Article

C^4 Space-Time.. a window to new Physics?

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Abstract: We explore the possibility to form a physical theory in C^4. We argue that the expansion of our usual 4-d real space-time to a 4-d complex space-time, can serve us to describe geometrically electromagnetism and unify it with gravity, in a different way that Kaluza-Klein theories do. Specifically, the electromagnetic field A_μ, is included in the free geodesic equation of C^4. By embedding our usual 4-d real space-time in the symplectic 8-d real space-time (symplectic R^8 is algebraically isomorphic to C^4), we derive the usual geodesic equation of a charged particle in gravitational field, plus new information which is interpreted. Afterwards, we explore the consequences of the formulation of a "special relativity" in the flat R^8.

1. Acknowledgments

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2. Introduction

This is the first paper of a series of papers [16] [17] [18] [19] [20], concerning a physical theory in an extended C^4 space-time. The most difficult problem in the present history of physics, is the hunt of a unified theory. A unified theory, which could incorporate general relativity and quantum theory and could explain the nature of dark energy and dark matter, as well. This task is on progress and several theories and suggestions exist in the literature of physics. But yet, a final satisfactory proposal is still missing. Of course, there are promising candidates, such as superstrings, loop quantum gravity and classic quantum gravity theories, which are still under development. At this point, we would like to suggest an alternative, which is pure geometric. We argue that an expansion of our usual 4-d real space-time to a 4-d complex space-time (or to the algebraically isomorphical symplectic 8-d real space-time), could be promising. In fact the extension to a complex space -time is not something new. A. Einstein has used several complex structures in order to unify gravity with electromagnetism [11], W. Pauli generalised the Kaluza-Klein theory to a six-dimensional space (3-d complex space) [12]

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and H. P. Soh , advised by A. Eddington, published a theory attempting to unifying gravitation and electromagnetism within a complex 4-dimensional Riemannian geometry [13]. Moreover, S. Hawking discussing mathematical models which involve imaginary time for the description of the Universe in [14], makes a comment [4] suggesting that the distinction between real and imaginary quantities is just a mind trap. In the past, several attempts were made in the direction of a classical unified field theory (UFT), (all our information about UFT can be found in a marvelous and extensive article by Hubert F. M. Goenner [23], [24]), where the basic idea was the generalization or the extension of the Riemannian geometry, in such a way that electromagnetism could be included in a geometrical way. The main efforts attempted were, Weyl's infinitesimal geometry in terms of gauge field, Kaluza-Klein's 5-d theories, Eddington's pire affine geometry, Schrodinger's affine geometry, Born's reciprocity theory, Sciama's attempt to define classical spin and finally Einstein's several attempts using one after the other possible geometries as mixed (metric-affine geometry), asymmetric or complex geometries. All these attempts were failed for many reasons, for instance Weyl's UFT failed to produce rational physical findings, but fortunately, his efforts and ideas guided scientists, through gauge invariance, to formulate quantum field theory. Schrodinger's affine geometry, which was a blend of a of Weyl's theory and Eddington's pure affine theory, was not gauge invariant plus his false intention to include mesons apart from the unification of gravitation and electromagnetism. On the other hand Eddington's affine geometry was mathematically difficult and his suggestions did not have the elegance and the presentation required. And then it was A. Einstein, who played with almost every possible geometric structure, during his hunt for UFT. In the period 1923-1933, he started investigating Eddington's affine geometry, Cartan's tere-parallel geometry, Kaluza's 5-d Riemannian geometry and finally a mixed one where he mixed affine geometry with a metric with a skew-symmetric part. In this mixed geometry, he argued, that the symmetric part of the metric tensor, is associated to inertia and gravitational field, while the assymetric part of the metric would be linked to electromagnetism, where the field equations should be derived as limiting case. Afterwards, in 1948, he started to relate mathematical objects to physical observables,such that, the anti-symmetric density play the role of an electromagnetic potential. , there were many problems in the interpretation, from the torsion tensor which appeared in the definition of the metric, to the two versions of UFT, a weak and a strong one, or to the spherically symmetric solution derived from A.Papapetrou, which did coincide asymptotically with the solution of Einstein-Maxwell equation. Moreover, a clear connection between geometric objects and observables could not be found and finally, there were many problems to pass from continuous to discrete mass.But afterwards, in 1945 tried something new and wrote about it

"What i now do will seem a bit crazy.... consider a space the 4 coordinates x_1, x_2, x_3, x_4 which are complex such that in fact it is an 8-d space...In place of the Riemannian metric another one of the form $g_i k$ obtains"

But eventually, in order to maintain the 4-d space, he merely abandon this idea and used only the field variables to be complex. In a same manner A. Eddington and H. P. Soh , used also a 4-d Riemannian geometry with real coordinates, but with a complex metric, where the real part of the metric corresponds with mass and gravitation, while the imaginary part corresponds with charge and electromagnetism. And afterwards, it was D.S.Sciama, who tried to give a new approach to UFT. Sharing the same opinion with A. Einstein, about a quantum theory derived directly from geometry, he returned to metric affine geometry. In order to geometrize the spin tensor, he abandoned the idea that the skew symmetric part of the connection is associated to electromagnetism but rather, with a classical spin angular momentum of matter. And finally, there where the Kaluza-Klein theory,

[4] "One might think this means that imaginary numbers are just a mathematical game having nothing to do with the real world. From the viewpoint of positivist philosophy, however, one cannot determine what is real. All one can do is find which mathematical models describe the universe we live in. It turns out that a mathematical model involving imaginary time predicts not only effects we have already observed but also effects we have not been able to measure yet nevertheless believe in for other reasons. So what is real and what is imaginary? Is the distinction just in our minds? "

where a fifth dimension was added, so that, this extra dimension would house the incorporation of the electromagnetism field into geometry. Eventually,Kaluza-Klein theory, also suffered with the problem of a static spherically symmetric solution. But all these attempts had bigger problems, that the above mentioned ones. The first of them, it was the lack of knowledge in that period i.e all these great scientists, did not know the existence of nuclear fields, they did not have the experimental data of today, nor the existence of what we call dark matter and dark energy fields. This lack of knowledge, did not give them the chance to properly connect and relate the mathematical objects of their theories to physical observables. Moreover, there was and there is, a certain belief that our usual 4-d space-time, should be derived as a limiting case or in reductive way or with compactification of the extra dimensions. In our attempt, fortunately, we take into account all our present knowledge in order to properly connect mathematical objects with observables, plus the fact, that we are willing to connect, even the extra dimensions with observables. Moreover, we asked ourselves, if Cosmos is not in reality 4-d and has a bigger dimension, how would a 4-d observer, would observe this higher dimensional space? Through this question, we do not any longer want to take limits or other similar techniques, but rather to embed our usual 4-d space-time to this higher dimensional space. We argue, that in our approach, we would have the possibility to get in touch with several existing problems of theoretical physics and as well, to give us the opportunity to seek for new physical phenomena. Two new terms will arise after the embedding procedure, apart from gravity, where these new terms are directly connected to geometry and could be linked with the problem of dark fields. In addition, through the embedding procedure, scales will arise, where a uniform scale is recognized as a bound in energy scale,which also looks like a geometric description of Higg's mechanism and the anti-symmetric part of the metric tensor give us enough room, in order to include nuclear forces. Especially, for the possibility to include nuclear forces as well, we give a clearer look by the point of view of symmetry groups in [18]. Finally the generalized spacial relativity that is formulated as a consequence of C^4 or R^8 and the two time consideration, suggests that there also exist a second invariant "velocity", apart the usual speed of light, that totally changes our beliefs about the propagation of information and everything to it and leads to new physical phenomena. Our main approach, is to repeat all the steps that were made in the past, but now not for the 4-d real space-time, but for the 4-d complex space-time. Specifically, we want to establish a theory of mechanics, a theory of "special relativity" and a "general relativity", directly in C^4. The extra dimensions of this formulation, can be served as additional degrees of freedom, which could can help us to describe geometrically the property of mass and "sources" in general. We want to present a "static" problem in C^4, which becomes "dynamic" after embedding our usual 4-d space-time in the 4-d complex space-time. Sources in general, will arise, as the lost information of this embedding. The advantage of such a consideration, is the ability to present a close theory, as it happens with mechanics and general relativity. Furthermore, we want to explore, the possibility to re-establish quantum theory, as a classic mechanics theory in C^4, giving us this way, the ability to alter the axiomatic demands of quantum theories, to axiomatic definitions of usual mechanics theory.

3. Method behind the choice of a C^4 space-time

There are two successful theories that are capable to describe the basic and elementary "forces" in Nature, General Relativity (GR) for gravity and Standard Model (SM) for electromagnetism, weak and strong nuclear interactions. We would like to find a method originated from these two theories, in order not to see or find a way to unify them, but rather to seek for a new frame, from which those two theories could arise. A nice way to start discussing about gravity and GR is the principle of general covariance, a principle that in our opinion expresses deep philosophical issues concerning the description, the existence and the understanding of the Universe or Cosmos. This principle as it is expressed in [21], is the idea that "every physical quantity must be describle by a geometric object and that the laws of physics must all be expressible as geometric relationships between these geometric objects". This principle, was originally expressed by Felix Klein (Erlanger program) and it

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was A. Einstein who successfully used in GR. Geometric objects are in general tensors such as vectors (1-tensors), metric tensor (2-tensor), Riemann-Christoffel 3-tensor R^{i}_{klm} etc, which exist independently of coordinate systems or reference frames but in general expressible by them. This way GR is usually called as a geometric theory and it has its foundation on three axioms

1. There is a metric tensor
2. The metric tensor fulfills the Einstein field equation

$$G_{ij} = 8\pi T_{ij}$$

3. All special relativistic laws pf physics are valid in local Lorentz frames of metric

Then curvature in geometry manifests itself as gravitation as the energy momentum tensor T_{ij}, is the "average' of curvature expressed by Einstein's tensor G_{ij}. Based on the above mentioned, we would like to impose a question as a new way of investigation. Can we expand the relationship between the energy-momentum tensor T_{ij} and geometry described by G_{ij}, to a new principle that even T_{ij} is not connected by relationship to geometry but T_{ij} can be described by geometry itself? Or in an other way, if T_{ij} generates an average curvature described by Einstein's tensor, can we find a higher dimensional space, let us call it X, implying this way a new extension of our usual 4-d real space-time to space-time X, where now T_{ij} can be described or connected with a new "average" curvature defined in the new expanding part of X and then the generalized "Einstein" tensor (this generalised "Einstein" tensor will follow the dimensionality of this space X, for instance if X is n dimensional then, this "generalised Einstein" tensor will be a matrix $n \times n$), let us call it for now $G^{'}_{ij}$, of the extended space-time X, fulfills the field equation

$$G^{'}_{ij} = 0$$

which suggests that the "average" curvature of this space-time X is 0 and the energy-momentum tensor is incorporated purely geometrical in $G^{'}_{ij}$? If the answer is yes, then $G^{'}_{ij} = 0$ is nothing else, than an equilibrium equation (not of "Poisson type" but rather a "Laplace type" equation) which means that Universe or Cosmos is an expanding dynamic system in equilibrium state , governed by the geometry of the space-time or manifold X, through a Ricci flow. Meanwhile, those extra dimensions can be seen also as additional degrees of freedom, from which the entities of our usual energy-momentum tensor expressed as matter, charges, currents or even undescribed by GR physical quantities such as pure quantum characteristics (spin, isospin, colours, etc), or sources in general could be defined or described pure geometrically. Of course extra dimensions and extension to a higher dimensional space-time is not something new, it was originally proposed by T. Kaluza and O. Klein and afterwards from string theories in general, where a 10+1 dimensional space is proposed as the necessary "arena" for the description of M-theory. At this part, we would like to examine, of there are any clues, from our well known and accepted theories, which could inform us about the type and dimensionality of space-time X. We think that there is no better candidate than the Standard Model (SM). Let us focus only in the electromagnetic part of SM, for simplicity, where the gauge symmetry is the abelian group U(1)and the covariant derivative associated with it, is

$$D_{\mu} = \partial_{\mu} - iA_{\mu}$$

There is a lot for someone to discuss about gauge theories, involving symmetry groups, Lie groups, Lie manifolds, tangent bundles, submersions etc, but we would like to focus on a different and more simple path. We want to examine this covariant derivative in a strict, in the beginning, mathematical or geometrical way and afterwards, we will try to evaluate the physical meaning and interpretation. In this covariant derivative, obviously ∂_{μ} is a 4-d real vector or vector field (the basic tangent vector of 4-d real space-time) and A_{μ} is a 4-d real vector or vector field as well. Now, if we consider that D_{μ} is a vector or even better a tangent vector, in the sense of a geometrical description, in which space does D_{μ} belongs to? The answer is very simple but awkward

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$$D_\mu \text{ belongs in a } C^4 \text{ space}$$

due to the fact that the complex number "i" lies in the covariant derivative between the two 4-d real vectors! But what does it means, is it just a mathematical tric or can we give physical meaning? We argue that not only C^4 has physical meaning and interpretation but rather this is the key or clue we were looking for space X. We suggest a new extension of our usual 4-d real space-time to a 4-d complex space-time, or a we will see further to its geometrically equivalent 8-d real symplectic space. We shall see in [18], that the choice of a C^4 space-time by the beginning, will explain not only how, but why as well, as concerned the choice of a complex field φ in quantum field theories and additionally, the causality of the existence of the symmetries described by the unitary groups $U(1)$, $SU(2)$, $SU(3)$ in gauge theories. This way we would not need to start by a God given field and God given symmetries (as R. Penrose comments in "The Road To Reality"), as they would arise naturally as properties of the choice of a C^4 space-time. For instance, we will see that the Hermitian metric, associated with C^4, is invariant under transformations described by the group $GL(4, C)$ which is isomorphic as

$$GL(4, C) \simeq SO(4, 4) \cap U(4)$$

Furthermore, we will see in [18], that $U(4)$ breaks simultaneously after embedding our usual space-time in C^4 into desired unitary groups, giving us this way, the chance to explain the phenomenon of spontaneous symmetry breaking, as the causality of this embedding. Specifically, in [18] we show that the symmetry group for nuclear and electromagnetic field should expanded to an extension of SM as $\frac{SU(4)}{SU(2)} \times SU(2) \times U(1)$, where the quotient is not anymore a group but rather a coset (orbit space), that is called in the literature of mathematics as Stieffel manifold, that contains an su(3) algebra,as a subalgebra. Furthermore, this coset is isomorphical to the product of spheres $S^7 \times S^5$ which is clearly "bigger" than the group SU(3)and as a result, it gives us room to seek for unexplained phenomena linked to strong nuclear field. In addition, all together the product naturally leads to an extension of SM. It is our desire throughout all our consideration to answer not only the "how" in physics , but the "why" as well. Additionally, if the choice of a C^4 space-time is valid, it could also explain the great success of quantum theories in general, as they would appear to have already used the fact of a complex space-time. Complex geometrical structures are not something new in physics, as we have already seen in the introduction, and already such structures are used in the sense of Kahler and Calabi-Yao manifolds. Moreover, symplectic and complex geometries are suggested as new tools in the connection of Yang-Mills theories and geometry in [22]. But our suggestion is not only some additional dimensions, serving as additional degrees of freedom, but we want to propose to give direct physical interpretation to these ones. There are two ways

1. We must find some physical quantities that will be related to this extra dimensions. Or, are there any physical necessities that could be introduced by the beginning and C^4 could be the right framework?
2. We can start with just a usual vector of C^4

   $$z_i = x_i + iy_i$$

   where x_i, $i = 0, 1, 2, 3$ are the usual coordinates of 4-d real space-time and leave y_i without any physical interpretation, in the beginning, and let the mathematical processing to lead us to a desired and suitable interpretation.

We have chosen the second way, due to the fact, that we can build and establish a more concrete framework and examine step by step the arisen structures and this way we keep in touch with the well known physical theories. Finally, from GR the key was geometry, from gauge theories and SM the key was C^4 space and the combination lead us to this consideration in the search of new physics (if someone believes in such a hunt)

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"We suggest to investigate geometrically C^4 space-time. In C^4 space-time there must be a unified field (Gravity, electromagnetism, etc) which is a property of this 4-d complex space-time itself, as gravitational field is a property of our usual 4-d real space-time"

As a consequence, in the next paragraph, we will start with a pure geometrical picture, by investigating the elementary length in a curved C^4 space-time and afterwards, we will give the physical interpretation of these extra dimensions as a natural consequence of geometry processing. The key in order to take back our usual well known theories which are expressed in the "language" of a 4-d real space-time, will be the embedding of our usual 4-d space-time, in the 4-d complex space-time. Moreover, in this paper, we investigate the flat cases of C^4 and R^8, which leads to an extended special relativity and a second invariant constant is introduces, while the symmetry group $SO(8)$ is connected with the signatures $(4,4)$, $(8,0)$, $(0,8)$ through Cartan's principle of triality. The field equations of the unified field in curved C^4 space-time is investigated in the second paper of this series [17]. In the third and fourth paper [18], [19] by releasing the end point of the action's integral, we pass to Hamilton-Jacobi equations and we argue that the covariant derivative of SM is nothing else than a part of the Hamilton-Jacobi derivative as it comes straightforward, from the problem of least action, derived directly from the geometry of the curved C^4 space-time and the usual symmetries and groups of SM are related with the symmetry of this action, which is invariant as we shall see, under transformations of the group $GL(4,C)$ and $U(4)$. Afterwards, in the fourth paper, complex time will help us to overcome the problems of the ADM formalism and express a suitable Hamilton- Jacobi equation for the curved C^4, defining this way a super-energy tensor connected to the complex time. Finally in the fifth paper [20], we introduce 1-linear forms, which could help us to describe fermions pure geometrically.

4. Geometry in C^4

There are several geometrical structures that we can equip a C^4 space such complex, almost complex, Hermitian, holomorphic,Kahler, Kalabi-Yao, etc. From these structures, we have chosen the Hermitian one because it is the most natural extension of the Riemann's spaces in a complex space. Specifically, we can define an elementary length of the type

$$ds^2 = G_{ij} dz^i d\bar{z}^j + hc \tag{1}$$

where G_{ij} is a Hermitian metric tensor (in analogy to a symmetric metric tensor in Riemann's spaces). It is obvious, that we treat to C^4 space as

$$C^4 \simeq X \times iY \simeq R^4 \times iR^4 \tag{2}$$

where $x^i \in X$ and $y^i \in Y$. Many authors write the Hermitian metric tensor $G_{i\bar{j}}$ instead of G_{ij} but we will keep the notation without the bra, in order to make the notation more simple. We can proceed by introducing the elements of the C^4 space as

$$z_i = x_i + iy_i \tag{3}$$

where $x_i \in R^4(X)$, $y_i \in R^4(Y)$. The x_i, y_i must be of the same type which means that x_0 and y_0 are both time-like while x_1, x_2, x_3 and y_1, y_2, y_3 are space-like. The corresponding Cauchy derivative will be

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$$\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i}) \tag{4}$$

In addition, the metric tensor of C^4 will be a Hermitian 4 x 4 metric G_{ij}

$$G_{ij} = g_{ij} + iI_{ij} \tag{5}$$

with g_{ij} its symmetric and I_{ij} its anti-symmetric part. Obviously, g_{ij} plays the role of the metric tensor in X and Y consisting only of terms without any mixing of variables in X and Y, while I_{ij} contains only such mixing terms. If we introduce Eq. (3) in Eq. (1) we will move from the C^4 space to an R^8 space equipped with a symplectic geometry where the elementary length will then be

$$ds^2 = g_{ij}dx^i dx^j + g_{ij}dy^i dy^j + I_{ij}(dx^i dy^j - dy^j dx^i) \tag{6}$$

where g_{ij} is our common symmetric metric tensor and I_{ij} is a symplectic antisymmetric tensor. In the case that I_{ij} vanishes, we fall naturally in the case of a Riemann's space of type R^{2n} where $n = 4$. The Hermitian metric tensor has become in the case of real representation

$$G_{ij} = \begin{pmatrix} g_{ij} & I_{ij} \\ -I_{ij} & g_{ij} \end{pmatrix}$$

The symplectic term in Eq. (6) can be written also as

$$ds^2 = g_{ij}dx^i dx^j + g_{ij}dy^i dy^j + 2I_{ij}dx^i dy^j \tag{7}$$

because $I_{ij}dy^i dx^j = I_{ji}dx^i dy^j = -I_{ij}dy^i dx^j$. Our next step is to generalise the usual Christoffel symbols $\Gamma_{k,ij}$ to Christoffel symbols $\widehat{\Gamma}_{k,ij}$ with respect to the Hermitian metric tensor G_{ij}. So, we have to compute the partial derivatives $\frac{\partial G_{jk}}{\partial z^i}, \frac{\partial G_{ki}}{\partial z^j}, \frac{\partial G_{ij}}{\partial z^k}$ with respect to the Cauchy's derivative as

$$\frac{\partial G_{jk}}{\partial z^i} = \frac{1}{2}(\frac{\partial G_{jk}}{\partial x^i} - i\frac{\partial G_{jk}}{\partial y^i}) = \frac{1}{2}\Big((\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial I_{jk}}{\partial y^i}) + i(\frac{\partial I_{jk}}{\partial x^i} - \frac{\partial g_{jk}}{\partial y^i})\Big) \tag{8}$$

thus, the Christoffel symbols $\widehat{\Gamma}_{k,ij}$ are

$$\widehat{\Gamma}_{k,ij} = \Gamma^{(x)}_{k,ij} + \Delta^{(x)}_{k,ij} - i(\Gamma^{(y)}_{k,ij} + \Delta^{(y)}_{k,ij}) \tag{9}$$

or in real representation R^8

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$$\widehat{\Gamma}_{k,ij} = \left(\Gamma^{(x)}_{k,ij} + \Delta^{(x)}_{k,ij}, -\Gamma^{(y)}_{k,ij} + \Delta^{(y)}_{k,ij}\right) \tag{10}$$

where $\Gamma_{k,ij}$ are the usual Christoffel symbols with respect to the symmetric tensor g_{ij}, $\Delta_{k,ij}$ are the "Christoffel symbols" with respect to the antisymmetric tensor I_{ij} and by (x), (y) we denote the kind of the coordinates to which we find the partial derivative. As concerned the $\Delta_{k,ij}$ symbols it is easy to see that

$$\Delta^{(x)}_{k,ij} = -\Delta^{(x)}_{k,ji} \tag{11}$$

$$\Delta^{(y)}_{k,ij} = -\Delta^{(y)}_{k,ji} \tag{12}$$

which means, that they are antisymmetric with respect to the pair of indices ij . Now we can proceed to find the geodesics through the variation of an action of the form

$$\delta S = \delta \int ds \tag{13}$$

for ds as defined by Eq. (7) which can be written also as

$$\delta S = \delta \int \left(g_{ij}u^i u^j + g_{ij}v^i v^j + 2I_{ij}u^i v^j\right) ds \tag{14}$$

where $u^i = \dfrac{dx^i}{ds}$ and $v^i = \dfrac{dy^i}{ds}$. After some calculus we derive the pair of geodesic equations

$$\left(g_{kj}\frac{du^j}{ds} + \Gamma^{(x)}_{k,ij}u^i u^j\right) + \left(I_{ki}\frac{dv^i}{ds} + 2\Delta^{(x)}_{k,ij}v^i u^j\right) + \frac{\partial I_{jk}}{\partial x^i}v^i u^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}v^i v^j = 0 \tag{15}$$

$$\left(g_{kj}\frac{dv^j}{ds} + \Gamma^{(y)}_{k,ij}v^i v^j\right) + \left(I_{ki}\frac{du^i}{ds} + 2\Delta^{(y)}_{k,ij}u^i v^j\right) + \frac{\partial I_{jk}}{\partial y^i}u^i v^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}u^i u^j = 0 \tag{16}$$

the first parenthesis in both equations reminds us our usual geodesic equation of the space-time R^4, while we have other terms that we want to link them to electromagnetism so that the equations (15), (16) could gives us the geodesic equation of a charged particle in gravitational field and hopefully new elements! It is obvious now, that we want to link the symplectic term I_{ij} (antisymmetric tensor) with a generalized field K_μ which will represent a generalized "electromagnetism" which could contain not only the electromagnetic field A_μ but the weak nuclear field W_μ and the strong nuclear field G_μ as well, giving us the opportunity to describe those fields purely geometrically in a larger extended space-time. We must remember that even the electromagnetic field A_μ is not a pure geometric object of our usual space-time, but rather added (ad-hoc) to the geometric action (derived by the elementary length of R^4) by a term

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$$-\int \frac{q}{c} A_i dx^i \tag{17}$$

The term $I_{ij}v^iv^j$ in Eq. (14) can be also seen as

$$I_{ij}u^iv^jds = I_{ij}\frac{dx^i}{ds}\frac{dy^j}{ds}ds = -I_{ji}\frac{dy^j}{ds}\frac{dx^i}{ds}ds = -\big(I_{ji}\frac{dy^j}{ds}\big)dx^i \tag{18}$$

It is obvious that we could immediately recognize as

$$A_i = I_{ji}\frac{dy^j}{ds} \tag{19}$$

but, these could be premature and as we have mentioned above we want to identify a "generalized unified electromagnetism" K_i firstly, but Eq. (19) can give us some clue. We introduce the anti-symmetric tensor K_{ij} defined as

$$K^{(x)}_{jk} = \frac{\partial K_k}{\partial x^j} - \frac{\partial K_j}{\partial x^k} \tag{20}$$

where $K_j = I_{ji}\dot{y}^i = -I_{ij}\dot{y}^i$ then Eq. (20) becomes

$$K^{(x)}_{jk} = \frac{\partial K_k}{\partial x^j} - \frac{\partial K_j}{\partial x^k} = \big(\frac{\partial I_{ki}}{\partial x^j} - \frac{\partial I_{ij}}{\partial x^k}\big)v^i \tag{21}$$

or with respect to Δ symbols

$$K^{(x)}_{jk} = \big(2\Delta^{(x)}_{k,ij} + \frac{\partial I_{jk}}{\partial x^i}\big)v^i \tag{22}$$

this way, the first pair of the geodesic equations can be written

$$\big(g_{kj}\frac{du^j}{ds} + \Gamma^{(x)}_{k,ij}u^iu^j + K^{(x)}_{jk}u^j\big) + I_{ki}\frac{dv^i}{ds} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}v^iv^j = 0 \tag{23}$$

The term in the parenthesis starts to look like the desired one, but we must remember that we have the second pair also which becomes

$$\big(g_{kj}\frac{dv^j}{ds} + \Gamma^{(y)}_{k,ij}v^iv^j + K^{(y)}_{jk}v^j\big) + I_{ki}\frac{du^i}{ds} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}u^iu^j = 0 \tag{24}$$

The tensor $K^{(y)}_{jk}$, $K^{(x)}_{jk}$ are nothing else than the "Christoffel symbols" $\Delta^{(y)}_{k,ij}$, $\Delta^{(x)}_{k,ij}$ multiplied by a velocity! This way, the analogue of the symmetric metric tensor "field" g_{ij} is the anti-symmetric tensor I_{ij} "field" and not the K_i (or A_i which is a sub case) as we have suspected as far now in the usual context of physics. Moreover, the 2-form K_{ij} (or F_{ij} for the sub case) is not equivalent with the curvature 2-form Riemann-Christoffel tensor R_{ij}. On the contrary the equivalence of K_{ij} is between the Christoffel symbols Γ. From our point of view, this is the reason that we have failed to unify successfully gravity and electromagnetism. Even in the case of the Kaluza -Klein theories, the g_{ij} was put in equal foot with the "field" A_i. As we have seen in our consideration g_{ij} and K_i are different with respect a velocity. And that was the reason that Kaluza-Klein theories where merely successful. This situation was merely saved, due to the fact that the variation of the action was taken with respect to the "field" A_i itself and not with respect a field analogue to the metric tensor, as we have done so far in our consideration. It is important to note though, that we could form "fields" with respect to the metric tensor g_{ij} in the same way as we have done for the "fields" K_i, combining the g_{ij} with a velocity, or even form a 2 tensor with respect to g_{ij} in the same way that we have done for K_{ij}, combining the Γ with a velocity. But all these, will be investigated later.

5. Embeding R^4 in R^8

The main problem of the pair of geodesic equations (22), (23) is that they express some physics in the symplectic space R^8 which is very different from our usual space R^4. Specifically, these equations should be valuable only to R^8 observers! Unfortunately, we are 4-d dimensional observers and our physical theories are expressed in the mathematical language of a 4-d real space. In order to identify the observables of the 8-d space we can embed our usual 4-d space-time in the 8-d extended space-time. This way, it seems that 4-d observers live in one of the projection spaces of C^4 and by embedding the one projection R^4 in C^4 or R^8 symplectic space, we will recover the lost information. But, before the embedding we must clarify some important issues about the flat cases and the signature problem. The flat Hermitian metric tensor can take the following signatures (1,1,1,1), (-1,-1,-1,-1,), (1,1,-1,-1), (1,1,1,-1) and (-1,1,1,1) where the 2 first two are Hermitian, while the other two are pseudo-Hermitian, which gives in the real representation the signatures (8,0), (0,8), (4,4), (6,2), (2,6) accordingly and similarly the first two are Euclidean, while all the others are pseudo-Euclidean. The signatures (8,0), (0,8) share a duality property and (6,2), (2,6) as well. But there is a unique property that comes as first time in 8-d real spaces, the Cartan's triality property, which states that the three signatures (8,0), (4,4), (0,8) are all correlated (for more information about triality see Appendix 1). By Cartan's principle of triality we will try not only to choose the right signature but also to explain the choice of the 8-d space (according to Duff's viewpoint in [3] a fundamental theory of everything should explain not only the dimensionality but the signature of the space-time as well). In fact, we will be able to provide an independent signature framework in the same spirit general relativity provides a coordinate independent description. For that reason, we have the right to pick one of those three signatures and we have chosen the (4,4) one, due to the fact that it can be splitted to (1+3,3+1) signature, giving us the opportunity to present our usual Minkowski's space as we shall see below. For clarity, we must emphasize that Hermitian geometry will only provide us with the signatures (8,0) and (0,8), the (4,4) one which comes from a pseudo-hermitian geometry, can be used only as a consequence of Cartan's property of triality and if used, we must automatically change the sign of the second g_ij in equation (6) or (7) in the general case of the Hermitian geometry, from (+) to (-) by hand. Specifically, the signature (4,4) stands for in the flat case

$$ds^2 = dx_0^2 + \mathbf{dx^2} - dy_0^2 - \mathbf{dy^2} \tag{25}$$

where bold means 3-d. We can split the signature if we change place between x_0 and y_0 as

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$$ds^2 = -dy_0^2 + \mathbf{dx^2} + dx_0^2 - \mathbf{dy^2} \tag{26}$$

The term $-dy_0^2 + \mathbf{dx^2}$ defines our usual Minkoskwi tensor n_{ij} with signature $(-1, 1, 1, 1)$. Moreover we would like to add some comments about the embedding procedure. In order to proceed with embedding, we must pass from the initial coordinate x_i and y_i that describe R^8, to a re-expression containing only x_i that describe the embedded space R^4. This way the y_i coordinates must be re-expressed with respect to the coordinates of the embedded space. The lost information referred to coordinates y_i, will be recovered, as we can see from equations of (33), (34) with additional terms in the final expression of the metric tensor. Now, we can proceed to the embedding which is a standard mathematic topic, similar to the parametrisation, equations (27)-(34) are part of this mathematical topic as it exists in the literature of diferrential geometry.

We start once again by equation (6) derived earlier in this section

$$ds^2 = g_{ij}dx^i dx^j + g_{ij}dy^i dy^j + I_{ij}(dx^i dy^j - dy^j dx^i) \tag{27}$$

If R^4 is embedded in R^8 and N_{ij} is the metric tensor of R^4 , then in R^4 we have

$$ds^2 = N_{ij}dx^i dx^j \tag{28}$$

We will write the metric tensor in R^8 using Greek indices α, β

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{\alpha\beta}dy^\alpha dy^\beta + 2I_{\alpha\beta}dx^\alpha dy^\beta \tag{29}$$

The elementary length ds of R^4 is the same in R^8 and as a result

$$N_{ij}dx^i dx^j = g_{\alpha\beta}dx^\alpha dx^\beta + g_{\alpha\beta}dy^\alpha dy^\beta + 2I_{\alpha\beta}dx^\alpha dy^\beta \tag{30}$$

If $y^\alpha = y^\alpha(x^0, x^1, x^2, x^3)$ and $dy^\alpha = \dfrac{\partial y^\alpha}{\partial \varrho}dx^\varrho$ we have

$$N_{ij}dx^i dx^j = g_{\alpha\beta}dx^\alpha dx^\beta + g_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\varrho}dx^\varrho \frac{\partial y^\beta}{\partial x^\mu}dx^\varrho + 2I_{\alpha\beta}dx^\alpha \frac{\partial y^\beta}{\partial x^\mu}dx^\mu \tag{31}$$

Because, now we refer to the variables x^i, we can replace the Greek indices by Latin i,j wherever needed and therefore

$$N_{ij}dx^i dx^j = g_{ij}dx^i dx^j + g_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}dx^i dx^j + 2I_{i\beta}\frac{\partial y^\beta}{\partial x^j}dx^i dx^j \tag{32}$$

which actually means that

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$$N_{ij} = g_{ij} + g_{\alpha\beta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}} + 2I_{i\beta}\frac{\partial y^{\beta}}{\partial x^{j}} \tag{33}$$

or even

$$N_{ij} = g_{ij} + g_{\alpha\beta}\frac{\partial y^{\alpha}}{\partial x^{i}}\frac{\partial y^{\beta}}{\partial x^{j}} + I_{i\alpha}\frac{\partial y^{\alpha}}{\partial x^{j}} + I_{j\alpha}\frac{\partial y^{\alpha}}{\partial x^{i}} \tag{34}$$

The pair of the geodesic equation (22),(23) becomes as one as

$$N_{ij}\frac{d^2x^j}{ds^2} + \widehat{\Gamma}_{i,jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{35}$$

where

$$\widehat{\Gamma}_{i,jk} = \frac{1}{2}(\frac{\partial N_{ki}}{\partial x^j} + \frac{\partial N_{ij}}{\partial x^k} - \frac{\partial N_{jk}}{\partial x^i}) \tag{36}$$

It is important to simplify a little bit the above mentioned equation by introducing a special case of the embedding functions

1. $y^{\alpha'} = \lambda\delta^{\alpha'}_{\varrho}x^{\varrho}$ for $\alpha' = 1,2,3$ and $y^0 = y^0(x^0)$. As we can see the space-like functions are linear while the time-like function is free and can be (as we can see in our next paper [18]) of the form $y_0 = Ae^{Bx_0}$. After some calculus, the metric tensor N_{ij} can be written as

$$N_{ij} = (1+\lambda^2)g_{ij} + \lambda D_{ij}\frac{\partial y^0}{\partial x^0} + 2E_{ij}(\frac{\partial y^0}{\partial x^0})^2 + M_{ij}\frac{\partial y^0}{\partial x^0} \tag{37}$$

This equation holds if our space is locally Euclidean, but if we want our space locally to have the desired signature (4,4) as we have mentioned, it will take the form

$$N_{ij} = (1-\lambda^2)g_{ij} + \lambda D_{ij}\frac{\partial y^0}{\partial x^0} - 2E_{ij}(\frac{\partial y^0}{\partial x^0})^2 - M_{ij}\frac{\partial y^0}{\partial x^0} \tag{38}$$

and if we want to split the signature in (1+3, 3+1) we just have to interchange x_0 with y_0. This way g_{ij} is our usual metric tensor and locally it is the Minkowsky's metric tensor. Moreover the tensors D_{ij}, E_{ij}, M_{ij} are

$$D_{ij} = \begin{pmatrix} 2g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & 0 & 0 & 0 \\ g_{20} & 0 & 0 & 0 \\ g_{03} & 0 & 0 & 0 \end{pmatrix}$$

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$$M_{ij} = \begin{pmatrix} 0 & I_{01} & I_{02} & I_{03} \\ I_{10} & 0 & 0 & 0 \\ I_{20} & 0 & 0 & 0 \\ I_{03} & 0 & 0 & 0 \end{pmatrix}$$

$$E_{ij} = g_{00}\delta_i^0\delta_j^0 \tag{39}$$

E_{ij} can be nicely combined with M_{ij}, in order to form the scalar quantity of electromagnetism! If we proceed in the calculation of $\widehat{\Gamma}_{i,jk}$ with respect to the tensors D_{ij}, E_{ij}, M_{ij} we can see that it breaks into pieces as

- our usual Christoffel symbols formed by the first term of Eq. (38) which means that they are formed by g_{ij}
- some peculiar " Christoffel symbols " formed by the second term of Eq. (37) D_{ij} which are g_{ij} related and have the form

$$\Gamma^{(D)}_{i,jk} = (\frac{\partial g_{k0}}{\partial x^j} + \frac{\partial g_{j0}}{\partial x^k})\delta_i^0 + (\frac{\partial g_{i0}}{\partial x^k} - \frac{\partial g_{k0}}{\partial x^i})\delta_j^0 + (\frac{\partial g_{i0}}{\partial x^j} - \frac{\partial g_{j0}}{\partial x^i})\delta_k^0 \tag{40}$$

  the first parenthesis is symmetric while the other two are antisymmetric, which is in contrast to the behaviour of our usual Christoffel symbols.
- the "Christoffel symbols" with respect to the antisymmetric tensor I_{ij} that we have called them as $\Delta_{i,jk}$

$$\Gamma^{(M)}_{i,jk} = \Delta_{i,jk} = (\frac{\partial I_{k0}}{\partial x^j} + \frac{\partial I_{j0}}{\partial x^k})\delta_i^0 + (\frac{\partial I_{i0}}{\partial x^k} - \frac{\partial I_{k0}}{\partial x^i})\delta_j^0 + (\frac{\partial I_{i0}}{\partial x^j} - \frac{\partial I_{j0}}{\partial x^i})\delta_k^0 \tag{41}$$

  it is peculiar but the $\Gamma^{(D)}_{i,jk}$, $\Gamma^{(M)}_{i,jk} = \Delta_{i,jk}$ have exactly the same form, except the fact that the first one is with respect to the symmetric g_{ij} while the second one with respect to the antisymmetric I_{ij}.

All these terms will appear in the geodesic equation. Afterwards, we can express some cases concerning Eq. (38) . Firstly, it is interesting to note that in the case that $i, j \neq 0$ we have

$$N_{ij} = (1-\lambda^2)g_{ij} \tag{42}$$

and for $i, j = 0$ we have

$$N_{00} = (1-\lambda^2)g_{00} + \lambda g_{00}\frac{\partial y^0}{\partial x^0} - 2g_{00}(\frac{\partial y^0}{\partial x^0})^2 = ((1-\lambda^2) + \lambda\frac{\partial y^0}{\partial x^0} - 2(\frac{\partial y^0}{\partial x^0})^2)g_{00} \tag{43}$$

Equation (43) expresses energies, which means that the parenthesis in front g_{00} is a coupling constant. This term has a maximum in the scale $\frac{\lambda}{2} = \frac{\partial y^0}{\partial x^0}$ suggesting that at this point the scale λ is unified with $\frac{\partial y^0}{\partial x^0}$ and that we cannot override this scale, all the permitted scales are only below this scale! If $\lambda > 0$ the term in parenthesis becomes $1 - \frac{\lambda^2}{2}$, but if $\lambda < 0$ this term becomes $1 - \frac{5}{2}\lambda^2$. It somewhat peculiar but it looks like we have a geometrical description of Higg's mechanism (without the interaction term that comes from φ^4 and can be recovered from the other papers) and that we have the possibility to enter in the area of high energy physics. We must proceed with the interpretation of Eq. (38) term by term in order to clarify what this energy scales mean.

- the first term of Eq. (38) is $(1 - \lambda^2)g_{ij}$ where g_{ij} is our usual metric tensor of the 4-d space-time and will we see in the next paper of this series [17] that expresses gravity and is connected with ordinary masses. Moreover, we will see that λ stands for Planck scale as it will be derived from general relativity. In this case, λ is fixed as it happens in General Relativity, but in the next case, the scale will be time depended.
- the last term represents the "unified generalised electromagnetism" as we have mentioned. But for $y^{\alpha'} = \lambda \delta^{\alpha'}_{\varrho} x^{\varrho}$ for $\alpha' = 1, 2, 3$ that we are studying, this should be our well known electromagnetism, due to the linearity of the embedding functions! Specifically, in this case the electromagnetic field tensor F_{ij} should stand for

$$F_{ij} = \frac{\partial y^0}{\partial x^0}\left(\frac{\partial I_{k0}}{\partial x^j} - \frac{\partial I_{j0}}{\partial x^k}\right)\frac{dy^0}{ds} \tag{44}$$

  where $\frac{\partial y^0}{\partial x^0} = \frac{q}{c}$.
- the second term has a scale as the product of the scale of the first term and the last one. Moreover, the $\Gamma^{(D)}_{i,jk}$ have the same behaviour with the $\Delta_{i,jk}$ but with respect to the symmetric tensor g_{ij}. It looks like this term both "gravitates" and "electromagnitates" in behavioral way! It is a hybrid between those two fundamental elementary fields. We propose to interpretate or connect this field to what we use to call as dark field (or for the linear case and only "dark electromagnetism")!
- finally the third term that has only one element $E_{ij} = g_{00}\delta^0_i\delta^0_j$ (scalar), share the scale of electromagnetism squared. We shall see later that it is invariant to any transformation that generalises $y^{\alpha'} = \lambda \delta^{\alpha'}_{\varrho} x^{\varrho}$ for $\alpha' = 1, 2, 3$, which can be interpreted as dark energy field.

2. If we write y^{α} around a point $(x^0_0, x^1_0, x^2_0, x^3_0)$, where $\vec{x}_0 = (x^1_0, x^2_0, x^3_0)$ is a steady point or pole, we can have for the embedding functions

$$y^{\alpha'} = y^{\alpha'}(x_0, \vec{x}_0) + \frac{\partial y^{\alpha'}(x_0, \vec{x}_0)}{\partial x^{\gamma}}(x^{\gamma} - x^{\gamma}_0) + ... \tag{45}$$

for $\alpha' = 1, 2, 3$ and $\gamma = 1, 2, 3$. If we keep only the two first terms of the expansion and if we set

$$\varepsilon^{\kappa}_{\lambda} = \begin{cases} 0, & \kappa = \lambda \\ 1, & \kappa \neq \lambda \end{cases} \tag{46}$$

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the final embedding functions are

$$y^{\alpha'} = y^{\alpha'}(x_0, \vec{x_0}) + c^{\alpha}_{\gamma}(x^{\gamma} - x^{\gamma}_0)\varepsilon^{\alpha}_0 \tag{47}$$

for $\alpha = 1,2,3,4$ and $\gamma = 1,2,3$. We have the following cases as concerning the indices i,j

- for $i, j = 1,2,3$ and for locally (4,4,) signature

$$N_{ij} = g_{ij} - g_{\alpha\beta}\frac{\partial y^{\alpha}}{\partial x^i}\frac{\partial y^{\beta}}{\partial x^j} + I_{i\alpha}\frac{\partial y^{\alpha}}{\partial x^j} + I_{j\alpha}\frac{\partial y^{\alpha}}{\partial x^i} \tag{48}$$

or

$$N_{ij} = g_{ij} - g_{\alpha\beta}c^{\alpha}_i c^{\beta}_j + I_{i\alpha}c^{\alpha}_i + I_{j\alpha}c^{\beta}_j \tag{49}$$

We have to mention that in contrast to $y^{\alpha'} = \lambda\delta^{\alpha'}_{\varrho}x^{\varrho}$ for $\alpha' = 1,2,3$ embedding transformations that we have studied earlier, we have terms generated by I_{ij}.

- $i = 0$ and $j = 1,2,3$ we have

$$N_{0j} = g_{0j} - g_{\alpha\beta}\Big(\frac{\partial y^{\alpha}_0}{\partial x^0} + \frac{\partial c^{\alpha}_{\gamma}}{\partial x^0}(x^{\gamma} - x^{\gamma}_0)\Big)c^{\beta}_j - g_{0\beta}\frac{\partial y^0}{\partial x^0}c^{\beta}_j + I_{0\alpha}c^{\alpha}_j + ...$$

$$I_{j\alpha}\Big(\frac{\partial y^{\alpha}_0}{\partial x^0} + \frac{\partial c^{\alpha}_{\gamma}}{\partial x^0}(x^{\gamma} - x^{\gamma}_0)\Big) + I_{j0}\frac{\partial y^0}{\partial x^0} \tag{50}$$

and if $x^{\gamma} \longrightarrow x^{\gamma}_0$ the above equation takes the simpler form

$$N_{0j} = g_{0j} - g_{\alpha\beta}\frac{\partial y^{\alpha}_0}{\partial x^0}c^{\beta}_j - g_{0\beta}\frac{\partial y^0}{\partial x^0}c^{\beta}_j + I_{0\alpha}c^{\alpha}_j + I_{j\alpha}\frac{\partial y^{\alpha}_0}{\partial x^0} + I_{j0}\frac{\partial y^0}{\partial x^0} \tag{51}$$

where in this equation we have time dependence for all the terms in contrast to the previous case $y^{\alpha'} = \lambda\delta^{\alpha'}_{\varrho}x^{\varrho}$ for $\alpha' = 1,2,3$ embedding transformations that we have studied earlier. This way even the scale for g_{ij} is time depended.

- finally the case $i, j = 0$ leads to

$$N_{00} = g_{00} - g_{\alpha\beta}\frac{\partial y^{\alpha}}{\partial x^0}\frac{\partial y^{\beta}}{\partial x^0} - 2g_{\alpha 0}\frac{\partial y^{\alpha}}{\partial x^0}\frac{\partial y^0}{\partial x^0} - 2g_{00}\Big(\frac{\partial y^0}{\partial x^0}\Big)^2 + 2I_{0\alpha} \tag{52}$$

if $x^{\gamma} \longrightarrow x^{\gamma}_0$ this equation take the form

$$N_{00} = g_{00} - g_{\alpha\beta}\frac{\partial y^{\alpha}_0}{\partial x^0}\frac{\partial y^{\beta}_0}{\partial x^0} - 2g_{\alpha 0}\frac{\partial y^{\alpha}_0}{\partial x^0}\frac{\partial y^0_0}{\partial x^0} - 2g_{00}\Big(\frac{\partial y^0_0}{\partial x^0}\Big)^2 + 2I_{0\alpha}\frac{\partial y^{\alpha}_0}{\partial x^0} \tag{53}$$

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we can see that again the term $E_{00} = 2g_{00}(\frac{\partial y^0}{\partial x^0})^2$ unchanged from the previous case $y^{\alpha'} = \lambda\delta^{\alpha'}_{\varrho} x^{\varrho}$ for $\alpha' = 1,2,3$. The last term splits into three scales for the $\alpha = 1,2,3$ where this term, as we have mentioned, expresses the "unified generalised electromagnetism". This split is exactly why we have called it this way. It would be formidable if we could interprate (in a first approach) this term as electromagnetism, weak nuclear field and strong unified nuclear in a unified pure geometrical way. Moreover, the two first terms that are gravity and ordinary mass related, splits into three scale where each one them splits into three sub-scales. The third term involves three energy scale splitting as the last term does, too. These energy scales will help us in the third paper of this series [18] to enter in the area of particle physics. For $a = 0$, we have a uniform scale involving all the terms and is the same with the previous case. The case for $a = 0$, can serce us, as a base scale, which can be seen, as the vacuum state. Moreover, the spliting of the scales for $i = 1,2,3$, c an serve us to form different subscales, that could be connected with the mass hierarchy problem abd as well, the existing number of families in Nature. Moreover, we must say that before the embedding, C^4 space had an original symmetry (as we shall see in the third paper [18]) which after the embedding has broken into several symmetries. This is exactly what we call in standard model and Higg's mechanism, spontaneous symmetry breaking. Of course, it is not spontaneous at all! There is a cause, the difference between how a 8-d observer and a 4-d one, observes Cosmos. The symmetry that is connected to our usual g_{ij} tensor is what we used to call external symmetries, while all the others , involving the g_{ij} connected with y_i and the I_{ij} involving both x_i andy_i,are what we use to call "internal". These symmetries, will be fyrher distinguished to global and local. But all these things will be extensively studied in the third paper of this series. Another comment for this paragraph is that the final case should be better be studied, involving not two but three parts, taking in account these way a term that is totally nonlinear and these non-linearity is that accompanies non-abelian theories. In addition, in [17], we argue that dark matter behaves as $\varepsilon_{d.matter} \sim \frac{l}{mr} \to m \sim R$, where l is constant that must be identified, with $w = -\frac{2}{3}$ in equation of state. We think that this proposed energy density, agrees with a certain belief on this matter that "the galaxial halo of dark matter are expanded with smoothly decreasing densities, such that the matter is increasing with respect to distance". As concerned dark energy, in [18], we solved the equation in the flat case (without embedding in the beginning) and we argue that we can provide a satisfactory explanation about the cosmological constant problem, based on the embedding function between T and t(this embedding was derived for the flat case from the solution).

6. Interpretation of the coordinates

The introduction of a C^4 as an extended space-time, automatically leads to the question, what is the physical interpretation of the coordinates of this space. We must admit that we have used more dimensions than four, but we do not wish to treat them as strings theories do. We want to connect the extra dimensions with already existing physical variables. Let us consider an element of C^4 space as

$$z^i = (z^0, z^1, z^2, z^3) = x^i + iy^i = (x^0, x^1, x^2, x^3) + i(y^0, y^1, y^2, y^3) \tag{54}$$

As we have mentioned, x_i, y_i must be of the same type which means that x_0 and y_0 are both time-like while x_1, x_2, x_3 and y_1, y_2, y_3 are space-like. If x_1, x_2, x_3 are our usual length, width and height, time can be x_0 or even y_0. In the case that time is y_0 we could define an imaginary time! But before messing with times, it is wiser to see what happens with y_1, y_2, y_3. Let us consider an elementary particle, in

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order to describe it, we must introduce a lot of information concerning its basic characteristics such as mass value, charge , spin weak isospin, colour, flavour and what ever else is still hidden. All these characteristics are not well defined, but rather ad-hoc properties that came by logic, observation and inspiration. Now, if we go back to the geodesic equation of the first embedding functions, there is a term as

$$(1-\lambda^2)\left(g_{kj}\frac{du^j}{ds}+\Gamma_{k,ij}u^i u^j\right) \tag{55}$$

and another term as

$$F_{ij}u^j\frac{dy^0}{ds} \tag{56}$$

We can observe that $(1-\lambda^2)$ stands exactly at the point that a mass term should be and that $\frac{dy^0}{ds}$ where charge q should be. These terms appeared as an echo of the information that we lost through the embedding, or just the pay back of y^i. This way, we can say that we have a sort of geometrisation for mass (from the g_{ij} part) and geometrisation of "charges" (from the I_{ij} part). This geometrisation will reflect to the equivalence principle. Specifically, before embedding, we have a C^4 or a symplectic R^8 space-time. Let us consider the case that I_{ij} vanishes. Then, there is an equivalence between velocities and accelerations of the two projection spaces $X \simeq R^4$ and $Y \simeq R^4$. But, space Y will reflect after the embedding to the definition of inertial mass, which finally in the second paper [17] will give us the equivalence principle, as a consequence. Let us now generalise the picture, we will use the the 3-d space that is defined by y^i in order to define geometrically the characteristics that elementary particles have. We like to call y^i as mass-like vectors (in the third paper [18], we can see the connection of y_i with mass eigenstates and that is the reason we called them mass-like) and the space that they are define as mass space. So, if y^i are mass-like, we need a physical quantity that is mass linked. In general relativity exists such a quantity the Schwarzschild radius r_g.

$$r_g = 2\frac{G}{c^2}m \longrightarrow r_g\frac{c^2}{G} = 2m \tag{57}$$

where m is the mass of a body. Every physical entity has a Schwarzschild radius . For instance for the Sun $r_g = 2,95\times 10^3$, for Earth $r_g = 8,87\times 10^{-3}$ and for an electron $r_g = 1,353\times 10^{-57}$. The study of a massive object through Schwarzschild radius or its mass is equivalent. Thus, it is worth to try relate the geometrical space Y with the mass property. To this end let us write $y_i = r_i$

$$\parallel r_i \parallel = \sqrt{r_1^2+r_2^2+r_3^2} = \frac{1}{4}r_g^2 = \frac{G^2}{c^4}m^2 \tag{58}$$

leading to a mass-related vector

$$(r_1,r_2,r_3) = \frac{G}{c^2}(m_1,m_2,m_3) \tag{59}$$

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where

$$\parallel m \parallel = m = \frac{1}{4}\frac{G}{c^2} r_g \tag{60}$$

Re-expressing r_i in spherical coordinates we get :

$$(r_1, r_2, r_3) \longrightarrow (r_g, \Theta, \Phi) = (\frac{G}{c^2} m, \Theta, \Phi) \tag{61}$$

where the angles Θ , Φ are related to mass states and therefore could be linked in the future to PMNS, CKM matrices in the context of a field theoretical description, combined with the scales of the previous paragraph. A vector in R^8 can be written as

$$\vec{k} = (x_1, x_2, x_3, ct, \frac{G}{c^2} m_1, \frac{G}{c^2} m_2, \frac{G}{c^2} m_3, T) \tag{62}$$

and setting G=c=1

$$\vec{k} = (x_1, x_2, x_3, t, m_1, m_2, m_3, T) \tag{63}$$

or even in C^4

$$\vec{k} = (x_1, x_2, x_3, t) + i(m_1, m_2, m_3, T) \tag{64}$$

At this part, in order to keep contact with the standard notation we perform a weak rotation in (t,T) subspace writing the metric as

$$dk^2 = dx_1^2 + dx_2^2 + dx_3^2 + dT^2 - dm_1^2 - dm_2^2 - dm_3^2 - dt^2 \tag{65}$$

giving a signature of (4,4). Our next step is to give a physical interpretation to the second time-like coordinate T. If we consider that T (which has units of meters) is the "cosmic" radius R(t) then $v = \frac{dT}{dt} = \frac{dR(t)}{dt}$ is the Hubble velocity. If such a picture is valid, we could set $T = \frac{c}{H(t)}$ where H(t) is the Hubble constant. This way

$$dT = d\left(\frac{c}{H(t)}\right) = -\frac{c}{H^2(t)} dH(t) \tag{66}$$

Writing Eq. (64) without the $d\vec{m}$ term we have:

$$dk^2 = d\vec{x}^2 + dT^2 - c^2 dt^2 \tag{67}$$

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which looks like the De-Sitter metric and models the De-Sitter's Universe in vacuum without mass. This way, the peculiar situation where we have two qualitatively different observers, one travelling in space and another travelling under the cosmic expansion, attains a simple interpretation. Let us add here that two-time approaches became recently very popular in the context of string or M-theory [2] [4] [5] [6] [7]. But we have to note that two times physics also means as we have seen a complex time, which is after all the basis of our consideration. This approach gives us many advantages, but it totally alters the way that we must look, understand and approach physically and philosophically Cosmos. Already, S. Hawking had refereed to this subject many times. If a complex time exists, Cosmos is much more different than we have thought. Our usual image, as 4-d observers (this is where we have written our usual theories) is that Cosmos looks like a giant "ring bell". But if time is complex, Cosmos will be actually a "sphere" inside the C^4 space. If such a hypothesis holds, we were driven to another paradox, comparable with the one of Ptolemy. It is very different what things seem to be, to what things actually are. Many times in our history senses have tricked us. Moreover, a singularity problem in the C^4 space, will have totally different meaning and require different approach, compared to a singularity problem in our usual 4-d space-time.

7. Special relativity in R^8

Let us now start working on the flat metric with signature (4,4)

$$ds^2 = d\vec{r}^2 + dT^2 - f^2 d\vec{m}^2 - c^2 dt^2 \tag{68}$$

where $f = \frac{G}{c^2}$. Our next step is to formulate the associated "special relativity" in R^8, compatible with all the above mentioned considerations. The first step is to write an action S.

$$S = \int_t L dt \tag{69}$$

and try to obtain a link to Einstein's special relativity action. To this end we apply the transformation $T \longleftrightarrow it$. Then

$$ds^2 = \Big(\frac{1}{c^2}\Big(\frac{d\vec{r}}{dt}\Big)^2 + \frac{1}{c^2}\Big(\frac{dT}{dt}\Big)^2 - \frac{1}{c^2}\Big(\frac{d(f\vec{m})}{dt}\Big)^2 - 1\Big)c^2 dt^2 \tag{70}$$

Introducing the notation

$$\vec{u} = \frac{d\vec{r}}{dt}, v = \frac{dT}{dt}, \vec{w} = \frac{d(f\vec{m})}{dt} \tag{71}$$

for the derivatives, the metric becomes

$$ds^2 = \Big(\frac{u^2}{c^2} + \frac{v^2}{c^2} - \frac{w^2}{c^2} - 1\Big)c^2 dt^2 \tag{72}$$

then the Lagrangian of a free point-particle is written

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$$S = -\int_t Dc\sqrt{1 - \frac{u^2}{c^2} + \frac{w^2}{c^2} - \frac{v^2}{c^2}}dt \tag{73}$$

where the constant D has dimensions of momentum. The canonical momenta are

$$p_u = \frac{\partial L}{\partial u} = -\frac{D^2 u}{L} \tag{74}$$

$$p_w = \frac{\partial L}{\partial w} = \frac{D^2 w}{L} \tag{75}$$

$$p_v = \frac{\partial L}{\partial v} = -\frac{D^2 v}{L} \tag{76}$$

while the Hamiltonian H is

$$H = p_u u + p_v v + p + w - L = -\frac{D^2 c^2}{L} \tag{77}$$

leading to

$$H = \frac{Dc}{\sqrt{1 - \frac{u^2}{c^2} + \frac{w^2}{c^2} - \frac{v^2}{c^2}}} \tag{78}$$

We can make the following observations concerning this Hamiltonian

1. If $\frac{w^2}{c^2} - \frac{v^2}{c^2} = 0 \longrightarrow f dm = dt \longrightarrow dm = \frac{1}{f}dT \longrightarrow m = \frac{1}{f}T + b$
   where m is the magnitude $m = |\vec{m}|$ and b is a constant. We can also write:

$$\int_{m_o}^{m} dm = \frac{1}{f}\int_{T_o}^{T} dT \longrightarrow m - m_o = \frac{1}{f}(t - t_o) \tag{79}$$

2. If $\vec{u} = \vec{w} = 0$ then $\frac{d\vec{m}}{dt} = 0 \longrightarrow m = m_o$
3. If $\frac{w^2}{c^2} - \frac{v^2}{c^2} = 0$ or $\vec{u} = \vec{w} = 0$ holds, the Hamiltonian coincides with the usual Hamiltonian of Einstein's special relativity for $D = m_o c$. The only free parameters are m_o and c
4. We have to give an interpretation to the velocity $\vec{w} = f\frac{\vec{m}}{dt}$. Let us write again the metric

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$$dk^2 = d\vec{r}^2 + dT^2 - f^2 d\vec{m}^2 - c^2 dt^2 \tag{80}$$

Rotating in the (t,T) plane we get:

$$dk^2 = d\vec{r}^2 - c^2 dt^2 - f^2 d\vec{m}^2 + dT^2 \tag{81}$$

Since the light speed is constant, $d\vec{r}^2 - c^2 dt^2$ is an invariant quantity. For $dT^2 - f^2 d\vec{m}^2$ a similar invariant quantity should occur

$$dT^2 - f^2 d\overrightarrow{m}^2 = \left(1 - f^2 \frac{dm^2}{dT^2}\right) dT^2 = f^2 \left(\frac{1}{f^2} dT^2 - d\overrightarrow{m}^2\right) \tag{82}$$

The equation $\left(\frac{T}{f}\right)^2 - (m_1^2 + m_2^2 + m_3^2) = 0$ defines a cone (not a light-cone) in space $M^{3,4}$ (we refer to space Y as mass space M). Setting $m = |\vec{m}| = \sqrt{m_1^2 + m_2^2 + m_3^2}$ then $\frac{m}{T} = \frac{1}{f}$. From the relation $\frac{1}{f^2} dT^2 - d\vec{m}^2 = 0$ we have $\frac{dm}{dt} = \frac{1}{f}$ where the quantity $\frac{m}{T}$ is a linear density. If T is the "Cosmos"(Universe) radius, then we get that this linear density (Cosmos' linear density) is an invariant. The above consideration holds on the cone. Consequently

$$d\vec{m} = \frac{1}{f} dT \longrightarrow m = \frac{1}{f} T + m_o \longrightarrow T = f(m - m_o) \tag{83}$$

then

$$d\vec{m} = \frac{1}{f} dT \longrightarrow \frac{\vec{m}}{dt} = \frac{1}{f} \frac{dT}{dt} \longrightarrow f \frac{\vec{m}}{dt} = \frac{dT}{dt} \longrightarrow \vec{u} = \vec{w} \tag{84}$$

which also holds on the cone. Then

$$H = \frac{Dc}{\sqrt{1 - \frac{u^2}{c^2} + \frac{w^2}{c^2} - \frac{v^2}{c^2}}} = \frac{Dc}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{85}$$

on the cone of space $M^{3,4}$. As a result, the equation $H = \frac{Dc}{\sqrt{1 - \frac{u^2}{c^2}}}$ is valid only on the cone of space $M^{3,4}$ or Einstein's special relativity is valid only on the cone of $M^{3,4}$. This way, we

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obtain a generalisation of Einstein's special relativity. This generalised picture gives us of course Einstein's special relativity plus information about matter and Cosmos' radius.

Axiom(Invariance principle): The linear density of Cosmos $\frac{d\vec{M}}{dT}$ (M is the mass of Cosmos) is constant and independent from observers in M^4. The quantity $\left(\frac{T}{f}\right)^2 - (m_1^2 + m_2^2 + m_3^2)$ is an invariance of space $M^{3,4}$ or in differential form the metric $ds_M^2 = dT^2 - f^2 d\vec{m}^2$ is invariant. Moreover $ds_R^2 = d\vec{r}^2 - c^2dt^2$ is invariant in space $R^{3,4}$. Since the variables are not mixed (flat space) the total length $ds^2 = ds_R^2 + ds_M^2$ is invariant, as well. Then, ds^2 must be invariant for all observers in R^8.

Theorem: For any quadratic form in R^n there is a group of linear transformations of space R^n that leave the associated quadratic form invariant. In the case of R^8 this group is $SO(4,4)$ or $SO(3+1,1+3)$. The linear transformations of this group are the transformations that the observers of R^8 must use in order to communicate with each other so the quadratic form will remain unchanged. This way, the "pseudo-distance" between two different points of R^8 must be the same for all observers of R^8.
Now we must "evaluate" the constant f. We have already mentioned that f is $\frac{G}{c^2}$ and we have to figure out the consistency of this choice. Let us consider two different states of Cosmos. The first state is when Cosmos was in Planck state while the second is "now". In the first one, Cosmos is considered as the theoretical Planck particle with mass m_P and length-radius l_P. Then $\frac{M}{T} = \frac{m_P}{l_P} = \frac{G}{c^2}$. In the second one Cosmos is considered to have a mass $10^{52}kgr$ and radius 10^{26} then $\frac{T}{M} \simeq \frac{10^{26}}{10^{52}} \simeq 10^{-26} \simeq \frac{G}{c^2}$. As a conclusion, these two different and far apart states lead to $f = \frac{G}{c^2}$. Of course all the above statements are valid and applicable to a Cosmos that is flat and looks as a De-Sitter Cosmos. Note that this way the coordinates of M^3 are expressed as $\frac{G}{c^2}m$, which is the Schwarzschild's radius and must be interpreted with care! We must also say that f is a global invariance and all the above results holds in R^8, while M and T are quantities concerning Cosmos. Thus, T=constant defines hyper surfaces of R^8. Additionally m_0 ($m_0 \in R$) describes a mass moving in the usual space-time originating from the sub-space M^3. Different subspaces of R^7 express different $m_1, m_2, ...$ that move inside different subspaces of the usual space-time, forming different "cosmic lines" for different masses m_i, which are connected through usual Lorentz transformations. As a conclusion, we have a local invariance, which is realized through the invariance of c and m_o. This picture extends Einstein's special relativity.

5. We considered what happens in the signature $(3+1,1+3)$ where we saw the existence of two cones. Trying a similar analysis for the signature (4,4) the (-) sign between the spaces M^4, R^4 will lead to three different "leave-spaces" which are separated since $S0(4,4)$ is not simply connected. We do not have cones of the type we are familiar with. For instance, if we are in $R^{1,3}$ we descend one dimension and we can find the cone as a hyper surface in $R^{1,3}$ ($c^2t^2 = x^2 + y^2 + z^2$). In our case, we have two spaces and we have to descend not one dimension but a whole dimensional space ($x_1^2 + x_2^2 + x_3^2 + T^2 = m_1^2 + m_2^2 + m_3^2 + c^2t^2$). We have to descend from R^8 to R^4 or M^4. This way, we have a "cone" like structure that cannot be handled as usual. We cannot formulate a "velocity" in order to proceed as we know. However, there is an alternative way through Casimir's and Pauli-Lubanki's invariants from which we can extract the existing invariance principle. If p_μ, $\mu = 1,2,..,8$ is the pure momentum vector then the expression $p_\mu p^\mu$ is an invariant

$$(p_R, p_M)(p^R, p^M) = p_R p^R - p_M p^M = -D^2 \tag{86}$$

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where D has units of momentum $[kgr\dfrac{m}{sec}] = [m\dfrac{kgr}{sec}]$. A mass m that moves in the space R^4 is described by vectors of the type $(\vec{r}, t)$ and velocities that have the general form $\vec{u} = \dfrac{1}{c^2}\dfrac{d\vec{r}}{dt}$ where c is an invariant . A "length" l that moves in the space M^4 is described by vectors of the type $(\vec{m}, t)$ and velocities that have the general form $\vec{w} = \dfrac{G}{c^3}\dfrac{d\vec{m}}{dt}$ where $\dfrac{c^3}{G}$ has dimensions $[\dfrac{kgr}{sec}]$ being an invariant, too. Of course this two evolutions must be equivalent for consistency reasons. Let us discuss what does a local observer in R^4 and M^4 experiences. Let us represent local observers of usual space as (SO) and local observers of "mass" space as (MO). An (SO) observes a Cosmos with diameter $\simeq 10^{52}$ m and he needs $\simeq 10^{18}$ sec to fully trespass it with velocity c. On the other hand, (MO) observes a Cosmos with diameter $\simeq 10^{53}$ kgr and he needs 10^{18} sec to fully trespass it with velocity c^3/G. So the trespass time is the same for the two observers. This situation is more correct in Planck's picture. What does a velocity of $[\dfrac{kgr}{sec}]$ means? Unfortunately we are used to think velocity in $[\dfrac{m}{sec}]$ and a $[\dfrac{kgr}{sec}]$ "velocity" seems irrational. In order to understand the differences between the two velocities let us consider the following case. Let us imagine two (SO) observers in the space of Milky way and Andromeda ($2.5 \cdot 10^6$ light years distance) respectively. In order to communicate they must sent a signal. If this signal travels with velocity c it will need $2.5 \cdot 10^6$ years to trespass this distance. On the other hand, this space is almost empty (one hydrogen atom per cubic meter or mass of 1 kgr distributed in this area). Two (MO) observers can communicate in 10^{-34} sec by sending signals with $\dfrac{c^3}{G}$ velocity. An (MO) signal can travel between galaxies extremely "fast", almost instantaneously. Although all observers (MO, SO) need the same time to trespass all Cosmos, the time needed to trespass local structures in Cosmos may vary tremendously between the two different kinds of observers, due to the difference between how masses and the distances between them are distributed in Cosmos. We have huge concentrations of mass in small areas and small concentrations in huge areas. Thus, specific information travelling with velocity c^3/G could lead to correlations during the Planck period which may explain the horizon and isotropy problems.

6. The elementary length leads us two three possible cases, the first one is $ds^2 > 0$, the second one is $ds^2 < 0$ and the third one $ds^2 = 0$. The question is what these three cases will represent if we apply not for the flat metric tensor but for a spherical symmetrical metric tensor, in the same spirit as we apply in the usual context of general relativity with the Schwarzschild metric which of course leads us to black holes. What must happen in order to pass from the first case $ds^2 > 0$ to $ds^2 = 0$ and afterwards to $ds^2 < 0$? What energy barrier we must overseen and is it possible? Can this energy scale that is required in order to make the passages, linked to Chandrasekhar limit? This are some questions that is worth to investigate in the future, giving us the chance to enter into a black hole. The most certain fact is that through our consideration, black holes do not have an information paradox any more, because of the existence of C^4 space. The information that we think is lost, is there inside the Y space and then the geometry of C^4 must be taken literally, in order to enter and investigate the interior of a black hole. The embedding, provide us only with the information taken from our projection space and tell us what we can observe from here. The horizon of the black hole, seems to be this "geometric" barrier.

Now we can continue to calculate the squared Hamiltonian as :

$$H^2 = \frac{D^2c^2}{1 - \dfrac{u^2}{c^2} + \dfrac{w^2}{c^2} - \dfrac{v^2}{c^2}} = D^2c^2\Big(1 + \frac{u^2}{c^2 - u^2 + w^2 - v^2}\Big) \tag{87}$$

or after some calculus

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$$H^2 = D^2c^2\Big(1 + \frac{u^2}{c^2 - u^2 + w^2 - v^2} - \frac{w^2}{c^2 - u^2 + w^2 - v^2} + \frac{v^2}{c^2 - u^2 + w^2 - v^2}\Big) \quad (88)$$

638 while conjugate momenta are

$$p_u^2 = \frac{D^2u^2}{c^2 - u^2 + w^2 - v^2} \quad (89)$$

$$p_w^2 = \frac{D^2w^2}{c^2 - u^2 + w^2 - v^2} \quad (90)$$

$$p_v^2 = \frac{D^2v^2}{c^2 - u^2 + w^2 - v^2} \quad (91)$$

639 As a result the squared Hamiltonian can be written

$$H^2 = D^2c^2 + p_u^2c^2 - p_w^2c^2 + p_v^2c^2 \quad (92)$$

640 or if the energy is conserved

$$E^2 = D^2c^2 + p_u^2c^2 - p_w^2c^2 + p_v^2c^2 \quad (93)$$

641 the first and the second terms on the right for $D = m_oc$ are the familiar terms of the Einstein's equation
642 of energy. Moreover, we can define the 8-d vector of energy-momentum as

$$\Big(p_{iu}, p_v, p_{iw}, \frac{H}{c}\Big) \quad (94)$$

643 The energy equation can be written also as

$$p_u^2 + p_v^2 - p_w^2 - \frac{H^2}{c^2} = -D^2 \quad (95)$$

644 where the left side of the equation coincides with the pseudo-measure of the 8-d vector of
645 energy-momentum.
646 ***Definition***: If $(A_1, B_1), (A_2, B_2)$ two 8-d vectors we define as the pseudo-internal product

$$(A_1, B_1) \bullet (A_2, B_2) = A_1A_2 - B_1B_2 \quad (96)$$

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where A_1, A_2, B_1, B_2 are 4-D vectors and A_1A_2, B_1B_2 Euclidean internal products. Then the pseudo-measure of an 8-d vector is

$$(A,B)^2 = A^2 - B^2 \tag{97}$$

where A^2, B^2 Euclidean measures.
As a conclusion, the square of the 8-d vector of the 8-d momentum is constant. If we use the action S we can write

$$p_{iu} = \frac{\partial S}{\partial x^i}, \quad p_{iw} = \frac{\partial S}{\partial y^i}, \quad p_{iv} = \frac{\partial S}{\partial T}, \quad H = -\frac{\partial S}{\partial t} \tag{98}$$

leading to the the Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial x^i}\right)^2 + \left(\frac{\partial S}{\partial T}\right)^2 - \frac{1}{\Lambda^2}\left(\frac{\partial S}{\partial m^i}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 + D^2 = 0 \tag{99}$$

if we set $y^i = \Lambda m^i$.

8. Angular -momentum

If $a = (a_i)$, $b = (b_i)$ are two n-dimensional vectors then the exterior product $a \times b = \tau_{ij}$ is a second rank antisymmetric tensor with dimension 6. We can write this tensor as

$$\tau_{ij} = a_i b_j - a_j b_i \tag{100}$$

$$\tau_{ij} = 0 \tag{101}$$

$$\tau_{ij} = -\tau_{ji} \tag{102}$$

In the space $K = R^8 \equiv C^4$ or $K = R^4 + iM^4$ the vectors have the form

$$k = (\vec{r}, T, \vec{m}, t) \equiv \vec{r} + i\vec{m} + T + it = (\vec{r} + T) + i(\vec{m} + t) \tag{103}$$

If we keep only the "length-mass" part then we can define the total angular-momentum in K as

$$L = \vec{k} \times \vec{p}_k \tag{104}$$

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where $\vec{p}_k = \left(p_{iu}, p_v, p_{iw}, \frac{H}{c}\right)$

This tensor $L = (L_{ij})$ has $\frac{n(n+1)}{2} = \frac{6 \times 5}{2} = 15$ components and can be written as a matrix

$$L_{ij} = \begin{pmatrix} 0 & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ -l_{12} & 0 & l_{23} & l_{24} & l_{25} & l_{26} \\ -l_{13} & -l_{23} & 0 & l_{34} & l_{35} & l_{36} \\ -l_{14} & -l_{24} & -l_{34} & 0 & l_{45} & l_{46} \\ -l_{15} & -l_{25} & -l_{35} & -l_{45} & 0 & l_{56} \\ -l_{16} & -l_{26} & -l_{36} & -l_{46} & -l_{56} & 0 \end{pmatrix}$$

or

$$L_{ij} = \begin{pmatrix} L_R & L_{RM} \\ -L_{RM}^T & L_M \end{pmatrix}$$

where L_R is our usual angular-momentum tensor in R^3, the L_M is the angular-momentum in M^3 and the L_{RM} is the mixture between them. The L_M can be interpreted as classical spin while the mixed L_{RM} as the interaction between angular-momentum and classical spin the same way that in quantum physics we have the spin-orbit coupling.

9. Poincare group

Before constructing the Poincare group in R^8 let us recall its structure as it appears in Minkowskian R^4 space-time. It consists of translations (P), rotations (J) and boosts (K). Specifically we have

1. translations (displacements) in time and space (P) which form the Abelian Lie group of translations in spacetime
2. rotations (J) in space which form the non Abelian Lie group of three dimensional rotations
3. boosts (K) which are transformations that connect two uniformly moving bodies

The symmetries J, K consist the homogeneous Lorentz group, while the semi-direct product of P and the Lorentz group, form the inhomogeneous Lorentz group or just the Poincare group. The Poincare group is a ten dimensional non-compact Lie group and actually is isometric to the group of Minkowski spacetime. We can write

$$\text{Poincare group} \cong ISO(3) \cong R^{1,3} \times SO(1,3) \tag{105}$$

where $SO(1,3)$ is the homogeneous Lorentz group and $ISO(1,3)$ the inhomogeneous one. Moreover if we set $J_i = -\varepsilon_{imn}\frac{M^{mn}}{2}$ and $K_i = M_{io}$

1. $[P_\mu, P_\nu] = 0$
2. $\frac{1}{i}[M_{\mu\nu}, P_\mu] = n_{\mu\rho}P_\nu - m_{\nu\rho}P_\mu$
3. $\frac{1}{i}[M_{\mu\nu}, P_{\rho\sigma}] = n_{\mu\rho}M_{\nu\sigma} - n_{\mu\sigma}M_{\nu\rho} - n_{\nu\rho}M_{\mu\sigma} + n_{\nu\sigma}M_{\mu\rho}$

where P is the generator of translations, M the generator of Lorentz transformations. The third relation is the homogeneous Lorentz group. Let us now form the Poincare group in the 8 dimensional

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space with signature (4.4). First of all we need to set our notation. We have two different indices with small letters $i, j = 0, 1, 2, 3$ and capital letters $I, J = R, M$ indicating the space in which we refer (using R for the usual length space and M for the mass space). From the Lagrangian we can observe that we have Galilean transformations for R^4, Galilean transformations for M^4 and Lorentzian transformations between R^4, M^4. In the case $(3+1, 1+3) \cong (4, 4)$ from the Lagrangian we have Lorentzian transformations in R^4, Lorentzian transformations in M^4 and Galilean ones between R^4, M^4. We find

1. $[P_I\mu, P_J\nu] = 0$
2. $\frac{1}{i}[M_{I\mu\nu}, P_{J\mu}] = \delta_{IJ}(n_{I\mu\rho}P_{I\nu} - m_{J\nu\rho}P_{J\mu})$
3. $\frac{1}{i}[M_{IJ\mu\nu}, P_{RS\rho\sigma}] = n_{MR\mu\rho}M_{NS\nu\sigma} - n_{MS\mu\sigma}M_{NR\nu\rho} - n_{NR\nu\rho}M_{MS\mu\sigma} + n_{NS\nu\sigma}M_{MR\mu\rho}$

where $M_{II} = M_I$, $M_{JJ} = M_J$, $P + II = P_I$, $P_{JJ} = P_J$ and δ_{IJ} is one for $I = J$ and zero for $I \neq J$. The flat metrics are for the cases:

1. $\text{sgn}(n_{I\mu\nu}) = (1, 1, 1, -1)$ and $\text{sgn}(n_{J\mu\nu}) = (-1, -1, -1, 1)$
2. $\text{sgn}(n_{I\mu\nu}) = (1, 1, 1, 1)$ and $\text{sgn}(n_{J\mu\nu}) = (-1, -1, -1, -1)$

The complete structure of the Poincare group can be found in Appendix A. Furthermore, in our usual space-time the Killing's vectors of Minkowski space-time have general solution $\xi_\mu = c_\mu + b_{\mu\gamma}x^\gamma$ where c_μ, $b_{\mu\gamma}$ are constants. The Minkowski's metric tensor has 10 unique components due to his symmetrical form. As a conclusion, it has ten linearly independent Killing vectors fields which corresponds to the 10 generators of the Poincare algebra. In the same spirit, in our case, the 8 dimensional real space, the flat metric N_{ij} is symmetric and has 36 unique components. Respectively, the 8 dimensional real space has 35 linearly independent Killing vectors which will correspond to the generators of the Poincare group, as it listed above. The Poincare group of the 8 dimensional space equipped with the metric tensor N_{ij} with signature $(4, 4)$ is represented by 36 generators. Especially, we have 6 generators from the R^3 part, 6 generators from the M^3 part and $2 \times 2 \times 4 = 16$ generators from the $R^3 \times M^3$ (mixed components) and 8 generators determined by the dimension. The Poincare group can be written as

$$\text{Poincare group} \cong ISO(4, 4) \cong R^{4,4} \times SO(4, 4) \tag{106}$$

The group $SO(4, 4)$ has $\frac{7 \times 8}{2} = 28$ generators plus 8 generators from the $R^{4,4}$ (displacements). There is a connection of the algebra of those 36 generators of the Poincare group, to the algebra of the groups $U(6)$ (has 36 generators) or $Sp(4)$ ($n(2n+1)$ generators, for $n = 4$ we have 36 generators). Both $U(6)$ and $Sp(4)$ are compact Lie group and it would be interesting to match the $ISO(4, 4)$ algebra to an algebra of a compact simply connected group.

10. Conclusion

We have explored the first steps of the formulation of a physical theory in C^4. Specifically, we have found the geodesic in C^4 and symplectic R^8. Furthermore, we have embed the usual 4-d real space-time in the symplectic R^8, in order to compare findings. We argued that the embedded geodesic equation, can describe the problem of a charged particle in gravitational field, with the advantage that we have not added ad-hoc the field A_μ, but rather A_μ was defined naturally from the the geometry of the symplectic R^8 space-time. Masses and "charges" where presented as the causality of this embedding and include the lost information. The key of this process, is the distinction of the initial Hermitian metric tensor $G_{\mu\nu}$, into a symmetric part $g_{\mu\nu}$ and to an anti-symmetric part $I_{\mu\nu}$. Moreover, we have enough room, not only to describe the field A_μ, but W_μ and G_μ as well. Afterwards, we have explored the flat case of R^8, in order to formulate a "special relativity" and what are the consequences and interpretations of this consideration, we have formed the Hamilton-Jacobi equations, the Poincare symmetry group $ISO(4,4)$ and the angular-momentum. In the next paper [17], we will proceed with the field equations in C^4, in the same spirit as general relativity does in the usual 4-d real space-time. This way, we will try to present a geometrical definition for the energy-momentum tensor $T_{\mu\nu}$.

11. Appendix A

The full Poincare noncovariant form for signature (1,3) are

1. $[J_m, P_n] = i\varepsilon_{mnk}P_k$
2. $[J_i, P_o] = 0$
3. $[K_i, P_k] = in_{ik}P_o$
4. $[K_i.P_o] = -iP_i$
5. $[J_m, J_n] = i\varepsilon_{mnk}J_k$
6. $[J_m, K_n] = i\varepsilon_{mnk}K_k$
7. $[K_m, K_n] = -i\varepsilon_{mnk}J_k$

On the other hand the Poincare group for the Galilean case in 4 dimensions are

1. $[J_m, P_n] = i\varepsilon_{mnk}P_k$
2. $[J_i, P_o] = 0$
3. $[K_i, P_k] = 0$
4. $[K_i.P_o] = iP_i$
5. $[J_m, J_n] = i\varepsilon_{mnk}J_k$
6. $[J_m, K_n] = i\varepsilon_{mnk}K_k$
7. $[K_m, K_n] = 0$

This way the Poincare group for (3+1,1+3) is represented

1. $[J_{Rm}, J_{Rn}] = i\varepsilon_{mnk}J_{Rk}$
2. $[J_{Rm}, K_{Rn}] = i\varepsilon_{mnk}K_{Rk}$
3. $[K_{Rm}, K_{Rn}] = -i\varepsilon_{mnk}J_{Rk}$
4. $[J_{Mm}, J_{Mn}] = i\varepsilon_{mnk}J_{Mk}$
5. $[J_{Mm}, K_{Mn}] = i\varepsilon_{mnk}K_{Mk}$
6. $[K_{Mm}, K_{Mn}] = -i\varepsilon_{mnk}J_{Mk}$
7. $[J_{Mm}, J_{Rn}] = i\varepsilon_{mnk}J_{MRk}$
8. $[J_{Rm}, K_{Mn}] = i\varepsilon_{mnk}K_{MRk}$
9. $[J_{Mm}, K_{Rn}] = i\varepsilon_{mnk}J_{MRk}$
10. $[K_{Mm}, K_{Rn}] = 0$
11. $[J_{Rm}, P_{Rn}] = i\varepsilon_{mnk}P_{Rk}$
12. $[J_{Ri}, P_{Ro}] = 0$
13. $[K_{Ri}, P_{Rk}] = in_{ik}P_{Ro}$
14. $[K_{Ri}.P_{Ro}] = -iP_{Ri}$
15. $[J_{Mm}, P_{Mn}] = i\varepsilon_{mnk}P_{Mk}$
16. $[J_{Mi}, P_{Mo}] = 0$
17. $[K_{Mi}, P_{Mk}] = in_{ik}P_{Mo}$
18. $[K_{Mi}.P_{Mo}] = -iP_{Mi}$

while for the (4,4) case:

1. $[J_{Rm}, J_{Rn}] = i\varepsilon_{mnk}J_{Rk}$
2. $[J_{Rm}, K_{Rn}] = i\varepsilon_{mnk}K_{Rk}$
3. $[K_{Rm}, K_{Rn}] = 0$
4. $[J_{Mm}, J_{Mn}] = i\varepsilon_{mnk}J_{Mk}$
5. $[J_{Mm}, K_{Mn}] = i\varepsilon_{mnk}K_{Mk}$
6. $[K_{Mm}, K_{Mn}] = 0$
7. $[J_{Mm}, J_{Rn}] = i\varepsilon_{mnk}J_{MRk}$

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8. $[J_{Rm}, K_{Mn}] = i\varepsilon_{mnk}K_{MRk}$
9. $[J_{Mm}, K_{Rn}] = i\varepsilon_{mnk}J_{MRk}$
10. $[K_{Mm}, K_{Rn}] = -i\varepsilon_{mnk}J_{MRk}$
11. $[J_{Rm}, P_{Rn}] = i\varepsilon_{mnk}P_{Rk}$
12. $[J_{Ri}, P_{Ro}] = 0$
13. $[K_{Ri}, P_{Rk}] = 0$
14. $[K_{Ri}.P_{Ro}] = iP_{Ri}$
15. $[J_{Mm}, P_{Mn}] = i\varepsilon_{mnk}P_{Mk}$
16. $[J_{Mi}, P_{Mo}] = 0$
17. $[K_{Mi}, P_{Mk}] = 0$
18. $[K_{Mi}.P_{Mo}] = iP_{Mi}$

12. Appendix B

In 1925 E.Cartan in his original paper "Le principe de dualite et la theorie des groupes simples et semi-simples" discovered that 8-d space has a unique property. Cartan's original statement [9] is: "Given an element A of SO(8) then there exist elements B and C of SO(8), unique up to sign, such that for any two Cayley numbers x and y in R^8, $A(x)B(y) = C(xy)$ where $A(x)$ denotes the action of A on the vector x and $A(x)B(y)$ denotes the product of the Cayley numbers $A(x)$, $B(y)$. The passage from A to B is induced by an explicit outer automorphism of order 3 of the Lie algebra so(8) of SO(8) and the passage from A to C is induced by an explicit outer automorphism of order 2 of so(8). These outer automorphisms leave fixed each element of the Lie subalgebra g_2 of the exceptional Lie group G_2 of all automorphism of the Cayley algebra."
But the automorphisms of the Lie algebra so(8) (which is the first algebra of the series $D_4, D_5, ..$) lifts to an automorphism of the Lie group Spin(8) which is the universal cover of SO(8). The fixed point of that automorphism is the exceptional group G_2. The definition of Spin(n) group is:
Definition: The group Spin(n) is

$$Spin(Q) = s \in CL(Q)_O : ss^* = 1, sVs^* \subseteq V$$

where V is a vector space and CL(Q) the Clifford geometric algebra of Q. Thus Cartan's statement can be generalised as it is presented in [9]:
"From the theory of Clifford algebras one obtains two non equivalent real spin representations, $\Delta_i : Spin(8k) \longrightarrow SO(2^{4K-1})$, $I = 1, 2$ for $K \geq 1$. The vector representation is by definition the universal covering homomorphism $\Delta_0 : Spin(8k) \longrightarrow SO(8K)$ determined up to an homomorphism of $SO(8k)$. The center of Spin(8) is $Z_2 \oplus Z_2$ which has three elements of order two ω_0, ω_1, ω_2 such that ω_i generates the kernel of Δ_i for $i = 0, 1, 2$. Any automorphism of Spin(8k) that is induced by an outer automorphism of SO(8k) fixes Δ_0 and interchanges Δ_1 and Δ_2. If $k = 1$ then each Δ_i maps Spin(8) onto SO(8) hence each Δ_i may be viewed as a covering homomorphism. Furthermore, the group of homomorphisms of Spin(8) modulo the subgroup of inner automorphisms is isomorphic to S_3, the permutation group of $0, 1, 2$ and permutes the Δ_i as can be seen from the Dynkin diagram of SO(8). In this case, for each permutation ijk of $0, 1, 2$ there is an embedding

$$Spin(8) \longmapsto SO(8) \times SO(8) \times SO(8)$$

defined by the correspondence $\xi \mapsto (\Delta_i(\xi), \Delta_j(\xi), \Delta_k(\xi))$, $\xi \in Spin(8)$. This statement is essentially the principle of triality". The proofs were presented in [10].

It is time to see what kind of "structures" have have this triality property. We can find two different uses of triality [1,3].

1. We have a triality property between $(4, 4)$, $(8, 0)$, $(0, 8)$ signatures that we will call signature's triality, which is a S_3 symmetry (every two of the signatures automatically concludes the other) and has a symmetrical Dykin diagram D_4. So from the signature's triality the three signatures $(4, 4)$, $(8, 0)$, $(0, 8)$ are all correlated and equivalent, which means that the three ones can be

"unified" and can be seen as one. This is the most extraordinary and useful fact in order to find an independed signature framework to work. The $(8,0)$, $(0,8)$ signatures provide us a pure octonionioc structure while the (4,4) a real one. We have to mention that only those three signatures share the triality property.

2. We have a triality property between vector and spinor spaces that we will call internal triality. Let us consider a vector space V, S^+ chiral spinor space and S^- antichiral spinor space, then we have the internal triality which unifies V, S^+, S^- to one form giving us the ability to define representations from one space to another; every two of them automatically concludes the other (S_3 symmetry and a D_4 Lie algebra with a Dykin diagram D_4). Each one of V, S^+, S^- is invariant under SO(8) and the unified form under Spin(8). Of course V is 8 dimensional space. Specifically if we define (V, g), (S^+, s^+), (S^-, s^-) vector space and spinor spaces respectively, where g is the metric tensor and s^+, s^- analogous tensors (charge conjugate matrices) in order to lower-raise spinor indices we can define a trilinear form which is the unification of those three spaces. This is why the principle of triality is used in M-theory, due to the ability to unify bosonic-fermionic structures (more details see in [1,2]). In the context of classical dynamics considered here, this interpretation of triality is not used. However, it is relevant for the study of quantum mechanical behaviour.

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