On \( p \)-common best proximity point results for \( S \)-weakly contraction in complete metric spaces

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Abstract

In this paper, we introduced many new notions and new contraction named as \( S \)-weakly contraction after that we obtained the \( p \)-common best proximity point theorems for different types of contractions in the setting of complete metric spaces by using weak \( P_p \)-property and proved the uniqueness of these points. Also we presented some examples to prove the validity of our results.

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1 Introduction

Banach Contraction Principle [4] is very familiar theorem that helps out in the branch of fixed point theory to describe the tools for finding a solution to non-linear equations of the type $Ux = x$ if given mapping $U$ is a self-mapping defined on any non-empty subset of metric space or any other relevant framework. If the given mapping $U$ is non-self then it is possible that given mapping has no solution $Ux = x$. Then in those cases we try to find those points for that non-self mapping $U$ which give us the close solution to the equation $Ux = x$, with this idea we approach towards the best approximation problems and then we obtain the solution which is not optimal but is approximate solution to the equation $Ux = x$. With the help of these approximate solutions we attain the target to find the solution which is optimal because the error $d(x, Ux)$ is minimum and $d(x, Ux) = d(A, B)$ and that optimal approximate solution is called best proximity point for given mapping which is non-self. To find out the best proximity point it is necessary that we should have only one non-self mapping with the help of that mapping we can find best proximity point. But, whenever we have more than one non-self mappings in a problem and we have to find the optimal solution for those mappings defined on same subsets of any space, then that type of optimal solution is known as common best proximity point for given mappings.

The basic purpose of this paper is to construct some new theorems with new notions and contractions and with the help of these new result we will describe common best proximity point for given mappings in metric spaces then furthermore, we will establish some examples for the justification of our results. Given results are more general than earlier ones.
2 Preliminaries

Definition 2.1. [2] Let $X$ be a metric space, $A$ and $B$ two nonempty subsets of $X$. Define
\[
d(A, B) = \inf \{d(a, b) : a \in A, b \in B\},
\]
\[
A_0 = \{a \in A : \text{there exists some } b \in B \text{ such that } d(a, b) = d(A, B)\},
\]
\[
B_0 = \{b \in B : \text{there exists some } a \in A \text{ such that } d(a, b) = d(A, B)\}.
\]

Definition 2.2. [3] Let $(A, B)$ be a pair of nonempty subsets of a metric space $(X, d)$ with $A_0 \neq \emptyset$. Then the pair $(A, B)$ is said to have the weak $P$-property if and only if for any $x_1, x_2, x_3, x_4 \in A_0$,
\[
d(x_1, fx_3) = d(A, B) \quad \text{and} \quad d(x_2, fx_4) = d(A, B) \quad \Rightarrow \quad d(x_1, x_2) \leq d(fx_3, fx_4).
\]

Definition 2.3. [6] Given a non-self mapping $f : A \to B$, then an element $x^*$ is called best proximity point of the mappings if this condition satisfied:
\[
d(x^*, fx^*) = d(A, B),
\]
where $BPP(f)$ denotes the set of best proximity points of $f$.

Definition 2.4. [5] Given a non-self mappings $f : A \to B$ and $g : A \to B$ then an element $x^*$ is called common best proximity point of the mappings if this condition satisfied:
\[
d(x^*, fx^*) = d(A, B) = d(x^*, gx^*).
\]

Definition 2.5. [7] Let $(X, d)$ be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called $w$-distance on $X$ if the following are satisfied:
1. $p(x, z) \leq p(x, y) + p(y, z)$, for any $x, y, z \in X$;
2. for any $x \in X$, $p(x, .) : X \to [0, \infty)$ is lower semi continuous;
3. for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ if $d(x, y) \leq \epsilon$.

Definition 2.6. [7] Let $(X, d)$ be a metric space. A set valued mapping $T : X \to X$ is called weakly contractive if there exists a $w$-distance $p$ on $X$ and $r \in [0, 1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$ there is $y_2 \in Tx_2$ with $p(y_1, y_2) \leq rp(x_1, x_2)$.
Lemma 2.1. [6] Let \( \{x_n\} \) be a sequence in \( X \) such that \( d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1}) \) for all \( n \in \mathbb{N} \) and \( 0 \leq k < 1 \), then \( \{x_n\} \) is a Cauchy sequence.

Theorem 2.1. [6] Let \((A, B)\) be a pair of non empty closed subsets of a complete metric space \((X, d)\) and let \( S : A \to B \) and \( T : A \to B \) such that \( A_0 \) is non empty. Assume that the following conditions are satisfied:

1. The pair \((A, B)\) has weak \( P \)-property;
2. \( d(Sx, Ty) \leq kd(x, y) \) for \( 0 \leq k < 1 \).

Then there exists a unique common best proximity point \( x \) to the pair \((S, T)\) that is \( d(x, Sx) = d(x, Tx) = d(A, B) \).

Theorem 2.2. [6] Let \((A, B)\) be a pair of non-empty closed subsets of a complete metric space \((X, d)\) and let \( S : A \to B \) and \( T : A \to B \) such that \( A_0 \) is non empty. Assume that the following conditions are satisfied:

1. The pair \((A, B)\) has weak \( P \)-property;
2. \( S \) and \( T \) are continuous;
3. \( d(Sx, Ty) \leq k \in [d(x, Sx) + d(y, Tx) - 2d(A, B)] \) for \( 0 \leq k < 1 \).

Then there exists a unique common best proximity point \( x \) to the pair \((S, T)\) that is \( d(x, Sx) = d(x, Tx) = d(A, B) \).

Theorem 2.3. [8] A \( C \)-contraction defined on a complete metric space has a unique fixed point that is if \( T : X \to X \), where \((X, d)\) is a complete metric space, satisfies

\[
d(Tx, Ty) \leq \alpha [d(x, Ty) + d(y, Tx)]
\]

where \( 0 < \alpha < 1 \) and \( x, y \in X \), then \( T \) has a unique fixed point.

3 On \( p \)-common best proximity point theorems for \( S \)-weakly contractive mappings

Definition 3.1. Let \( X \) be a metric space, \( A \) and \( B \) two nonempty subsets of \( X \). Define

\[
p(A, B) = \inf \{ p(a, b) : a \in A, b \in B \},
\]

\[
A_{0,p} = \{ a \in A : \text{there exists some } b \in B \text{ such that } p(a, b) = p(A, B) \},
\]

\[
B_{0,p} = \{ b \in B : \text{there exists some } a \in A \text{ such that } p(a, b) = p(A, B) \}.
\]
Definition 3.2. Let \((X, d)\) be a metric space. Then a function \(p : X \times X \to [0, \infty)\) is called \(w_s\)-distance on \(X\) if the following are satisfied:

1. \(p(x, z) \leq p(x, y) + p(y, z), \text{ for any } x, y, z \in X;\)

2. \(p(x, y) \geq 0, \text{ for any } x, y \in X;\)

3. if \(\{x_m\}\) and \(\{y_m\}\) be any sequences in \(X\) such that \(x_n \to x, \ y_n \to y\) as \(n \to \infty\), then \(p(x_n, y_n) \to p(x, y)\) as \(x \to \infty;\)

4. for any \(\epsilon > 0\), there exists \(\delta > 0\) such that \(p(z, x) \leq \delta\) and \(p(z, y) \leq \delta\) for any \(z \in X\).

Definition 3.3. Let \((X, d)\) be a metric space and \(A, B \subseteq X\) and \(A_{0,p} \neq \emptyset\). A set valued mapping \(T : A \to B\) with \(T(A_0, p) \subseteq B_{0,p}\) is called \(S\)-weakly contractive or \(P_p\)-contractive if there exists a \(w_s\)-distance \(p\) on \(A\) and \(r \in [0, 1)\) such that for any \(x_1, x_2 \in A\) and \(y_1, y_2 \in T x_1\) in \(B\) there is \(y_2 \in T x_2\) in \(B\) with \(p(y_1, y_2) \leq r p(x_1, x_2).\)

Definition 3.4. Let \((A, B)\) be a part of nonempty subsets of a metric space \((X, d)\) with \(A_{0,p} \neq \emptyset\). Then the pair \((A, B)\) is said to have weak \(P_p\)-property if and only if for any \(x_1, x_2 \in A_{0,p}\) and \(y_1, y_2 \in B_{0,p}\)

\[
\begin{align*}
p(x_1, y_1) &= p(A, B) & \quad (x_1, y_1) \\
p(x_2, y_2) &= p(A, B) & \quad (x_2, y_2)
\end{align*}
\]

\[
\Rightarrow p(x_1, x_2) \leq p(y_1, y_2).
\]

Definition 3.5. Given non-self mappings \(f : A \to B\) and \(g : A \to B\) then an element \(x^*\) is called \(p\)-common best proximity point of the mappings if this condition satisfied:

\[p(x^*, f x^*) = p(A, B) = p(x^*, g x^*).\]

Lemma 3.1. Let \(\{x_n\}\) be a sequence in \(X\) such that \(p(x_{n+1}, x_n) \leq k p(x_n, x_{n-1})\) for all \(n \in N\) and \(0 < k < 1\), then \(\{x_n\}\) is a Cauchy sequence.

Proof. We have, \(p(x_{n+1}, x_n) \leq k p(x_n, x_{n-1}) \leq k^2 p(x_{n-1}, x_{n-2}) \leq \ldots \leq k^n p(x_1, x_0)\). Let \(m > n \geq n_0\) for some \(n_0 \in N\). Then

\[
p(x_m, x_n) \leq p(x_m, x_{m-1}) + p(x_{m-1}, x_{m-2}) + \ldots + p(x_{n+1}, x_n)
\]

\[
\leq (k^{m-1} + k^{m-2} + \ldots + k^n)p(x_1, x_0)
\]

\[
\leq (k^n + k^{n+1} + \ldots)p(x_1, x_0)
\]

\[
= \frac{k^n}{1 - k} d(x_1, x_0) \to 0 \text{ as } n \to \infty, \text{ and as } 0 \leq k < 1.
\]

This implies \(\{x_n\}\) is a Cauchy sequence. \(\square\)
Theorem 3.1. Let \((X, d)\) be a metric space, \(A, B\) are nonempty closed subsets of \(X\) and let \(T : A \to B\) and \(U : A \to B\) be a continuous set valued \(S\)-weakly contractive or \(p_p\)-contractive mappings with \((A, B)\) satisfies the weak \(P_p\)-property where \(p\) is the \(w_s\)-distance, then there exists \(p\)-common best proximity point.

**Proof.** Since \(T\) and \(U\) are \(S\)-weakly-contractive mappings, so \(A_{0,p}\) is nonempty and \(T(A_{0,p}) \subseteq B_{0,p}\) and \(U(A_{0,p}) \subseteq B_{0,p}\), we take \(x_0 \in A_{0,p}\), there exists \(x_1 \in A_{0,p}\) such that

\[
p(x_1, Tx_0) = p(A, B).
\]

and similarly

\[
p(x_1, Ux_0) = p(A, B).
\]

Again, since \(T(A_{0,p}) \subseteq B_{0,p}\) and \(U(A_{0,p}) \subseteq B_{0,p}\), there exists \(x_2 \in A_{0,p}\) such that

\[
p(x_2, Tx_1) = p(A, B).
\]

Also,

\[
p(x_2, Ux_1) = p(A, B).
\]

Repeating this process, we get a sequence \(\{x_n\}\) in \(A_{0,p}\) satisfying

\[
p(x_{n+1}, Tx_n) = p(A, B) = p(x_{n+1}, Ux_n),
\]

for any \(n \in \mathbb{N}\).

Since \((A, B)\) has weak \(P_p\)-property, we have that

\[
p(x_n, x_{n+1}) \leq p(Tx_{n-1}, Tx_n)
\]

and

\[
p(x_n, x_{n+1}) \leq p(Ux_{n-1}, Ux_n),
\]

for any \(n \in \mathbb{N}\).

Note that \(T\) and \(S\) are \(S\)-weakly-contractive mappings and \((A, B)\) has weak \(P_p\)-property, so for any \(n \in \mathbb{N}\), we have that

\[
p(x_n, x_{n+1}) \leq p(Tx_{n-1}, Tx_n) \\
\leq rp(x_{n-1}, x_n) \\
< p(x_{n-1}, x_n).
\]
also
\[ p(x_n, x_{n+1}) \leq p(Ux_{n-1}, Ux_n) \]
\[ \leq rp(x_{n-1}, x_n) \]
\[ < p(x_{n-1}, x_n). \]

where \(0 \leq r < 1\).
\[ \Rightarrow p(x_n, x_{n+1}) < p(x_{n-1}, x_n), \]
so \{p(x_n, x_{n+1})\} is strictly decreasing sequence of nonnegative real numbers. Suppose that there exists \(n_0 \in \mathbb{N}\) such that \(p(x_{n_0}, x_{n_0+1}) = 0\). In this case,
\[ 0 = p(x_{n_0}, x_{n_0+1}) = p(Tx_{n_0-1}, Tx_{n_0}) = p(Ux_{n_0-1}, Ux_{n_0}), \]
and consequently
\[ Tx_{n_0-1} = Tx_{n_0}, \]
and
\[ Ux_{n_0-1} = Ux_{n_0}, \]
Therefore,
\[ p(A, B) = p(x_{n_0}, Tx_{n_0-1}) = p(x_{n_0}, Tx_{n_0}) = p(x_{n_0}, Ux_{n_0}). \]

Note that \(x_{n_0} \in A_0, U, Tx_{n_0-1} \in B_0\), and \(x_{n_0} = Tx_{n_0-1}, x_{n_0} = Ux_{n_0-1}\), for any \(n_0 \in \mathbb{N}\), so \(A \cap B\) is nonempty, then \(p(A, B) = 0\). Thus in this case, there exists unique \(p\)-common best proximity point, i.e. there exists unique \(x^*\) in \(A\) such that \(p(x^*, Tx^*) = p(A, B) = p(x^*, Ux^*)\).

In the contrary case, suppose that \(p(Tx_{n_0}, Tx_{n_0-1}) > 0\) and \(p(Ux_{n_0}, Ux_{n_0-1}) > 0\) this implies that \(p(x_n, x_{n+1}) > 0\) for any \(n \in \mathbb{N}\). Since \{p(x_n, x_{n+1})\} is strictly decreasing sequence of nonnegative real numbers and hence there exists \(k \geq 0\) such that
\[ \lim_{n \to \infty} p(x_n, x_{n+1}) = k. \]

We have to show that \(k=0\). Let \(k \neq 0\) and \(k > 0\), then from
\[ p(x, y) = \lim_{n \to \infty} p(x_n, x_{n+1}) \]
and
\[ p(x, y) \leq \lim \inf_{n \to \infty} p(x, x_{n+1}) \leq 0, \]
we have
\[ \lim_{n \to \infty} p(x_n, x_{n+1}) = 0. \]
for any \( n \in \mathbb{N} \). Which yields that
\[ \lim_{n \to \infty} p(x_{n-1}, x_n) = 0. \]
Hence \( k = 0 \) and this contradicts our assumption that \( k > 0 \). Therefore,
\[ \lim_{n \to \infty} p(x_n, x_{n+1}) = 0. \]
Since \( p(x_{n+1}, Tx_n) = p(A, B) \) for any \( n \in \mathbb{N} \), for fixed \( p, q \in \mathbb{N} \), we have
\[ p(x_p, Tx_{p-1}) = p(x_q, Tx_{q-1}) = p(A, B) \]
and since \((A, B)\) satisfies weak \( P_p\)-property, so
\[ p(x_p, x_q) \leq p(Tx_{p-1}, Tx_{q-1}) \text{ and } p(x_p, x_q) \leq p(Ux_{p-1}, Ux_{q-1}). \]
Now we have to show that \( \{x_n\} \) is a Cauchy sequence. By previous Lemma, we conclude that \( \{x_n\} \) is a Cauchy sequence in \( A \). Since \( \{x_n\} \subseteq A \) and \( A \) is closed subset of a complete metric space \((X, d)\). There is \( x^* \in A \) such that \( x_n \to x^* \) as \( n \to \infty \). Since \( T \) and \( U \) are continuous, so we have
\[ Tx_n \to Tx^*. \]
\[ \Rightarrow p(x_{n+1}, Tx_n) \to p(x^*, Tx^*) \]
and
\[ Ux_n \to Ux^*. \]
\[ \Rightarrow p(x_{n+1}, Ux_n) \to p(x^*, Ux^*). \]
Taking into account that \( \{p(x_{n+1}, Tx_n)\} \) and \( \{p(x_{n+1}, Ux_n)\} \) are constant sequences with a value \( p(A, B) \), we deduce
\[ p(x^*, Tx^*) = p(A, B) = p(x^*, Ux^*), \]
i.e., \( x^* \) is \( p \)-common best proximity point of \( T \).
For uniqueness of \( p \)-common best proximity point.
Since \( p \) is a \( w \)-distance also \( T \) and \( U \) are \( P_p \)-contractives then \( p(Tx, Ty) \leq rp(x, y) \) for every \( x, y \in A \) of \( X \). We suppose that given mappings \( T \) and \( S \)
has two distinct \( p \)-common best proximity points \( x_0, x_1 \in A \),
that is \( p(x_0, Tx_0) = p(x_0, Ux_0) = p(A, B) \), and \( p(x_1, Tx_1) = p(x_1, Ux_1) = p(A, B) \). Since \( T \) and \( U \) have \( P_p \)-property, then
\[
p(x_0, x_1) = p(Tx_0, Tx_1) \leq rp(x_0, x_1),
\]
and
\[
p(x_0, x_1) = p(Ux_0, Ux_1) \leq rp(x_0, x_1),
\]
which shows
\[
p(x_0, y_0) \leq rp(x_0, y_0).
\]
It contradicts towards our assumption and so we get \( x_0 = y_0 \).
Therefore, there exists unique \( p \)-common best proximity point for the pair \((S, U)\).

4 Characterizations related to \( p \)-contractive type mappings

Now we are in a position to show the results for different \( p \)-contractive type mappings.

**Theorem 4.1.** Let \((A, B)\) be a pair of non empty closed subsets of a complete metric space \( X \) and let \( S : A \to B \) and \( T : A \to B \) such that \( A_{0,p} \) is non empty and \( S, T(A_{0,p}) \subseteq B_{0,p} \). Assume that the following conditions are satisfied:

1. The pair \((A, B)\) has weak \( P_p \)-property;
2. \( p(Sx, Ty) \leq kp(x, y) \) for \( 0 \leq k < 1 \).

Then there exists a unique \( p \)-common best proximity point \( x \) to the pair \((S, T)\) that is \( p(x, Sx) = p(x, Tx) = p(A, B) \).

**Proof.** We consider \( x_0 \in A_{0,p} \) as \( A_{0,p} \) is non empty, since \( Sx_0 \in S(A_{0,p}) \subseteq B_{0,p} \), then by definition of \( A_{0,p} \) we can find \( x_1 \in A_{0,p} \), such that \( p(x_1, Sx_0) = \)
\(p(A, B)\). Again \(Tx_1 \in T(A_{0,p}) \subseteq B_{0,p}\), we find \(x_2 \in A_{0,p}\) such that \(p(x_2, Tx_1) = p(A, B)\). Since \(x_2 \in A_{0,p}\) and \(S(A_{0,p}) \subseteq B_{0,p}\), we have \(x_3 \in A_{0,p}\) such that \(p(x_3, Sx_2) = p(A, B)\). In this manner we can get \(x_4 \in A_{0,p}\) such that \(p(x_4, Tx_3) = p(A, B)\) as \(T(A_{0,p}) \subseteq B_{0,p}\) and \(Tx_3 \in B_{0,p}\). Repeated process, we obtain a sequence \(\{x_n\}\) in \(A_{0,p}\) satisfying \(p(x_{2n}, Tx_{2n-1}) = p(A, B)\), for all \(n \in N\) and \(p(x_{2n-1}, Sx_{2n-2}) = p(A, B)\), for all \(n \in N\) Since \((A, B)\) has weak \(P_p\)-property, we obtain that

\[
p(x_{2n}, x_{2n-1}) \leq p(Tx_{2n-1}, Sx_{2n-2}) = p(Sx_{2n-2}, Tx_{2n-1})
\]

for any \(n \in N\) and

\[
p(x_{2n+1}, x_{2n}) \leq p(Sx_{2n}, Tx_{2n-1}) = p(Tx_{2n-1}, Sx_{2n})
\]

for any \(n \in N\). Now \(p(x_{2n+2}, x_{2n+1}) \leq p(Sx_{2n}, Tx_{2n+1}) \leq kp(x_{2n}, x_{2n+1})\). Again \(p(x_{2n+1}, x_{2n}) \leq p(Sx_{2n}, Tx_{2n-1}) \leq kp(x_{2n}, x_{2n-1})\).

Hence, we get \(p(x_{n+1}, x_n) \leq kp(x_n, x_{n-1})\) for all \(n \in N\), where \(0 \leq k < 1\). Then by Lemma 3.1, \(\{x_n\}\) is a Cauchy sequence in \(A\). As \(A\) is closed subset of a complete metric space so \(A\) is complete. Hence there exists \(x \in A\) such that \(x_n \to x\) as \(n \to \infty\). Now we claim that \(p(Sx_n, Sx) = 0\) and \(p(Tx_m, Tx) = 0\) as \(n, m \to \infty\).

\[
p(Sx_n, Sx) \leq p(Sx_n, Tx_m) + p(Tx_m, Sx) \\
\leq k[p(x_n, x_m) + p(x_m, x)] \\
\to 0 \text{ as } n, m \to \infty.
\]

Similarly, one can show that \(p(Tx_m, Tx) = 0\). Now as \(n \to \infty\), we have

\[
p(x_{2n-1}, Sx_{2n-2}) = p(A, B)p(x, Sx) = p(A, B)
\]

and

\[
p(x_{2n}, Tx_{2n-1}) = p(A, B)p(x, Tx) = p(A, B).
\]

Therefore, \(p(x, Sx) = p(x, Tx) = p(A, B)\) that is \(x\) is a \(p\)-common best proximity point for the pair of mappings \((S, T)\). Now, we shall prove uniqueness of the \(p\)-common best proximity point to the pair of mappings \((ST)\). Let us consider another \(p\)-common best proximity point \(y\) for the pair of mappings \((S, T)\) then

\[
p(y, Sy) = p(y, Ty) = p(A, B).
\]
Then by weak $P_p$-property,
\[ p(x, Sx) = p(x, Tx) = p(A, B), \]
and
\[ p(y, Sy) = p(y, Ty) = p(A, B) \]
imply
\[ p(x, y) \leq p(Sx, Ty) \leq kp(x, y) \]
or
\[ p(x, y) \leq p(Sx, Sy) \leq p(Sx, Ty) + p(Ty, Sy) \leq k[p(x, y) + p(y, y)] = kp(x, y) \]
or
\[ p(x, y) \leq p(Tx, Ty) \leq p(Tx, Sy) + p(Sy, Ty) \leq k[p(x, y) + p(y, y)] = kp(x, y). \]

As $0 \leq k < 1$, in any of the above three cases, we conclude a contradiction. Hence there exists a unique $p$-common best proximity point to the pair $(S, T)$ that is $p(x, Sx) = p(x, Tx) = p(A, B)$. \qed

5 \hspace{1cm} C_p\text{-contractive mapping}

\textbf{Theorem 5.1.} Let $(A, B)$ be a pair of non empty closed subsets of a complete metric space $(X, d)$ and let $S: A \to B$ and $T: A \to B$ such that $A_{0,p}$ is non empty, $S, T(A_{0,p}) \subseteq B_{0,p}$ and $B_{0,p}$ is closed. Assume that the following conditions are satisfied:

1. The pair $(A, B)$ has weak $P_p$-property;
2. $S$ and $T$ are continuous;
3. $p(Sx, Ty) \leq \frac{k}{2}[p(x, Ty) + p(y, Sx) - 2p(A, B)]$ for $0 \leq k < 1$.

Then there exists a unique $p$-common best proximity point $x$ to the pair $(S, T)$ that is $p(x, Sx) = p(x, Tx) = p(A, B)$.

\textit{Proof.} Since $A_{0,p} \neq \emptyset$ and the pair $(A, B)$ satisfies weak $P_p$-property, also $B_{0,p}$ is closed. We have $S(A_{0,p}) \subseteq B_{0,p}$ and $T(A_{0,p}) \subseteq B_{0,p}$. Let us define an
operator $PA_{0,p} : S(A_{0,p}) \rightarrow A_{0,p}$, by $PA_{0,p}y = \{x \in A_{0,p} : p(x,y) = p(A,B)\}$. Since the pair $(A,B)$ has weak $P_p$-property, then

$$p(PA_{0,p}(Sx), Sx) = p(A,B)$$

and

$$p(PA_{0,p}(Sy), Sy) = p(A,B).$$

imply that

$$p(PA_{0,p}(Sx)PA_{0,p}(Sy)) \leq p(Sx, Sy)$$

$$\leq \frac{k}{2}[p(x, Sy) + p(y, Sx) - 2p(A,B)]$$

$$\leq \frac{k}{2}[p(x, PA_{0,p}(Sy)) + p(PA_{0,p}(Sy), Sy) + p(y, PA_{0,p}(Sy)) + p(PA_{0,p}(Sy), Sx)]$$

for any $x, y \in A_{0,p}$ and $0 \leq k < 1$. This gives that $PA_{0,p}oS : A_{0,p} \rightarrow A_{0,p}$ is $C_p$-contractive mapping from complete metric subspace $A_{0,p}$ into itself then by [8], we can see that $PA_{0,p}oS$ has a unique $p$-fixed point say $x_1$. That is $PA_{0,p}oSx_1 = x_1 \in A_{0,p}$, which implies that $p(x_1, Sx_1) = p(A,B)$. In the same fashion, we can take a mapping $PA0 : T(A_{0,p}) \rightarrow A_{0,p}$ and see that $x^2$ is unique $p$-fixed point of $PA_{0,p}oT$. That is $PA_{0,p}oTx_2 = x_2 \in A_{0,p}$, which implies that $p(x_2, Tx_2) = p(A,B)$. Now we will show that $x_1 = x_2$. Since $(A,B)$ satisfies weak $P_p$-property, then $p(x_1, Sx_1) = p(A,B)$ and $p(x_2, Tx_2) = p(A,B)$ imply that

$$p(x_1, x_2) \leq p(Sx_1, Tx_2)$$

$$\leq \frac{k}{2}\{p(x_1, Tx_2) + p(x_2, Sx_1) - 2p(A,B)\}$$

$$\leq \frac{k}{2}\{p(x_1, x_2) + p(x_2, Tx_2) + p(x_2, x_1) + p(x_1, Sx_1) - 2p(A,B)\}$$

$$= \frac{k}{2}\{p(x_1, x_2)\}$$

$$= kp(x_1, x_2)$$

which shows that $x_1 = x_2 = x$ (say). Therefore

$$p(x, Sx) = p(x, Tx) = p(A,B).$$
Now we prove the uniqueness of the $p$-common best proximity point theorem. Let $y$ be another $p$-common best proximity point for the pair of mappings $(S,T)$. Then

$$p(x, Sx) = p(x, Tx) = p(A, B).$$

Then by weak $P_p$-property, we have

$$p(x, y) \leq p(Sx, Ty) \leq \frac{k}{2} \{p(x, Ty) + p(y, Sx) - 2p(A, B)\}$$

or

$$p(x, y) \leq p(Sx, Sy) \leq \frac{k}{2} \{p(x, Ty) + p(y, Sx) - 2p(A, B)\} + \frac{k}{2} \{p(y, Ty) + p(y, Sx) - 2p(A, B)\}$$

or

$$p(x, y) \leq p(Tx, Ty) \leq \frac{k}{2} \{p(x, Ty) + p(y, Tx) - 2p(A, B)\} + \frac{k}{2} \{p(y, Sx) + p(y, Ty) - 2p(A, B)\}$$

or

$$p(x, y) \leq p(Sx, Sy) \leq \frac{k}{2} \{p(x, Ty) + p(y, Sx) - 2p(A, B)\} + \frac{k}{2} \{p(y, Ty) + p(y, Sx) - 2p(A, B)\}$$

As $0 \leq k < 1$, in any of the above three different situations we conclude that $x = y$. Hence there exists a unique $p$-common best proximity point $x$ to the pair $(S, T)$ that is

$$p(x, Sx) = p(x, Tx) = p(A, B).$$
Example 5.1. Consider $X = \mathbb{R}^2$, with the $p$-distance defined as $p((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. Let $A = \{(x, 1) : 0 \leq x < \infty\}$ and $B = \{(x, 0) : 0 \leq x < \infty\}$. Obviously, $p(A, B) = 1$ and $A, B$ are nonempty subsets of $X$, take $A_{0,p} = A$ and $B_{0,p} = B$.

We define $S : A \to B$ as:

$$S(x, 1) = (\frac{x + 1}{3}, 0),$$

where $(x, 1) \in A$.

Let $T : A \to B$ defined as:

$$T(x, 1) = (\frac{x + 1}{4}, 0).$$

Then, we see that $S(A_{0,p}) \subseteq B_{0,p}$ and $T(A_{0,p}) \subseteq B_{0,p}$. Also, the pair $(A, B)$ has weak $P_p$-property as:

$$p((x_1, 1), (y_1, 1)) = \sqrt{(1 - 0^2) + (x_1 - y_1)^2} = p(A, B) = 1,$$

and

$$p((x_2, 1), (y_2, 1)) = \sqrt{(1 - 0^2) + (x_2 - y_2)^2} = p(A, B) = 1,$$

then one can easily obtain $x_1 = y_1$ and $x_2 = y_2$, hence $p((x_1, 1), (x_2, 1)) = |x_1 - x_2| = |y_1 - y_2| \leq p((y_1, 0), (y_2, 0))$. Furthermore, $p((0, 1), (0, 2)) = 1 = p(A, B)$ and $p((0, 1), (0, 0)) = 1 = p(A, B)$, implies that $p((0, 1), (0, 0)) = 1 = p(A, B)$. Thus, the given pair $(A, B)$ satisfies the weak $P_p$-property but not $P_p$-property.

Next, for any different $x, y$, let us suppose two elements $(x_1, 1), (x_2, 1) \in A$,

$$p(S(x_1, 1), (x_2, 1)) = p((\frac{x_1 + 1}{3}, 0), (\frac{x_2 + 1}{4}, 0))$$

$$= \frac{x}{3} - \frac{y}{4} + \frac{1}{12}$$

$$\leq k|x - y|$$

$$\leq kp((x_1, 1), (x_2, 1))$$

for any $k \in [0, 1)$. If $x_1 = x_2$ then surely this satisfied. So every condition of the theorem is satisfied thus one can find the unique $p$-common best proximity point for given pair of mappings $(S, T)$. Hence, that $p$-common best proximity point is $(0, 1) \in A$. 

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Competing interests
The authors declared that they have no competing interests.

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