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ON BOUNDS OF THE SINE AND COSINE ALONG STRAIGHT LINES ON THE COMPLEX PLANE

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ABSTRACT. In the paper, the author discusses and computes bounds of the sine and cosine along straight lines on the complex plane.

1. MOTIVATIONS

In the theory of complex functions, the sine and cosine on the complex plane $\mathbb{C}$ are denoted and defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{-iz} + e^{-iz}}{2},$$

where $z = x + iy$ and $x, y \in \mathbb{R}$. When $z = x \in \mathbb{R}$, these two functions become $\sin x$ and $\cos x$ which satisfy the periodicity and boundedness

$$\sin(x + 2k\pi) = \sin x, \quad \cos(x + 2k\pi) = \cos x, \quad |\sin x| \le 1, \quad |\cos x| \le 1$$

for $k \in \mathbb{Z}$. On the other hand, when $z = iy$ for $y \in \mathbb{R}$,

$$\sin(iy) = \frac{e^{-y} - e^{y}}{2i} \to \pm i\infty \quad \text{and} \quad \cos(iy) = \frac{e^{-y} + e^{y}}{2} \to +\infty$$

as $y \to \pm\infty$. These imply that the sine and cosine are bounded on the real x-axis, but unbounded on the imaginary y-axis.

Motivated by the above boundedness, we naturally guess that the complex functions $\sin z$ and $\cos z$ for $z \in \mathbb{C}$ are

(1) bounded on all straight lines parallel to the real x-axis,
(2) unbounded on all straight lines whose slopes are not horizontal.

In this paper, we will verify the above guesses and compute bounds for $\sin z$ and $\cos z$ on all straight lines parallel to the real x-axis.

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2. Unboundedness of sine and cosine

Let $y=\alpha+\beta x$ for constants $\alpha\in\mathbb{R}$ and $\beta\ne0$. On this straight line on the complex plane, by the triangle inequality for complex numbers, we have

$$\begin{aligned}|\sin z|&=|\sin(x+i(\alpha+\beta x))|=\left|\frac{e^{i[x+i(\alpha+\beta x)]}-e^{-i[x+i(\alpha+\beta x)]}}{2i}\right|\\
&=\left|\frac{e^{[ix-(\alpha+\beta x)]}-e^{-[ix-(\alpha+\beta x)]}}{2i}\right|\ge\frac12\Big||e^{[ix-(\alpha+\beta x)]}|-|e^{-[ix-(\alpha+\beta x)]}|\Big|\\
&=\frac12\left|e^{-(\alpha+\beta x)}-e^{(\alpha+\beta x)}\right|\to+\infty,\quad x\to\pm\infty\end{aligned}$$

and

$$\begin{aligned}|\cos z|&=|\cos(x+i(\alpha+\beta x))|=\left|\frac{e^{i[x+i(\alpha+\beta x)]}+e^{-i[x+i(\alpha+\beta x)]}}{2}\right|\\
&=\left|\frac{e^{[ix-(\alpha+\beta x)]}+e^{-[ix-(\alpha+\beta x)]}}{2}\right|\ge\frac12\Big||e^{[ix-(\alpha+\beta x)]}|-|e^{-[ix-(\alpha+\beta x)]}|\Big|\\
&=\frac12\left|e^{-(\alpha+\beta x)}-e^{(\alpha+\beta x)}\right|\to+\infty,\quad x\to\pm\infty.\end{aligned}$$

Consequently, the functions $\sin z$ and $\cos z$ are not bounded along any straight line whose slope is not horizontal.

On the vertical line $x=\gamma$ for any constant $\gamma\in\mathbb{R}$ on the complex plane, by the triangle inequality for complex numbers, we have

$$\begin{aligned}|\sin z|&=|\sin(\gamma+iy)|=\left|\frac{e^{i(\gamma+iy)}-e^{-i(\gamma+iy)}}{2i}\right|\\
&\ge\frac12\Big||e^{i(\gamma+iy)}|-|e^{-i(\gamma+iy)}|\Big|=\frac12|e^{-y}-e^{y}|\to+\infty,\quad y\to\pm\infty\end{aligned}$$

and

$$\begin{aligned}|\cos z|&=|\cos(\gamma+iy)|=\left|\frac{e^{i(\gamma+iy)}+e^{-i(\gamma+iy)}}{2}\right|\\
&\ge\frac12\Big||e^{i(\gamma+iy)}|-|e^{-i(\gamma+iy)}|\Big|=\frac12|e^{-y}-e^{y}|\to+\infty,\quad y\to\pm\infty.\end{aligned}$$

Consequently, the functions $\sin z$ and $\cos z$ are not bounded along any vertical line.

3. Boundedness of the sine

On the horizontal straight line $y=\alpha$ for any constant $\alpha\in\mathbb{R}$ on the complex plane $\mathbb{C}$, by the triangle inequality for complex numbers, we have

$$\begin{aligned}|\sin z|&=|\sin(x+i\alpha)|=\left|\frac{e^{i(x+i\alpha)}-e^{-i(x+i\alpha)}}{2i}\right|\\
&=\left|\frac{e^{(ix-\alpha)}-e^{-(ix-\alpha)}}{2i}\right|=\frac12\left|\frac{e^{ix}}{e^{\alpha}}-e^{-ix}e^{\alpha}\right|\\
&\le\frac12\left(\left|\frac{e^{ix}}{e^{\alpha}}\right|+\left|e^{-ix}e^{\alpha}\right|\right)=\frac12\left(\frac{1}{e^{\alpha}}+e^{\alpha}\right)\end{aligned}$$

and

$$|\sin z|=|\sin(x+i\alpha)|=\left|\frac{e^{i(x+i\alpha)}-e^{-i(x+i\alpha)}}{2i}\right|$$

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$$
\begin{aligned}
&= \left|\frac{e^{(ix-\alpha)} - e^{-(ix-\alpha)}}{2i}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} - e^{-ix}e^{\alpha}\right| \\
&\geq \frac{1}{2}\left|\left|\frac{e^{ix}}{e^{\alpha}}\right| - \left|e^{-ix}e^{\alpha}\right|\right| = \frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right|.
\end{aligned}
$$

Therefore, it follows that

$$
\frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right| \leq |\sin(x+i\alpha)| \leq \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right), \quad x, \alpha \in \mathbb{R}.
$$

When $z = 2k\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$
\sin z = \sin(2k\pi + i\alpha) = \frac{e^{i(2k\pi+i\alpha)} - e^{-i(2k\pi+i\alpha)}}{2i} = -\frac{i}{2}\left(\frac{1}{e^{\alpha}} - e^{\alpha}\right).
$$

When $z = 2k\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
\sin z &= \sin\left(2k\pi + \frac{\pi}{2} + i\alpha\right) = \frac{e^{i(2k\pi+\pi/2+i\alpha)} - e^{-i(2k\pi+\pi/2+i\alpha)}}{2i} \\
&= \frac{e^{i(\pi/2+i\alpha)} - e^{-i(\pi/2+i\alpha)}}{2i} = \frac{e^{-\alpha} + e^{\alpha}}{2} = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).
\end{aligned}
$$

When $z = (2k+1)\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
\sin z &= \sin((2k+1)\pi + i\alpha) = \frac{e^{i((2k+1)\pi+i\alpha)} - e^{-i((2k+1)\pi+i\alpha)}}{2i} \\
&= \frac{e^{i(\pi+i\alpha)} - e^{-i(\pi+i\alpha)}}{2i} = \frac{1}{2i}\left(e^{\alpha} - \frac{1}{e^{\alpha}}\right) = -\frac{i}{2}\left(e^{\alpha} - \frac{1}{e^{\alpha}}\right).
\end{aligned}
$$

When $z = (2k+1)\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
\sin z &= \sin\left((2k+1)\pi + \frac{\pi}{2} + i\alpha\right) = \frac{e^{i((2k+1)\pi+\pi/2+i\alpha)} - e^{-i((2k+1)\pi+\pi/2+i\alpha)}}{2i} \\
&= \frac{e^{i(3\pi/2+i\alpha)} - e^{-i(3\pi/2+i\alpha)}}{2i} = -\frac{e^{-\alpha} + e^{\alpha}}{2} = -\frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).
\end{aligned}
$$

4. Boundedness of the cosine

On the horizontal straight line $y = \alpha$ for any constant $\alpha \in \mathbb{R}$ on the complex plane $\mathbb{C}$, by the triangle inequality for complex numbers, we have

$$
\begin{aligned}
|\cos z| = |\cos(x+i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} + e^{-i(x+i\alpha)}}{2}\right| \\
&= \left|\frac{e^{(ix-\alpha)} + e^{-(ix-\alpha)}}{2}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} + e^{-ix}e^{\alpha}\right| \\
&\leq \frac{1}{2}\left(\left|\frac{e^{ix}}{e^{\alpha}}\right| + \left|e^{-ix}e^{\alpha}\right|\right) = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|\cos z| = |\cos(x+i\alpha)| &= \left|\frac{e^{i(x+i\alpha)} + e^{-i(x+i\alpha)}}{2}\right| \\
&= \left|\frac{e^{(ix-\alpha)} + e^{-(ix-\alpha)}}{2}\right| = \frac{1}{2}\left|\frac{e^{ix}}{e^{\alpha}} + e^{-ix}e^{\alpha}\right|
\end{aligned}
$$

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$$\geq \frac{1}{2}\left|\left|\frac{e^{ix}}{e^{\alpha}}\right| - \left|e^{-ix}e^{\alpha}\right|\right| = \frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right|.$$

Therefore, it follows that

$$\frac{1}{2}\left|\frac{1}{e^{\alpha}} - e^{\alpha}\right| \leq |\cos(x+i\alpha)| \leq \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right), \quad x,\alpha \in \mathbb{R}.$$

When $z = 2k\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\cos z = \cos(2k\pi + i\alpha) = \frac{e^{i(2k\pi+i\alpha)} + e^{-i(2k\pi+i\alpha)}}{2} = \frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).$$

When $z = 2k\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}\cos z = \cos\left(2k\pi + \frac{\pi}{2} + i\alpha\right) &= \frac{e^{i(2k\pi+\pi/2+i\alpha)} + e^{-i(2k\pi+\pi/2+i\alpha)}}{2}\\ &= \frac{e^{i(\pi/2+i\alpha)} + e^{-i(\pi/2+i\alpha)}}{2} = \frac{ie^{-\alpha} - ie^{\alpha}}{2} = \frac{i}{2}\left(\frac{1}{e^{\alpha}} - e^{\alpha}\right).\end{aligned}$$

When $z = (2k+1)\pi + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}\cos z = \cos((2k+1)\pi + i\alpha) &= \frac{e^{i((2k+1)\pi+i\alpha)} + e^{-i((2k+1)\pi+i\alpha)}}{2}\\ &= \frac{e^{i(\pi+i\alpha)} + e^{-i(\pi+i\alpha)}}{2} = -\frac{1}{2}\left(\frac{1}{e^{\alpha}} + e^{\alpha}\right).\end{aligned}$$

When $z = (2k+1)\pi + \frac{\pi}{2} + i\alpha$ for $k \in \mathbb{Z}$, we have

$$\begin{aligned}\cos z = \cos\left((2k+1)\pi + \frac{\pi}{2} + i\alpha\right) &= \frac{e^{i((2k+1)\pi+\pi/2+i\alpha)} + e^{-i((2k+1)\pi+\pi/2+i\alpha)}}{2}\\ &= \frac{e^{i(3\pi/2+i\alpha)} + e^{-i(3\pi/2+i\alpha)}}{2} = \frac{-ie^{-\alpha} + ie^{\alpha}}{2} = -\frac{i}{2}\left(\frac{1}{e^{\alpha}} - e^{\alpha}\right).\end{aligned}$$

5. Conclusions

On the sloped straight line $y = \alpha + \beta x$ for $\alpha \in \mathbb{R}$ and $\beta \neq 0$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(x + i(\alpha + \beta x))$ and $\cos z = \cos(x + i(\alpha + \beta x))$ are unbounded.

On the vertical line $x = \gamma$ for any scalar $\gamma \in \mathbb{R}$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(\gamma + iy)$ and $\cos z = \cos(\gamma + iy)$ are unbounded.

On the horizontal line $y = \alpha$ for any constant $\alpha \in \mathbb{R}$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z = \sin(x + i\alpha)$ and $\cos z = \cos(x + i\alpha)$ are bounded by the double inequalities

$$|\sinh\alpha| \leq |\sin(x+i\alpha)| \leq \cosh\alpha, \quad x,\alpha \in \mathbb{R} \tag{1}$$

and

$$|\sinh\alpha| \leq |\cos(x+i\alpha)| \leq \cosh\alpha, \quad x,\alpha \in \mathbb{R} \tag{2}$$

whose equalities are attained at points

$$2k\pi + i\alpha, \quad 2k\pi + \frac{\pi}{2} + i\alpha, \quad (2k+1)\pi + i\alpha, \quad 2k\pi + \frac{3\pi}{2} + i\alpha$$

with concrete values

$$\sin(2k\pi + i\alpha) = \cos\left(2k\pi + \frac{3\pi}{2} + i\alpha\right) = i\sinh\alpha,$$

$$\sin\left(2k\pi+\frac{\pi}{2}+i\alpha\right)=\cos(2k\pi+i\alpha)=\cosh\alpha,$$
$$\sin((2k+1)\pi+i\alpha)=\cos\left(2k\pi+\frac{\pi}{2}+i\alpha\right)=-i\sinh\alpha,$$
$$\sin\left(2k\pi+\frac{3\pi}{2}+i\alpha\right)=\cos((2k+1)\pi+i\alpha)=-\cosh\alpha$$

for $k\in\mathbb{Z}$.

Letting $\alpha\to0$ in the double inequalities (1) and (2) leads to

$$0\le|\sin x|\le1\quad\text{and}\quad0\le|\cos x|\le1,\quad x\in\mathbb{R}.$$

On the horizontal belt zones $0\le A\le y\le B$ and $-B\le y\le -A\le0$ on the complex plane $\mathbb{C}$, the trigonometric functions $\sin z=\sin(x+iy)$ and $\cos z=\cos(x+iy)$ are bounded by the double inequality

$$|\sinh A|\le|\cos(x\pm iy)|\le\cosh B,\quad x\in\mathbb{R}.$$