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Cut-and-Project Schemes for Pisot Family Substitution Tilings

Jeong-Yup Lee [1,*], Shigeki Akiyama [2] and Yasushi Nagai [3]

[1] Department of Mathematics Education, Catholic Kwandong University, Gangneung, Gangwon 210-701, Korea and KIAS, 85 Hoegiro, Dongdaemun-gu, Seoul, 02455, Korea; jylee@cku.ac.kr
[2] Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki, Japan (zip:305-8571); akiyama@math.tsukuba.ac.jp
[3] Montanuniversität, Department Mathematik und Informationstechnologie, Lehrstuhl für Mathematik und Statistik, Franz Josef Strasse 18, A-8700 Leoben, Austria; yasushi.nagai@unileoben.ac.at
* Correspondence: jylee@cku.ac.kr; Tel.: +82-10-7726-8606

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Abstract: We consider Pisot family substitution tilings in $\mathbb{R}^d$ whose dynamical spectrum is pure point. There are two cut-and-project schemes(CPS) which arise naturally: one from the Pisot family property and the other from the pure point spectrum respectively. The first CPS has an internal space $\mathbb{R}^m$ for some integer $m \in \mathbb{N}$ defined from the Pisot family property, and the second CPS has an internal space H which is an abstract space defined from the property of the pure point spectrum. However it is not known how these two CPS's are related. Here we provide a sufficient condition to make a connection between the two CPS's. In the case of Pisot unimodular substitution tiling in $\mathbb{R}$, the two CPS's turn out to be same due to [5, Remark 18.5].

Keywords: Pisot substitution tilings; pure point spectrum; regular model set; algebraic coincidence

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0. Introduction

There has been a lot of study on Pisot substitution sequences or Pisot family substitution tilings on $\mathbb{R}^d$ which characterizes the property of pure point spectrum (see [2] and therein). There are two natural cut-and-project schemes(CPS) arising in this study. One CPS is constructed with an Euclidean internal space using the Pisot property [5]. We extend the idea of constructing the CPS from the Pisot property to the Pisot family property. The other CPS is made by constructing an abstract internal space from the property of pure point spectrum [4]. These two CPS's got developed independently from different aims of study. It is not known yet if these two CPS's have any relation to each other. Here we would like to provide how these two CPS's are related showing that two internal spaces are basically isomorphic to each other in Theorem 3.1.

In this paper we are mainly interested in primitive substitution tilings in $\mathbb{R}^d$ with pure point spectrum. It is shown in [17] that primitive substitution tilings with pure point spectrum always have finite local complexity (FLC). So it is not necessary to make an assumption of FLC in the consideration of pure point spectrum.

In section 1, we visit the basic definitions of the terms that we use. In section 2, we construct a natural CPS which arises from the property of Pisot family substitution. In section 3, we recall the other CPS constructed from the property of pure point spectrum. We show in Theorem 3.1 that the two CPS's are closely related showing that there is an isomorphism between two internal spaces of the CPS's under certain model set condition. In section 4, we raise a few questions for later study.

1. Preliminary

1.1. Tilings

We begin with a set of types (or colours) $\{1, \ldots, \kappa\}$, which we fix once and for all. A *tile* in $\mathbb{R}^d$ is defined as a pair $T = (A, i)$ where $A = \mathrm{supp}(T)$ (the support of T) is a compact set in $\mathbb{R}^d$, which is the closure of its interior, and $i = l(T) \in \{1, \ldots, \kappa\}$ is the type of T. We let $g + T = (g + A, i)$ for $g \in \mathbb{R}^d$. We say that a set P of tiles is a *patch* if the number of tiles in P is finite and the tiles of P have mutually disjoint interiors. A tiling of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \bigcup\{\mathrm{supp}(T) : T \in \mathcal{T}\}$ and distinct tiles have disjoint interiors. Given a tiling $\mathcal{T}$, a finite set of tiles of $\mathcal{T}$ is called $\mathcal{T}$-patch. We always assume that any two $\mathcal{T}$-tiles with the same colour are translationally equivalent (hence there are finitely many $\mathcal{T}$-tiles up to translations).

We will make use of the following notation:

$$F^{+r} := \{x \in \mathbb{R}^d : \mathrm{dist}(x, F) \le r\} \ \text{and} \ F^{-r} := \{x \in F : \mathrm{dist}(x, \partial F) \ge r\}. \tag{1.1}$$

A *van Hove sequence* for $\mathbb{R}^d$ is a sequence $\mathcal{F} = \{F_n\}_{n \ge 1}$ of bounded measurable subsets of $\mathbb{R}^d$ satisfying

$$\lim_{n \to \infty} \mathrm{Vol}((\partial F_n)^{+r}) / \mathrm{Vol}(F_n) = 0, \ \text{for all } r > 0. \tag{1.2}$$

1.2. Delone κ-sets

Recall that a Delone set is a relatively dense and uniformly discrete subset in $\mathbb{R}^d$. We say that $\mathbf{\Lambda} = (\Lambda_i)_{i \le \kappa}$ is a *Delone κ-set* in $\mathbb{R}^d$ if each Λ_i is Delone and $\mathrm{supp}(\mathbf{\Lambda}) := \bigcup_{i=1}^{\kappa} \Lambda_i \subset \mathbb{R}^d$ is Delone.

A Delone κ-set $\mathbf{\Lambda} = (\Lambda_i)_{i \le \kappa}$ is called *representable* (by tiles) if there exist tiles $T_i = (A_i, i), i \le \kappa$, so that $\{x + T_i : x \in \Lambda_i, i \le \kappa\}$ is a tiling of $\mathbb{R}^d$, that is, $\mathbb{R}^d = \bigcup_{i \le \kappa} \bigcup_{x \in \Lambda_i} (x + A_i)$, and the sets in this union have disjoint interiors.

1.3. Substitutions

Definition 1. Let $\mathcal{A} = \{T_1, \ldots, T_\kappa\}$ be a finite set of tiles in $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them *prototiles*. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of patches made of tiles each of which is a translate of one of T_i's. We say that $\omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is a *tile-substitution* (or simply *substitution*) with expansive map ϕ if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le \kappa$, such that

$$\omega(T_j) = \{u + T_i : \ u \in \mathcal{D}_{ij}, \ i = 1, \ldots, \kappa\} \tag{1.3}$$

with

$$\phi A_j = \bigcup_{i=1}^{\kappa} (\mathcal{D}_{ij} + A_i) \quad \text{for } j \le \kappa. \tag{1.4}$$

Here all sets in the right-hand side must have disjoint interiors; it is possible for some of the $\mathcal{D}_{ij}$ to be empty. The *substitution* $\kappa \times \kappa$ *matrix* S is defined by $\mathsf{S}(i, j) = \#\mathcal{D}_{ij}$. If $\mathsf{S}^m > 0$ for some $m \in \mathbb{N}$, we say that the substitution tiling $\mathcal{T}$ is *primitive*.

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A set of algebraic integers $\Theta = \{\theta_1, \cdots, \theta_r\}$ is a *Pisot family* if for any $1 \le j \le r$, every Galois conjugate γ of θ_j, with $|\gamma| \ge 1$, is contained in Θ. For $r = 1$, with θ_1 real and $|\theta_1| > 1$, this reduces to $|\theta_1|$ being a real Pisot number, and for $r = 2$, with θ_1 non-real and $|\theta_1| > 1$, to θ_1 being a complex Pisot number. We say that $\mathcal{T}$ is a *Pisot substitution tiling* if the expansive map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a Pisot expansive factor λ, and a *Pisot family substitution tiling* if the eigenvalues of the expansive map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ form a Pisot family.

1.4. Cut and project scheme

Definition 2. A *cut and project scheme* (CPS) consists of a collection of spaces and mappings as follows;

$$\begin{array}{ccccc} \mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times H & \xrightarrow{\pi_2} & H \\ & & \cup & & \\ & & \widetilde{L} & & \end{array} \tag{1.5}$$

where $\mathbb{R}^d$ is a real Euclidean space, H is a locally compact Abelian group, π_1 and π_2 are the canonical projections, $\widetilde{L} \subset \mathbb{R}^d \times H$ is a lattice, i.e. a discrete subgroup for which the quotient group $(\mathbb{R}^d \times H)/\widetilde{L}$ is compact, $\pi_1|_{\widetilde{L}}$ is injective, and $\pi_2(\widetilde{L})$ is dense in H.

For a subset $V \subset H$, we denote $\Lambda(V) := \{\pi_1(x) \in \mathbb{R}^d : x \in \widetilde{L}, \pi_2(x) \in V\}$.

A *model set* in $\mathbb{R}^d$ is a subset Λ of $\mathbb{R}^d$ for which, up to translation, $\Lambda(W^\circ) \subset \Gamma \subset \Lambda(W)$, W is compact in H, $W = \overline{W^\circ} \neq \emptyset$. The model set Λ is *regular* if the boundary $\partial W = W \backslash W^\circ$ of W is of (Haar) measure 0. We say that $\boldsymbol{\Lambda} = (\Lambda_i)_{i \le \kappa}$ is a *model* κ*-set* (resp. *regular model* κ*-set*) if each Λ_i is a model set (resp. regular model set) with respect to the same CPS.

Without loss of generality, we assume that H is generated by the windows W_i's, where $\Lambda_i = \Lambda(W_i)$ for all $i \le \kappa$. When H satisfies the following

$$\{t \in H : t + W_i = W_i \text{ for all } i \le \kappa\} = \{0\}, \tag{1.6}$$

we say that the windows W_i's have *irredundancy*.

1.5. Pure point spectrum

Let $X_{\mathcal{T}}$ be the collection of all primitive substitution tilings in $\mathbb{R}^d$ each of whose clusters is a translate of a $\mathcal{T}$-patch. We give a usual metric δ on tilings in such a way that two tilings are close if there is a large agreement on a big area with small shift (see [15,24,26]). Then $X_{\mathcal{T}} = \overline{\{-h + \mathcal{T} : h \in \mathbb{R}^d\}}$ where the closure is taken in the topology induced by the metric δ. We have a natural action of $\mathbb{R}^d$ on the dynamical hull $X_{\mathcal{T}}$ of $\mathcal{T}$ by translations and get a topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$. Let $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ be a measure preserving dynamical system with a unique ergodic measure μ. We consider the associated group of unitary operators $\{T_x\}_{x \in \mathbb{R}^d}$ on $L^2(X_{\mathcal{T}}, \mu)$:

$$T_x g(\mathcal{T}') = g(-x + \mathcal{T}').$$

Every $g \in L^2(X_{\mathcal{T}}, \mu)$ defines a function on $\mathbb{R}^d$ by $x \mapsto \langle T_x g, g \rangle$. This function is positive definite on $\mathbb{R}^d$, so its Fourier transform is a positive measure σ_g on $\mathbb{R}^d$ called the *spectral measure* corresponding to g. The dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ is said to have *pure point spectrum* if σ_g is pure point for every

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$g \in L^2(X_{\mathcal{T}}, \mu)$. We also say that $\mathcal{T}$ has pure point spectrum if the dynamical system $(X_{\mathcal{T}}, \mu, \mathbb{R}^d)$ has pure point spectrum.

2. Cut-and-project scheme for Pisot family substitution tilings

Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with expansion map ϕ. There is a standard way to choose distinguished points in the tiles of primitive substitution tiling so that they form a ϕ-invariant Delone set. They are called *control points*. A tiling $\mathcal{T}$ is called a fixed point of the substitution ω if $\omega(\mathcal{T}) = \mathcal{T}$.

Definition 3 ([23,28]). *Let $\mathcal{T}$ be a fixed point of a primitive substitution with expansion map ϕ. For each $\mathcal{T}$-tile T, fix a tile γT in the patch $\omega(T)$; choose γT with the same relative position for all tiles of the same type. This defines a map $\gamma : \mathcal{T} \to \mathcal{T}$ called the tile map. Then define the control point for a tile $T \in \mathcal{T}$ by*

$$\{c(T)\} = \bigcap_{n=0}^{\infty} \phi^{-n}(\gamma^n T).$$

The control points have the following properties:

(a) $T' = T + c(T') - c(T)$, for any tiles T, T' of the same type;
(b) $\phi(c(T)) = c(\gamma T)$, for $T \in \mathcal{T}$.

Control points are also fixed for tiles of any tiling $\mathcal{S} \in X_{\mathcal{T}}$: they have the same relative position as in $\mathcal{T}$-tiles. Note that the choice of control points is non-unique, but there are only finitely many possibilities, determined by the choice of the tile map. Let

$$\mathcal{C} := \mathcal{C}(\mathcal{T}) = \{c(T) : T \in \mathcal{T}\}$$

be a set of control points of the tiling $\mathcal{T}$ in $\mathbb{R}^d$. Let

$$\Xi := \Xi(\mathcal{T}) = \bigcup_{i=1}^{\kappa} (\mathcal{C}_i - \mathcal{C}_i)$$

where $\mathcal{C}_i$ is the set of control points of tiles of type i. Equivalently, Ξ is the set of translation vectors between two $\mathcal{T}$-tiles of the same type.

Let us assume that ϕ is diagonalizable over $\mathbb{C}$ and the eigenvalues of ϕ are algebraically conjugate with multiplicity one. For a complex eigenvalue λ of ϕ, the 2×2 diagonal block $\begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix}$ is similar to a real 2×2 matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = S^{-1} \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} S, \tag{2.1}$$

where $\lambda = a + ib$, $a, b \in \mathbb{R}$, and $S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$. So we can assume, by appropriate choice of basis, that ϕ is diagonal with the diagonal entries equal to λ corresponding to real eigenvalues, and diagonal 2×2 blocks of the form $\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$ corresponding to complex eigenvalues $a_j + ib_j$.

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Without loss of generality, we can assume that ϕ is a diagonal matrix.

We recall the following theorem. The theorem is not in the form as shown here, but one can readily note that from the proof of [16, Thm.4.1].

Theorem 2.1. *[16, Thm.4.1] Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with expansion map ϕ. Assume that $\mathcal{T}$ has FLC, ϕ is diagonalizable, and all the eigenvalues of ϕ are algebraically conjugate with multiplicity one. Then there exists an isomorphism $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$\sigma\phi = \phi\sigma \quad \text{and} \quad k \cdot \mathbb{Z}[\phi]\alpha \subset \sigma(\mathcal{C}(\mathcal{T})) \subset \mathbb{Z}[\phi]\alpha\,,$$

where $\alpha = (1, 1, \ldots, 1) \in \mathbb{R}^d$ and $k \in \mathbb{Z}$.

Let us assume now that $\mathcal{T}$ has FLC, ϕ is diagonalizable, the eigenvalues of ϕ are all algebraically conjugate with multiplicity one, and there exists at least one other algebraic conjugate different from eigenvalues of ϕ. Suppose that ϕ has e number of real eigenvalues, and f number of 2×2 blocks of the form of complex eigenvalues where $d = e + 2f$. Let all the algebraic conjugates of eigenvalues of ϕ be real numbers $\lambda_1, \ldots, \lambda_s$ and complex numbers $\lambda_{s+1}, \overline{\lambda}_{s+1}, \ldots, \lambda_{s+t}, \overline{\lambda}_{s+t}$. Let $m := s + 2t$ and write $\lambda_{s+t+i} = \overline{\lambda}_{s+i}$ for $i = 1, \ldots t$ for the convenience. Let us consider a space $\mathbb{K}$, where

$$\mathbb{K} := \mathbb{R}^{s-e} \times \mathbb{C}^{t-f} \backsimeq \mathbb{R}^{m-d}.$$

Let us consider a following map

$$\Psi : \mathbb{Z}[\phi]\xi \to \mathbb{K}, \tag{2.2}$$

$$P(\phi)\xi \mapsto (P(\lambda_{e+1}), \ldots, P(\lambda_s), P(\lambda_{s+f+1}), \ldots, P(\lambda_{s+t})), \tag{2.3}$$

where $P(x)$ is a polynomial over $\mathbb{Z}$. Let us construct a new cut and project scheme :

$$\begin{array}{ccccc}
\mathbb{R}^d & \xleftarrow{\pi_1} & \mathbb{R}^d \times \mathbb{K} & \xrightarrow{\pi_2} & \mathbb{K} \\
 & & \cup & & \\
L & \longleftarrow & \widetilde{L} & \longrightarrow & \Psi(L) \\
 & & & & \\
x & \longleftarrow\!\!\shortmid & (x, \Psi(x)) & \longmapsto & \Psi(x)\,,
\end{array} \tag{2.4}$$

where π_1 and π_2 are canonical projections, $L = \langle \mathcal{C}_i \rangle_{i \le \kappa}$ and $\widetilde{L} = \{(x, \Psi(x)) : x \in L\}$. It is clear to see that $\pi_1|_{\widetilde{L}}$ is injective. We now show that $\pi_2(\widetilde{L})$ is dense in $\mathbb{K}$ and $\widetilde{L}$ is a lattice in $\mathbb{R}^d \times \mathbb{K}$.

Lemma 4. *$\widetilde{L}$ is a lattice in $\mathbb{R}^d \times \mathbb{K}$.*

Proof. Since $\mathbb{Z}[\phi]\alpha$ is a free $\mathbb{Z}$-module of rank m and $m \times m$ matrix $A = (\lambda_i^{j-1})_{i,j \in \{1,\ldots,m\}}$ is non-degenerate by the Vandermonde determinant, the natural embedding combining all conjugates; $f : \mathbb{Z}[\phi]\alpha \to \mathbb{R}^s \times \mathbb{C}^t \simeq \mathbb{R}^d \times \mathbb{K}$ gives a lattice $f(\mathbb{Z}[\phi]\alpha)$ in $\mathbb{R}^d \times \mathbb{K}$. Consequently $\widetilde{L}$ is isomorphic to a free $\mathbb{Z}$-submodule of $f(\mathbb{Z}[\phi]\alpha)$ due to the theory of elementary divisors. From Theorem 2.1, $\widetilde{L}$ is isomorphic to a full rank $\mathbb{Z}$-submodule of $f(\mathbb{Z}[\phi]\alpha)$, that is, a sub-lattice of $f(\mathbb{Z}[\phi]\alpha)$.Thus the

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claim is shown. The case with complex conjugates can be shown in a similar manner, taking care of embeddings $\mathbb{C}$ to $\mathbb{R}^2$. $\square$

Lemma 5. *$\Psi(L) = \pi_2(\widetilde{L})$ is dense in $\mathbb{K}$.*

Proof. As in the proof of Lemma 4, we showed that $\widetilde{L}$ is a sub-lattice of $f(\mathbb{Z}[\phi]\alpha)$, it suffices to prove that $\Psi(\mathbb{Z}[\phi])$ is dense in $\mathbb{K}$. We prove the totally real case, i.e., $\lambda_i \in \mathbb{R}$ for all i. By [27, Theorem 24], $\Psi(\mathbb{Z}[\phi])$ is dense if

$$\sum_{i=d+1}^{m} x_i \lambda_i^{j-1} \in \mathbb{Z} \quad (j = 1, \ldots, m)$$

implies $x_i = 0$ for $i = d+1, \ldots, m$. The condition is equivalent to

$$\xi A \in \mathbb{Z}^m$$

with $\xi = (x_i) = (0, \ldots, 0, x_{d+1}, \ldots, x_m) \in \mathbb{R}^m$ in the terminology of Lemma 4. Multiplying the inverse of A, we see that entries of ξ must be Galois conjugates. As ξ has at least one zero entry, we obtain $\xi = 0$ which shows $x_i = 0$ for $i = d+1, \ldots, m$. In fact, this discussion is using the Pontryagin duality that the $\Psi : \mathbb{Z}^m \to \mathbb{R}^{m-d}$ has a dense image if and only if its dual map $\widehat{\Psi} : \mathbb{R}^{m-d} \to \mathbb{T}^m$ is injective (see also [18, Chapter II, Section 1], [10] and [1]). The case with complex conjugates is similar. $\square$

3. Two cut-and-project schemes

3.1. ϕ-topology

Let $\mathcal{T}$ be a primitive substitution tiling on $\mathbb{R}^d$ with expansion map ϕ. Define

$$L := \langle \mathcal{C}_i \rangle_{i \leq \kappa}$$

be the group generated by $\mathcal{C}_i$, $i \leq \kappa$, where $\mathcal{C} = (\mathcal{C}_i)_{i \leq \kappa}$ is a control point set of $\mathcal{T}$ and

$$\mathcal{K} := \{x \in \mathbb{R}^d : \mathcal{T} + x = \mathcal{T}\}$$

be the set of periods of $\mathcal{T}$. We say that $\mathcal{T}$ admits an *algebraic coincidence* if there exist $M \in \mathbb{Z}_+$ and $\xi \in \mathcal{C}_i$ for some $i \leq \kappa$ such that $\xi + Q^M \Xi(\mathcal{T}) \subset \mathcal{C}_i$. It is known in [11] that $\mathcal{T}$ has pure point spectrum if and only if $\mathcal{T}$ admits an algebraic coincidence.

Under the assumption that $\mathcal{T}$ admits an algebraic coincidence, we introduce a topology on L and find a completion H of the topological group L such that the image of L is a dense subgroup of H. This enables us to construct a cut and project scheme (CPS) such that each point set $\mathcal{C}_i$, $i \leq \kappa$, arises from the CPS. From the following lemma, we know that the system $\{\alpha + \phi^n \Xi(\mathcal{T}) + \mathcal{K} : n \in \mathbb{Z}_+, \alpha \in L\}$ satisfies the topological properties for the group L to be a topological group ([19], [7] and [20]).

Lemma 6. *[11, Lemma 4.1] Let $\mathcal{T}$ be a primitive substitution tiling with an expansive map ϕ. Suppose that $\mathcal{T}$ admits an algebraic coincidence. Then the system $\{\phi^n \Xi(\mathcal{T}) + \mathcal{K} : n \in \mathbb{Z}_+\}$ serves as a neighbourhood base for $0 \in L$ of the topology on L relative to which L becomes a topological group.*

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We call the topology on L with the neighbourhood base $\{\alpha + \phi^n \Xi(\mathcal{T}) + \mathcal{K} : n \in \mathbb{Z}_+, \alpha \in L\}$ *ϕ-topology*. Let L_ϕ be the space L with ϕ-topology.

Let $L' = L/\mathcal{K}$. From [7, III. §3.4, §3.5] and Lemma 6, we know that there exists a complete Hausdorff topological group of L', which we denote by H, for which L' is isomorphic to a dense subgroup of the complete group H (see [4] and [14]). Furthermore there is a uniformly continuous mapping $\psi : L \to H$ which is the composition of the canonical injection of L' into H and the canonical homomorphism of L onto L' for which $\psi(L)$ is dense in H and the mapping ψ from L onto $\psi(L)$ is an open map, the latter with the induced topology of the completion H. One can directly consider H as the Hausdorff completion of L vanishing $\mathcal{K}$.

3.2. P_ϵ-topology

We introduce another topology on L which becomes equivalent to ϕ-topology under the assumption of algebraic coincidence.

Let $\{F_n\}_{n\in\mathbb{Z}_+}$ be a van Hove sequence and let $\mathcal{T}', \mathcal{T}''$ be two tilings in $\mathbb{R}^d$, where $\mathbf{\Lambda}' = (\Lambda'_i)_{i\le\kappa}$ and $\mathbf{\Lambda}'' = (\Lambda''_i)_{i\le\kappa}$ are representable Delone κ-sets of the tilings $\mathcal{T}', \mathcal{T}''$. We define

$$\rho(\mathcal{T}', \mathcal{T}'') := \lim_{n\to\infty} \sup \frac{\sum_{i=1}^{\kappa} \sharp((\Lambda'_i \triangle \Lambda''_i) \cap F_n)}{\mathrm{Vol}(F_n)}. \tag{3.1}$$

Here $\triangle$ is the symmetric difference operator. Let $P_\epsilon = \{x \in L : \rho(x + \mathcal{T}, \mathcal{T}) < \epsilon\}$ for each $\epsilon > 0$. If $\mathcal{T}$ admits an algebraic coincidence, then, for any $\epsilon > 0$, P_ϵ is relatively dense [3,4,15,22]. In this case the system $\{P_\epsilon : \epsilon > 0\}$ serves as a neighbourhood base for $0 \in L$ of the topology on L relative to which L becomes a topological group. We name P_ϵ-topology for this topology on L and denote the space L with P_ϵ-topology by L_P (see [3,4] for P_ϵ-topology under the name of autocorrelation topology).

Proposition 7. *[11, Prop. 4.6, 4.7] Let $\mathcal{T}$ be a primitive substitution tiling. Suppose that $\mathcal{T}$ admits an algebraic coincidence, then the mapping $\iota : x \mapsto x$ from L_ϕ onto L_P is topologically isomorphic.*

Remark 8. From Prop. 7, L_P is topologically isomorphic to L_ϕ. Thus the completion of L_P is topologically isomorphic to the completion H of L_ϕ. We will identify the former with H. Thus $\varphi := \psi \cdot \iota^{-1} : L_P \to H$ is uniformly continuous, $\varphi(L_P)$ is dense in H, and the mapping φ from L_P onto $\varphi(L_P)$ is an open map, the latter with the induced topology of the completion H. Therefore we can consider the CPS (1.5) with an internal space H which is a completion of L_P. Note that since $\mathcal{T}$ is repetitive, $\bigcap_{\epsilon>0} P_\epsilon = \mathcal{K}$ and $\mathcal{K} = \overline{\{0\}}$ in L_ϕ.

We observe that L_P and $\Psi(L)$ are all topologically isomorphic when the control point set $\mathcal{C}$ is a regular model κ-set in CPS(2.4).

Theorem 3.1. *Let $\mathcal{T}$ be a primitive Pisot family substitution tiling in $\mathbb{R}^d$ with an expansive map ϕ. Suppose that ϕ is diagonalizable, all the eigenvalues of ϕ are algebraic conjugates with multiplicity one, and there exists at least one algebraic conjugate λ of eigenvalues of ϕ for which $|\lambda| < 1$. If $\mathcal{C}$ is a regular model κ-set in CPS(2.4), then the internal space H which is the completion of L_ϕ with ϕ-topology is isomorphic to the internal space $\mathbb{K}$ which is constructed from using the conjugation map Ψ in (2.2).*

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PROOF. Since ϕ is an expansive map and satisfies the Pisot family condition, we first note that there is no algebraic conjugate γ of eigenvalues of ϕ with $|\gamma| = 1$.

We will show that if for $t \in L$, $\Psi(t)$ is close to 0 in $\mathbb{K}$, then $\rho(t + \mathcal{T}, \mathcal{T})$ is close to 0 in H. Since each point set $\mathcal{C}_i$ is a regular model set by the assumption where $\mathcal{C} = (\mathcal{C}_i)_{i \le \kappa}$ and $\mathcal{C}_i = \Lambda(W_i)$ in the CPS (2.4), for $t \in L$

$$\begin{aligned} \rho(t+\mathcal{T},\mathcal{T}) &= \lim_{n\to\infty} \sup \frac{\sum_{i=1}^{\kappa} \sharp(((t+\mathcal{C}_i) \triangle \mathcal{C}_i) \cap A_n)}{\mathrm{Vol}(A_n)} \\ &= \sum_{i=1}^{\kappa} \lim_{n\to\infty} \frac{\sharp(((t+\mathcal{C}_i) \triangle \mathcal{C}_i) \cap A_n)}{\mathrm{Vol}(A_n)} \\ &= \sum_{i=1}^{\kappa} (\theta(W_i \backslash (\Psi(t) + W_i)) + \theta(W_i \backslash (-\Psi(t) + W_i))), \end{aligned} \tag{3.2}$$

where θ is a Haar measure in $\mathbb{K}$(see [21, Thm. 1]).

Note that

$$\theta(W_i \backslash (s + W_i)) = \theta(W_i) - \mathbf{1}_{W_i} * \widetilde{\mathbf{1}_{W_i}}(s)$$

is uniformly continuous in $s \in \mathbb{K}$ (see [25, Subsec. 1.1.6]). So if $\Psi(t)$ converges to 0 in $\mathbb{K}$, then $\rho(t + \mathcal{T}, \mathcal{T})$ converges to 0 in $\mathbb{R}$.

On the other hand, suppose that $\{t_n\}$ is a sequence such that $\rho(t_n + \mathcal{T}, \mathcal{T}) \to 0$ as $n \to \infty$. Then for each $i \le \kappa$

$$\{\theta(W_i \backslash (\Psi(t_n) + W_i))\}_n \to 0 \text{ as } n \to \infty.$$

Note that for large enough n, $W_i \cap (\Psi(t_n) + W_i) \neq \emptyset$ and so $\Psi(t_n) \in W_i - W_i$ for all $i \le \kappa$. Since $W_i - W_i$ is compact, $\{\Psi(t_n)\}_n$ has a converging subsequence $\{\Psi(t_{n_k})\}_k$. For any such sequence define $t_0{}^* := \lim_{k\to\infty} \Psi(t_{n_k})$. Then

$$\theta(W_i \backslash (t_0{}^* + W_i)) = 0$$

and so $\theta(W_i{}^\circ \backslash (t_0{}^* + W_i)) = 0$ for each $i \le \kappa$. Thus $W_i{}^\circ \subset t_0{}^* + W_i$ and this implies $W_i \subset t_0{}^* + W_i$. On the other hand, $\lim_{k\to\infty} -\Psi(t_{n_k}) = -t_0{}^*$ and $\theta(W_i{}^\circ \backslash (-t_0{}^* + W_i)) = 0$. So $W_i \subset -t_0{}^* + W_i$. Hence $W_i \subset t_0{}^* + W_i \subset t_0{}^* - t_0{}^* + W_i$ and $W_i = t_0{}^* + W_i$. This equality is for each $i \le \kappa$. Since $\mathbb{K}$ is isomorphic to $\mathbb{R}^{m-d}$, each model set W_i has irredundancy. Thus $t_0{}^* = 0$. So all converging subsequences $\{\Psi(t_{n_k})\}_k$ converge to 0 and $\{\Psi(t_n)\}_n \to 0$ as $n \to \infty$.

This establishes the equivalence of the two topologies. By [7, Prop 5, III §3.3], there exists an isomorphism between H onto $\mathbb{K}$. □

The above theorem shows that the internal space H constructed from L_ϕ with ϕ-topology is isomorphic to Euclidean space $\mathbb{K}$(i.e. $\mathbb{R}^{m-d}$).

It is known in [5,9], [2, Thm. 3.6] that unimodular irreducible Pisot substitution tilings in $\mathbb{R}$ with pure point spectrum give rise to regular model sets. We give a precise statement below.

Theorem 3.2. *[5, Remark 18.5] Let $\mathcal{T}$ be a primitive substitution tiling in $\mathbb{R}$ with expansion factor β being a unimodular irreducible Pisot number. Then $\mathcal{T}$ has pure point spectrum if and only if for any $1 \le i \le \kappa$, each $\mathcal{C}_i$ is a regular model set in CPS (2.4).*

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Corollary 9. *Let $\mathcal{T}$ be a primitive Pisot substitution tiling in $\mathbb{R}$ with an expansion factor β. Assume that there exists at least one algebraic conjugate λ of β for which $|\lambda| < 1$. If $\mathcal{T}$ has pure point spectrum, then the internal space H which is the completion of L_ϕ with ϕ-topology can be realized by Euclidean space $\mathbb{R}^{m-1}$ where m is the degree of the characteristic polynomial of β.*

PROOF. By Theorem 3.2, it is known that for a primitive Pisot substitution tiling in $\mathbb{R}$, if $\mathcal{T}$ has pure point spectrum, then $\mathcal{C}$ is a regular model κ-set in CPS(2.4).

4. Further study

We are left with the following questions extending Theorem 3.1.

Question 10. *Can we replace the assumption of regular model κ-set by pure point spectrum? Another words, for a primitive Pisot family substitution tiling $\mathcal{T}$ in $\mathbb{R}^d$ with an expansion map ϕ, does the pure point spectrum of $\mathcal{T}$ imply that $\mathcal{C}$ is a regular model κ-set with an Euclidean internal space.*

Question 11. *Can the theorem be still extended into the case that the multiplicity of eigenvalues of ϕ is not one?*

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