([7]).

From now on, we may assume the constant vector \( C \) is non-zero and \( f \neq g \) unless otherwise stated. Similarly as before, the leading coefficient of the polynomial in the left side of equation (4.13) in \( t \) with functions of \( s \) as the coefficients is zero and we get

\[
\omega_2 R + \omega_1 R' = 0. \tag{4.19}
\]

Since \( \omega'_1 = \omega_2 \) in (4.12), we see that \( \omega_1 R \) is constant. Also, the coefficient of the term involving \( t^3 \) in (4.13) must be zero. With the help of (4.19), we get

\[
2\omega_2 Q' + \omega_3 Q R - \omega_1 u'R + \omega_1 Q'' = 0. \tag{4.20}
\]

If we examine the coefficient of the term involving \( t^2 \) in (4.13), using (4.19) and (4.20) we obtain

\[
\omega_1 Q^2 R' - 3\omega_2 u'Q + 2\omega_3 Q Q' - \omega_1 QQ'R - 3\omega_1 u'Q' - \omega_1 u''Q = 0. \tag{4.21}
\]

Furthermore, from the coefficient of the linear term in \( t \) in (4.13) with the help of (4.19), (4.20) and (4.21), we also get

\[
Q\{\omega_2 QQ' + \omega_3 u'Q - \omega_1 (u')^2 + \omega_1 (Q')^2\} = 0. \tag{4.22}
\]

Consider an open set \( V = \{ s \in \text{dom}(\alpha)|Q(s) \neq 0\} \). Suppose that \( V \) is not empty. Equation (4.22) gives that

\[
\omega_2 QQ' + \omega_3 u'Q - \omega_1 (u')^2 + \omega_1 (Q')^2 = 0. \tag{4.23}
\]

Moreover, considering the constant term with respect to \( t \) in (4.13) and using (4.19), (4.20) and (4.21), we obtain

\[
Q\{3u(u')^2 + 3u'QQ' - Q^2 Q'R - 3u(Q')^2 - u''Q^2 + Q^3 R'\}
- 3\omega_2 uQQ' - \omega_3 (3uu'Q + Q^2 Q') = 0.
\]

Hence, on the open subset \( V \) in \( \mathbb{R} \),

\[
\omega_1 \{3u(u')^2 + 3u'QQ' - Q^2 Q'R - 3u(Q')^2 - u''Q^2 + Q^3 R'\}
- 3\omega_2 uQQ' - \omega_3 (3uu'Q + Q^2 Q') = 0. \tag{4.24}
\]

Applying (4.21) and (4.23) to (4.24), we have

\[
2\omega_1 u'Q' + \omega_2 u'Q - \omega_3 QQ' = 0. \tag{4.25}
\]

On the other hand, since \( \omega_3 R = \omega_1 + \omega_2' \) in (4.12), (4.20) becomes

\[
(\omega_1 Q)' + \omega_1 Q - \omega_1 u'R = 0. \tag{4.26}
\]

Now, suppose the open subset \( V_1 = \{ s \in V|u'(s) \neq 0\} \) is not empty. Then (4.23) yields

\[
\omega_3 Q = \frac{1}{u'}\{\omega_1 (u')^2 - \omega_1 (Q')^2 - \omega_2 QQ'\}. \tag{4.27}
\]
Putting (4.27) into (4.25), $(u'^2 + Q'^2)(\omega_2 Q + \omega_1 Q') = 0$ and thus $\omega_2 Q + \omega_1 Q' = 0$. Therefore, $\omega_1 Q$ is constant on a component $C$ of $V_1$. From (4.26), we get $\omega_1 Q = \omega_1 u'R$.

If $\omega_1 \equiv 0$ on an open interval $\bar{I} \subset C$, the constant vector $C$ is zero on $M$, a contradiction. Thus, $\omega_1 \neq 0$ and so $Q = u'R$ on $C$. The fact that $\omega_1 Q$ and $\omega_1 R$ are constant on $C$ implies that $u'$ is a non-zero constant on $C$. Then, (4.21) and (4.25) are simplified as follows:

$$\omega_1 Q^2 R' + 2\omega_3 QQ' - \omega_1 QQ'R = 0,$$

$$\omega_1 u'Q' - \omega_3 QQ' = 0. \tag{4.29}$$

Putting $Q = u'R$ into (4.28), $\omega_3 Q' = 0$ is derived. Thus, (4.29) implies that $\omega_1 Q' = 0$ and so $Q' = 0$ on $C$. Hence, $Q$ and $R$ are both non-zero constants on $C$.

On the other hand, without difficulty, we can show that the torsion of the director vector field $\beta = \beta(s)$ viewing as a curve is zero and so $\beta$ is part of a plane curve which is a small circle on the unit sphere centered at the origin with the normal curvature -1 and the geodesic curvature $R$ on $C$. Without loss of generality, we may put

$$\beta(s) = \frac{1}{p}(\cos \psi s, \sin \psi s, R)$$

on $C$, where we have put $p = \sqrt{1 + R^2}$. Then, $u = (\alpha', \beta') = -\alpha'_1 \sin \psi s + \alpha'_2 \cos \psi s$, where $\alpha'(s) = (\alpha'_1(s), \alpha'_2(s), \alpha'_3(s))$. Therefore, on $C$, we get

$$u' = - (\alpha''_1 + \alpha'_2 p) \sin \psi s + (\alpha''_2 - \alpha'_1 p) \cos \psi s,$$

from which, we see that $u' = 0$ on $C \subset V_1$, contradiction. Hence, $V_1$ is empty and so $u' = 0$ on $V$. Then, (4.20), (4.23) and (4.25) can be respectively reduced to

$$2\omega_2 Q' + \omega_3 QR + \omega_1 Q'' = 0, \tag{4.30}$$

$$\omega_2 QQ' + \omega_1 (Q')^2 = 0, \tag{4.31}$$

$$\omega_3 QQ' = 0. \tag{4.32}$$

Suppose that $Q'(\tilde{s}_0) \neq 0$ at a point $\tilde{s}_0$ in $V$. From (4.31) and (4.32), $\omega_3 = 0$ and $\omega_1 Q$ is a constant on an open interval $\bar{J} \subset V$ containing $\tilde{s}_0$. Hence, $\omega_2 Q = 0$ is derived from (4.30). Therefore, $\omega_2 Q' = 0$ on $\bar{J}$. The third equation of (4.12) yields $\omega_1 = 0$. It follows that $\omega_2 = 0$. Since $C$ is a constant vector, $C$ is zero on $M$, a contradiction. So, $Q' = 0$ on $V$. Thus, $Q$ is non-zero constant on each component of $V$. If we consider (4.20) and (4.21), we have

$$\omega_3 R = 0 \quad \text{and} \quad \omega_1 R' = 0.$$ 

Since $R \neq 0$, $\omega_3 = 0$ on each component of $V$. By (4.19), $\omega_2 R = 0$, which yields that $C$ is zero on $M$. It is a contradiction. Hence, the open subset $V$ of $\Omega$ is empty and the function $Q$ is vanishing on $M$. Thus, $M$ is flat due to (4.3). Since the ruled surface $M$ is non-cylindrical, $M$ is one of an open part of a tangent developable surface or a conical surface. One of the authors proved that tangential developable surfaces do not have generalized 1-type Gauss map and a conical surface of $G$-type can be constructed by the given functions $f, g$ and the constant vector $C ([15])$. 

Consequently, we have

**Theorem 4.3.** Let $M$ be a non-cylindrical ruled surface in $E^3$ with generalized 1-type Gauss map. Then, $M$ is an open part of a plane, a helicoid, a right cone or a conical surface of $G$-type.

Summing up our results, we obtain the following classification theorem.

**Theorem 4.4.** (Classification) Let $M$ be a ruled surface in $E^3$ with generalized 1-type Gauss map. Then, $M$ is an open part of a plane, a circular cylinder, a cylinder over a base curve of an infinite-type satisfying (3.3), (3.7) and (3.8), a helicoid, a right cone or a conical surface of $G$-type.

**References**

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