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On p-common best proximity point results for $\mathcal{S}$-weakly contraction in complete metric spaces

Chayut Kongban[1], Somayya Komal[1] and Poom Kumam[1,2,*]

[1]KMUTTFixed Point Research Laboratory, Department of Mathematics,
Room SCL 802 Fixed Point Laboratory, Science Laboratory Building,
Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT),
126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand
Emails: chayut_kb@hotmail.com (C. Kongban) and
somayya.komal@mail.kmutt.ac.th (S. Komal)
[2]KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA),
Theoretical and Computational Science (TaCS) Center,
Science Laboratory Building, Faculty of Science,
King Mongkut's University of Technology Thonburi (KMUTT),
126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand.
Email: poom.kum@kmutt.ac.th

Abstract

In this paper, we introduced many new notions and new contraction named as $\mathcal{S}$-weakly contraction after that we obtained the p-common best proximity point theorems for different types of contractions in the setting of complete metric spaces by using weak P_p-property and proved the uniqueness of these points. Also we presented some examples to prove the validity of our results.
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*Corresponding author: poom.kum@kmutt.ac.th

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1 Introduction

Banach Contraction Principle [4] is very familiar theorem that helps out in the branch of fixed point theory to describe the tools for finding a solution to non-linear equations of the type $Ux = x$ if given mapping U is a self-mapping defined on any non-empty subset of metric space or any other relevant framework. If the given mapping U is non-self then it is possible that given mapping has no solution $Ux = x$. Then in those cases we try to find those points for that non-self mapping U which give us the close solution to the equation $Ux = x$, with this idea we approach towards the best approximation problems and then we obtain the solution which is not optimal but is approximate solution to the equation $Ux = x$. With the help of these approximate solutions we attain the target to find the solution which is optimal because the error $d(x, Ux)$ is minimum and $d(x, Ux) = d(A, B)$ and that optimal approximate solution is called best proximity point for given mapping which is non-self. To find out the best proximity point it is necessary that we should have only one non-self mapping with the help of that mapping we can find best proximity point. But, whenever we have more than one non-self mappings in a problem and we have to find the optimal solution for those mappings defined on same subsets of any space, then that type of optimal solution is known as common best proximity point for given mappings.s

The basic purpose of this paper is to construct some new theorems with new notions and contractions and with the help of these new result we will describe common best proximity point for given mappings in metric spaces then furthermore, we will establish some examples for the justification of our results. Given results are more general than earlier ones.

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2 Preliminaries

Definition 2.1. *[2] Let X be a metric space, A and B two nonempty subsets of X. Define*

$$\begin{aligned} d(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ A_0 &= \{a \in A : \textit{there exists some } b \in B \textit{ such that } d(a,b) = d(A,B)\}, \\ B_0 &= \{b \in B : \textit{there exists some } a \in A \textit{ such that } d(a,b) = d(A,B)\}. \end{aligned}$$

Definition 2.2. *[3] Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$.Then the pair (A,B) is said to have the weak P-property if and only if for any $x_1, x_2, x_3, x_4 \in A_0$,*

$$\left.\begin{aligned} d(x_1, fx_3) &= d(A,B) \\ d(x_2, fx_4) &= d(A,B) \end{aligned}\right\} \Rightarrow d(x_1,x_2) \leq d(fx_3, fx_4).$$

Definition 2.3. *[6] Given a non-self mapping $f : A \to B$, then an element x^* is called best proximity point of the mappings if this condition satisfied:*

$$d(x^*, fx^*) = d(A,B),$$

where $BPP(f)$ denotes the set of best proximity points of f.

Definition 2.4. *[5] Given a non-self mappings $f : A \to B$ and $g : A \to B$ then an element x^* is called common best proximity point of the mappings if this condition satisfied:*

$$d(x^*, fx^*) = d(A,B) = d(x^*, gx^*).$$

Definition 2.5. *[7] Let (X,d) be a metric space. Then a function $p : X \times X \to [0,\infty)$ is called w-distance on X if the following are satisfied:*

1. *$p(x,z) \leq p(x,y) + p(y,z)$, for any $x,y,z \in X$;*

2. *for any $x \in X$, $p(x,.) : X \to [0,\infty)$ is lower semi continuous;*

3. *for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ $d(x,y) \leq \epsilon$.*

Definition 2.6. *[7] Let (X,d) be a metric space. A set valued mapping $T : X \to X$ is called weakly contractive if there exists a w-distance p on X and $r \in [0,1)$ such that for any $x_1, x_2 \in X$ and $y_1 \in Tx_1$ there is $y_2 \in Tx_2$ with $p(y_1,y_2) \leq rp(x_1,x_2)$.*

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Lemma 2.1. *[6] Let $\{x_n\}$ be a sequence in X such that $d(x_{n+1}, x_n) \leq kd(x_n, x_{n-1})$ for all $n \in N$ and $0 \leq k < 1$, then $\{x_n\}$ is a Cauchy sequence.*

Theorem 2.1. *[6] Let (A, B) be a pair of non empty closed subsets of a complete metric space (X, d) and let $S : A \to B$ and $T : A \to B$ such that A_0 is non empty. Assume that the following conditions are satisfied:*

1. *The pair (A, B) has weak P-property;*
2. *$d(Sx, Ty) \leq kd(x, y)$ for $0 \leq k < 1$.*

Then there exists a unique common best proximity point x to the pair (S, T) that is $d(x, Sx) = d(x, Tx) = d(A, B)$.

Theorem 2.2. *[6] Let (A, B) be a pair of non-empty closed subsets of a complete metric space (X, d) and let $S : A \to B$ and $T : A \to B$ such that A_0 is non empty. Assume that the following conditions are satisfied:*

1. *The pair (A, B) has weak P-property;*
2. *S and T are continuous;*
3. *$d(Sx, Ty) \leq k \in [d(x, Sx) + d(y, Ty) - 2d(A, B)]$ for $0 \leq k < 1$.*

Then there exists a unique common best proximity point x to the pair (S, T) that is $d(x, Sx) = d(x, Tx) = d(A, B)$.

Theorem 2.3. *[8] A C-contraction defined on a complete metric space has a unique fixed point that is if $T : X \to X$, where (X, d) is a complete metric space, satisfies*

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)]$$

where $0 < \alpha < 1$. and $x, y \in X$, then T has a unique fixed point.

3 On p-common best proximity point theorems for $\mathcal{S}$-weakly contractive mappings

Definition 3.1. *Let X be a metric space, A and B two nonempty subsets of X. Define*

$$\begin{aligned} p(A, B) &= \inf\{p(a, b) : a \in A, b \in B\}, \\ A_{0,p} &= \{a \in A : \textit{there exists some } b \in B \textit{ such that } p(a, b) = p(A, B)\}, \\ B_{0,p} &= \{b \in B : \textit{there exists some } a \in A \textit{ such that } p(a, b) = p(A, B)\}. \end{aligned}$$

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Definition 3.2. *Let (X,d) be a metric space. Then a function $p : X \times X \to [0,\infty)$ is called w_s-distance on X if the following are satisfied:*

1. *$p(x,z) \leq p(x,y) + p(y,z)$, for any $x,y,z \in X$;*
2. *$p(x,y) \geq 0$, for any $x,y \in X$;*
3. *if $\{x_m\}$ and $\{y_m\}$ be any sequences in X such that $x_n \to x, \quad y_n \to y$ as $n \to \infty$, then $p(x_n,y_n) \to p(x,y)$ as $x \to \infty$;*
4. *for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ $d(x,y) \leq \epsilon$.*

Definition 3.3. *Let (X,d) be a metric space and $A,B \subseteq X$ and $A_{0,p} \neq \emptyset$. A set valued mapping $T : A \to B$ with $T(A_0,p) \subseteq B_{0,p}$ is called $\mathcal{S}$-weakly contractive or P_p-contractive if there exists a w_s-distance p on A and $r \in [0,1)$ such that for any $x_1,x_2 \in A$ and $y_1 \in Tx_1$ in B there is $y_2 \in Tx_2$ in B with $p(y_1,y_2) \leq rp(x_1,x_2)$.*

Definition 3.4. *Let (A,B) be a part of nonempty subsets of a metric space (X,d) with $A_{0,p} \neq \emptyset$. Then the pair (A,B) is said to have weak P_p-property if and only if for any $x_1,x_2 \in A_{0,p}$ and $y_1,y_2 \in B_{0,p}$*

$$\left.\begin{array}{ll} p(x_1,y_1) & = p(A,B) \\ p(x_2,y_2) & = p(A,B) \end{array}\right\} \Rightarrow p(x_1,x_2) \leq p(y_1,y_2).$$

Definition 3.5. *Given non-self mappings $f : A \to B$ and $g : A \to B$ then an element x^* is called p-common best proximity point of the mappings if this condition satisfied:*

$$p(x^*, fx^*) = p(A,B) = p(x^*, gx^*).$$

Lemma 3.1. *Let $\{x_n\}$ be a sequence in X such that $p(x_{n+1},x_n) \leq kp(x_n,x_{n-1})$ for all $n \in N$ and $0 \leq k < 1$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. We have, $p(x_{n+1},x_n) \leq kp(x_n,x_{n-1}) \leq k^2p(x_{n-1},x_{n-2}) \leq ... \leq k^np(x_1,x_0)$. Let $m > n \geq n_0$ for some $n_0 \in N$. Then

$$\begin{aligned} p(x_m,x_n) &\leq p(x_m,x_{m-1}) + p(x_{m-1},x_{m-2}) + ... + p(x_{n+1},x_n) \\ &\leq (k^{m-1} + k^{m-2} + ... + k^n)p(x_1,x_0) \\ &\leq (k^n + k^{n+1} + ...)p(x_1,x_0) \\ &= \frac{k^n}{1-k}d(x_1,x_0) \to 0 as n \to \infty, and as 0 \leq k < 1. \end{aligned}$$

This implies $\{x_n\}$ is a Cauchy sequence. □

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Theorem 3.1. *Let (X, d) be a metric space, A, B are nonempty closed subsets of X and let $T : A \to B$ and $U : A \to B$ be a continuous set valued S-weakly contractives or p_p-contractive mappings with (A, B) satisfies the weak P_p-property where p is the w_s-distance, then there exists p-common best proximity point.*

Proof. Since T and U are $\mathcal{S}$-weakly-contractive mappings, so $A_{0,p}$ is nonempty and $T(A_{0,p}) \subseteq B_{0,p}$ and $U(A_{0,p}) \subseteq B_{0,p}$, we take $x_0 \in A_{0,p}$, there exists $x_1 \in A_{0,p}$ such that

$$p(x_1, Tx_0) = p(A, B). \tag{1}$$

and similarly

$$p(x_1, Ux_0) = p(A, B). \tag{2}$$

Again, since $T(A_{0,p}) \subseteq B_{0,p}$ and $U(A_{0,p}) \subseteq B_{0,p}$, there exists $x_2 \in A_{0,p}$ such that

$$p(x_2, Tx_1) = p(A, B). \tag{3}$$

Also,

$$p(x_2, Ux_1) = p(A, B). \tag{4}$$

Repeating this process, we get a sequence $\{x_n\}$ in $A_{0,p}$ satisfying

$$p(x_{n+1}, Tx_n) = p(A, B) = p(x_{n+1}, Ux_n),$$

for any $n \in \mathbb{N}$.
Since (A, B) has weak P_p-property, we have that

$$p(x_n, x_{n+1}) \leq p(Tx_{n-1}, Tx_n)$$

and

$$p(x_n, x_{n+1}) \leq p(Ux_{n-1}, Ux_n),$$

for any $n \in \mathbb{N}$.
Note that T and S are $\mathcal{S}$-weakly-contractive mappings and (A, B) has weak P_p-property, so for any $n \in \mathbb{N}$, we have that

$$\begin{aligned} p(x_n, x_{n+1}) &\leq p(Tx_{n-1}, Tx_n) \\ &\leq rp(x_{n-1}, x_n) \\ &< p(x_{n-1}, x_n). \end{aligned}$$

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also

$$\begin{aligned} p(x_n, x_{n+1}) &\le p(Ux_{n-1}, Ux_n) \\ &\le rp(x_{n-1}, x_n) \\ &< p(x_{n-1}, x_n). \end{aligned}$$

where $0 \le r < 1$.

$$\Rightarrow p(x_n, x_{n+1}) < p(x_{n-1}, x_n),$$

so $\{p(x_n, x_{n+1})\}$ is strictly decreasing sequence of nonnegative real numbers. Suppose that there exists $n_0 \in \mathbb{N}$ such that $p(x_{n_0}, x_{n_0+1}) = 0$. In this case,

$$0 = p(x_{n_0}, x_{n_0+1}) = p(Tx_{n_0-1}, Tx_{n_0}) = p(Ux_{n_0-1}, Ux_{n_0}),$$

and consequently

$$Tx_{n_0-1} = Tx_{n_0},$$

and

$$Ux_{n_0-1} = Ux_{n_0},$$

Therefore,

$$p(A, B) = p(x_{n_0}, Tx_{n_0-1}) = p(x_{n_0}, Tx_{n_0}) = p(x_{n_0}, Ux_{n_0}).$$

Note that $x_{n_0} \in A_0$, $U, Tx_{n_0-1} \in B_0$, and $x_{n_0} = Tx_{n_0-1}$, $x_{n_0} = Ux_{n_0-1}$, for any $n_0 \in \mathbb{N}$, so $A \bigcap B$ is nonempty, then $p(A, B) = 0$. Thus in this case, there exists unique p-common best proximity point, i.e. there exists unique x^* in A such that $p(x^*, Tx^*) = p(A, B) = p(x^*, Ux^*)$.
In the contrary case, suppose that $p(Tx_{n_0}, Tx_{n_0-1}) > 0$ and $p(Ux_{n_0}, Ux_{n_0-1}) > 0$ this implies that $p(x_n, x_{n+1}) > 0$ for any $n \in N$. Since $\{p(x_n, x_{n+1})\}$ is strictly decreasing sequence of nonnegative real numbers and hence there exists $k \ge 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = k.$$

We have to show that k=0. Let $k \neq 0$ and $k > 0$, then from

$$p(x, y) = \lim_{n\to\infty} p(x_n, x_{n+1})$$

and

$$p(x, y) \le \liminf_{n\to\infty} p(x, x_{n+1}) \le 0,$$

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we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

for any $n \in \mathbb{N}$. Which yields that

$$\lim_{n\to\infty} p(x_{n-1}, x_n) = 0.$$

Hence $k = 0$ and this contradicts our assumption that $k > 0$. Therefore,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

Since $p(x_{n+1}, Tx_n) = p(A, B)$ for any $n \in \mathbb{N}$, for fixed $p, q \in \mathbb{N}$, we have

$$p(x_p, Tx_{p-1}) = p(x_q, Tx_{q-1}) = p(A, B)$$

and since (A, B) satisfies weak P_p-property, so

$$p(x_p, x_q) \leq p(Tx_{p-1}, Tx_{q-1}) \text{ and } p(x_p, x_q) \leq p(Ux_{p-1}, Ux_{q-1}).$$

Now we have to show that $\{x_n\}$ is a Cauchy sequence. By previous Lemma, we conclude that $\{x_n\}$ is a Cauchy sequence in A. Since $\{x_n\} \subseteq A$ and A is closed subset of a complete metric space (X, d). There is $x^* \in A$ such that $x_n \to x^*$ as $n \to \infty$. Since T and U are continuous, so we have

$$Tx_n \to Tx^*.$$

$$\Rightarrow p(x_{n+1}, Tx_n) \to p(x^*, Tx^*)$$

and

$$Ux_n \to Ux^*.$$

$$\Rightarrow p(x_{n+1}, Ux_n) \to p(x^*, Ux^*).$$

Taking into account that $\{p(x_{n+1}, Tx_n)\}$ and $\{p(x_{n+1}, Ux_n)\}$ are constant sequences with a value $p(A, B)$, we deduce

$$p(x^*, Tx^*) = p(A, B) = p(x^*, Ux^*),$$

i.e., x^* is p-common best proximity point of T.
For uniqueness of p-common best proximity point.
Since p is a w-distance also T and U are P_p-contractives then $p(Tx, Ty) \leq rp(x, y)$ for every $x, y \in A$ of X. We suppose that given mappings T and S

has two distinct p-common best proximity points $x_0, x_1 \in A$,
that is $p(x_0, Tx_0) = p(x_0, Ux_0) = p(A, B)$, and $p(x_1, Tx_1) = p(x_1, Ux_1) = p(A, B)$. Since T and U have P_p-property, then

$$\begin{aligned} p(x_0, x_1) &= p(Tx_0, Tx_1) \\ &\leq rp(x_0, x_1), \end{aligned}$$

and

$$\begin{aligned} p(x_0, x_1) &= p(Ux_0, Ux_1) \\ &\leq rp(x_0, x_1), \end{aligned}$$

which shows

$$p(x_0, y_0) \leq rp(x_0, y_0).$$

It contradicts towards our assumption and so we get $x_0 = y_0$.
Therefore, there exists unique p-common best proximity point for the pair (S, U). □

4 Characterizations related to p-contractive type mappings

Now we are in a position to show the results for different p-contractive type mappings.

Theorem 4.1. *Let (A, B) be a pair of non empty closed subsets of a complete metric space X and let $S : A \to B$ and $T : A \to B$ such that $A_{0,p}$ is non empty and $S, T(A_{0,p}) \subseteq B_{0,p}$. Assume that the following conditions are satisfied:*

1. *The pair (A, B) has weak P_p-property;*
2. *$p(Sx, Ty) \leq kp(x, y)$ for $0 \leq k < 1$.*

Then there exists a unique p-common best proximity point x to the pair (S, T) that is $p(x, Sx) = p(x, Tx) = p(A, B)$.

Proof. We consider $x_0 \in A_{0,p}$ as $A_{0,p}$ is non empty, since $Sx_0 \in S(A_{0,p}) \subseteq B_{0,p}$, then by definition of $A_{0,p}$ we can find $x_1 \in A_{0,p}$, such that $p(x_1, Sx_0) =$

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$p(A,B)$. Again $Tx_1 \in T(A_{0,p}) \subseteq B_{0,p}$, we find $x_2 \in A_{0,p}$ such that $p(x_2, Tx_1) = p(A,B)$. Since $x_2 \in A_{0,p}$ and $S(A_{0,p}) \subseteq B_{0,p}$, we have $x_3 \in A_{0,p}$ such that $p(x_3, Sx_2) = p(A,B)$. In this manner we can get $x_4 \in A_{0,p}$ such that $p(x_4, Tx_3) = p(A,B)$ as $T(A_{0,p}) \subseteq B_{0,p}$ and $Tx_3 \in B_{0,p}$. Repeated process, we obtain a sequence $\{x_n\}$ in $A_{0,p}$ satisfying $p(x_{2n}, Tx_{2n-1}) = p(A,B)$, for all $n \in N$ and $p(x_{2n-1}, Sx_{2n-2}) = p(A,B)$, for all $n \in N$ Since (A,B) has weak P_p- property, we obtain that

$$p(x_{2n}, x_{2n-1}) \leq p(Tx_{2n-1}, Sx_{2n-2}) = p(Sx_{2n-2}, Tx_{2n-1})$$

for any $n \in N$ and

$$p(x_{2n+1}, x_{2n}) \leq p(Sx_{2n}, Tx_{2n-1}) = p(Tx_{2n-1}, Sx_{2n})$$

for any $n \in N$. Now $p(x_{2n+2}, x_{2n+1}) \leq p(Sx_{2n}, Tx_{2n+1}) \leq kp(x_{2n}, x_{2n+1})$. Again $p(x_{2n+1}, x_{2n}) \leq p(Sx_{2n}, Tx_{2n-1}) \leq kp(x_{2n}, x_{2n-1})$.
Hence, we get $p(x_{n+1}, x_n) \leq kp(x_n, x_{n-1})$ for all $n \in N$, where $0 \leq k < 1$. Then by Lemma 3.1, $\{x_n\}$ is a Cauchy sequence in A. As A is closed subset of a complete metric space so A is complete. Hence there exists $x \in A$ such that $x_n \to x$ as $n \to \infty$. Now we claim that $p(Sx_n, Sx) = 0$ and $p(Tx_m, Tx) = 0$ as $n, m \to \infty$.

$$\begin{aligned} p(Sx_n, Sx) &\leq p(Sx_n, Tx_m) + p(Tx_m, Sx) \\ &\leq k[p(x_n, x_m) + p(x_m, x)] \\ &\to 0 \text{ as } n, m \to \infty. \end{aligned}$$

Similarly, one can show that $p(Tx_m, Tx) = 0$. Now as $n \to \infty$, we have

$$p(x_{2n-1}, Sx_{2n-2}) = p(A,B))p(x, Sx) = p(A,B)$$

and

$$p(x_{2n}, Tx_{2n-1}) = p(A,B)p(x, Tx) = p(A,B).$$

Therefore, $p(x, Sx) = p(x, Tx) = p(A,B)$ that is x is a p-common best proximity point for the pair of mappings (S,T). Now, we shall prove uniqueness of the p-common best proximity point to the pair of mappings (ST). Let us consider another p-common best proximity point y for the pair of mappings (S,T) then

$$p(y, Sy) = p(y, Ty) = p(A,B).$$

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Then by weak P_p-property,

$$p(x, Sx) = p(x, Tx) = p(A, B),$$

and

$$p(y, Sy) = p(y, Ty) = p(A, B)$$

imply

$$p(x, y) \leq p(Sx, Ty) \leq kp(x, y)$$

or

$$p(x, y) \leq p(Sx, Sy) \leq p(Sx, Ty) + p(Ty, Sy) \leq k[p(x, y) + p(y, y)] = kp(x, y)$$

or

$$p(x, y) \leq p(Tx, Ty) \leq p(Tx, Sy) + p(Sy, Ty) \leq k[p(x, y) + p(y, y)] = kp(x, y).$$

As $0 \leq k < 1$, in any of the above three cases, we conclude a contradiction. Hence there exists a unique p-common best proximity point to the pair (S, T) that is $p(x, Sx) = p(x, Tx) = p(A, B)$. □

5 C_p-contractive mapping

Theorem 5.1. *Let (A, B) be a pair of non empty closed subsets of a complete metric space (X, d) and let $S : A \to B$ and $T : A \to B$ such that $A_{0,p}$ is non empty, $S, T(A_{0,p}) \subseteq B_{0,p}$ and $B_{0,p}$ is closed. Assume that the following conditions are satisfied:*

1. *The pair (A, B) has weak P_p-property;*
2. *S and T are continuous;*
3. *$p(Sx, Ty) \leq \frac{k}{2}[p(x, Ty) + p(y, Sx) - 2p(A, B)]$ for $0 \leq k < 1$.*

Then there exists a unique p-common best proximity point x to the pair (S, T) that is $p(x, Sx) = p(x, Tx) = p(A, B)$.

Proof. Since $A_{0,p} \neq \emptyset$ and the pair (A, B) satisfies weak P_p-property, also $B_{0,p}$ is closed. We have $S(A_{0,p}) \subseteq B_{0,p}$ and $T(A_{0,p}) \subseteq B_{0,p}$. Let us define an

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operator $PA_{0,p} : S(\overline{A_{0,p}}) \to A_{0,p}$, by $PA_{0,p}y = \{x \in A_{0,p} : p(x,y) = p(A,B)\}$. Since the pair (A,B) has weak P_p-property, then

$$p(PA_{0,p}(Sx), Sx) = p(A,B)$$

and

$$p(PA_{0,p}(Sy), Sy) = p(A,B).$$

imply that

$$\begin{aligned} p(PA_{0,p}(Sx)PA_{0,p}(Sy)) &\leq p(Sx,Sy) \\ &\leq \frac{k}{2}[p(x,Sy) + p(y,Sx) - 2p(A,B)] \\ &\leq \frac{k}{2}[p(x,PA_{0,p}(Sy)) + p(PA_{0,p}(Sy),Sy) + p(y,PA_{0,p}(Sx)) + p(PA_{0,p}(S \\ &\leq \frac{k}{2}[p(x,PA_{0,p}(Sy)) + p(y,PA_{0,p}(Sx))]. \end{aligned}$$

for any $x, y \in \overline{A_{0,p}}$ and $0 \leq k < 1$. This gives that $PA_{0,p}oS : \overline{A_{0,p}} \to \overline{A_{0,p}}$ is C_p-contractive mapping from complete metric subspace $\overline{A_{0,p}}$ into itself then by [8], we can see that $PA_{0,p}oS$ has a unique p-fixed point say x_1. That is $PA_{0,p}oSx_1 = x_1 \in A_{0,p}$, which implies that $p(x_1, Sx_1) = p(A,B)$. In the same fashion, we can take a mapping $PA0 : T(\overline{A_{0,p}}) \to A_{0,p}$ and see that x^2 is unique p-fixed point of $PA_{0,p}oT$. That is $PA_{0,p}oTx_2 = x_2 \in A_{0,p}$, which implies that $p(x_2, Tx_2) = p(A,B)$. Now we will show that $x_1 = x_2$. Since (A,B) satisfies weak P_p-property, then $p(x_1, Sx_1) = p(A,B)$ and $p(x_2, Tx_2) = p(A,B)$ imply that

$$\begin{aligned} p(x_1,x_2) &\leq p(Sx_1, Tx_2) \\ &\leq \frac{k}{2}\{p(x_1,Tx_2) + p(x_2,Sx_1) - 2p(A,B)\} \\ &\leq \frac{k}{2}\{p(x_1,x_2) + p(x_2,Tx_2) + p(x_2,x_1) + p(x_1,Sx_1) - 2p(A,B)\} \\ &= \frac{k}{2}\{p(x_1,x_2)\} \\ &= kp(x_1,x_2) \end{aligned}$$

which shows that $x_1 = x_2 = x(say)$. Therefore

$$p(x,Sx) = p(x,Tx) = p(A,B).$$

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Now we prove the uniqueness of the p-common best proximity point theorem. Let y be another p-common best proximity point for the pair of mappings (S,T). Then

$$p(x,Sx)=p(x,Tx)=p(A,B).$$
$$p(y,Sy)=p(y,Ty)=p(A,B).$$

Then by weak P_p-property, we have

$$\begin{aligned}
p(x,y) &\leq p(Sx,Ty)\\
&\leq \frac{k}{2}\{p(x,Ty)+p(y,Sx)-2p(A,B)\}\\
&\leq \frac{k}{2}\{p(x,y)+p(y,Ty)+p(y,x)+p(x,Sx)-2p(A,B)\}\\
&= kp(x,y)
\end{aligned}$$

or

$$\begin{aligned}
p(x,y) &\leq p(Sx,Sy)\\
&\leq \{p(Sx,Ty)+p(Ty,Sy)\}\\
&\leq \frac{k}{2}\{p(x,Ty)+p(y,Sx)-2p(A,B)\}+\frac{k}{2}\{p(y,Ty)+p(y,Sy)-2p(A,B)\}\\
&\leq \frac{k}{2}\{p(x,y)+p(y,Ty)+p(y,x)+p(x,Sx)-2p(A,B)\}\\
&\quad +\frac{k}{2}\{p(y,Ty)+p(y,Sy)-2p(A,B)\}\\
&= kp(x,y)
\end{aligned}$$

or

$$\begin{aligned}
p(x,y) &\leq p(Tx,Ty)\\
&\leq \{p(Tx,Sy)+p(Sy,Ty)\}\\
&\leq \frac{k}{2}\{p(x,Sy)+p(y,Tx)-2p(A,B)\}+\frac{k}{2}\{p(y,Sy)+p(y,Ty)-2p(A,B)\}\\
&\leq \frac{k}{2}\{p(x,y)+p(y,Sy)+p(y,x)+p(x,Tx)-2p(A,B)\}\\
&= kp(x,y).
\end{aligned}$$

As $0\leq k<1$, in any of the above three different situations we conclude that $x=y$. Hence there exists a unique p-common best proximity point x to the pair (S,T) that is

$$p(x,Sx)=p(x,Tx)=p(A,B).$$

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□

Example 5.1. *Consider $X = \mathbb{R}^2$, with the with the p-distance defined as $p((x_1, y_1),(x_2, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$. Let $A = \{(x, 1) : 0 \leq x < \infty\}$ and $B = \{(x, 0) : 0 \leq x < \infty\}$. Obviously, $p(A, B) = 1$ and A, B are nonempty subsets of X, take $A_{0,p} = A$ and $B_{0,p} = B$.*
We define $S : A \to B$ as:

$$S(x, 1) = (\frac{x + 1}{3}, 0),$$

where $(x, 1) \in A$.
Let $T : A \to B$ defined as:

$$T(x, 1) = (\frac{x + 1}{4}, 0).$$

Then, we see that $S(\overline{A_{0,p}}) \subseteq B_{0,p}$ and $T(\overline{A_{0,p}}) \subseteq B_{0,p}$. Also, the pair (A, B) has weak P_p-property as:

$$p((x_1, 1),(y_1, 1)) = \sqrt{(1 - 0^2) + (x_1 - y_1)^2} = p(A, B) = 1,$$

and

$$p((x_2, 1),(y_2, 1)) = \sqrt{(1 - 0^2) + (x_2 - y_2)^2} = p(A, B) = 1,$$

then one can easily obtain $x_1 = y_1$ and $x_2 = y_2$, hence $p((x_1, 1),(x_2, 1)) = |x_1 - x_2| = |y_1 - y_2| \leq p((y_1, 0),(y_2, 0))$. Furthermore, $p((0, 1),(0, 2)) = 1 = p(A, B)$ and $p((0, 1),(0, 0)) = 1 = p(A, B)$,
implies that $p((0, 1),(0, 0)) = 1 = p(A, B)$. Thus, the given pair (A, B) satisfies the weak P_p-property but not P_p-property.
Next, for any different x, y, let us suppose two elements $(x_1, 1),(x_2, 1) \in A$,

$$\begin{aligned} p(S(x_1, 1),(x_2, 1)) &= p((\frac{x_1 + 1}{3}, 0),(\frac{x_2 + 1}{4}, 0)) \\ &= \frac{x}{3} - \frac{y}{4} + \frac{1}{12} \\ &\leq k|x - y| \\ &\leq kp((x_1, 1),(x_2, 1)) \end{aligned}$$

for any $k \in [0, 1)$. If $x_1 = x_2$ then surely this satisfied. So every condition of the theorem is satisfied thus one can find the unique p-common best proximity point for given pair of mappings (S, T). Hence, that p-common best proximity point is $(0, 1) \in A$.

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Competing interests
The authors declared that they have no competing interests.

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