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RANDOM FIXED POINT THEOREMS FOR GENERALIZED RANDOM $\alpha-\psi-$CONTRACTIVE MAPPINGS WITH APPLICATIONS TO STOCHASTIC DIFFERENTIAL EQUATION

CHAYUT KONGBAN[1], POOM KUMAM[1,2,§], JUAN MARTÍNEZ-MORENO[3]

[1]KMUTTFixed Point Research Laboratory, Department of Mathematics,
Room SCL 802 Fixed Point Laboratory, Science Laboratory Building,
Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT),
126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand

[2]KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA),
Theoretical and Computational Science Center (TaCS), Science Laboratory Building,
Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT),
126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand

[3] Department of Mathematics, Faculty of Experimental Science, University of Jaén,
Campus Las Lagunillas,s/n, 23071 Jaén, Spain

Email addresses: chayut_kb@hotmail.com (C. Kongban), poom.kum@kmutt.ac.th (P. Kumam),
jmmoreno@ujaen.es (J. Martínez-Moreno)

Abstract. In this paper, we prove some random fixed point theorems for generalized random $\alpha-\psi-$contractive mappings in a Polish space and, as some applications, we show the existence of random solutions of second order random differential equation.

Keywords : random fixed point, random $\alpha-$admissible with respect to η, generalized random $\alpha-\psi-$contractive mapping.

Mathematics Subject Classification: 47H10; 47H40.

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1. INTRODUCTION

Random fixed point theorems are stochastic generalization of a classical fixed point theorems. Random fixed point theorems for contraction mapping in aPolish space, i.e., a separable complete metric space, were proved by Špaček [22], Hanš [5, 6]. Some random fixed point theorems play amain role in developing theory of random differential and random integral equations (see, [2, 8, 15]). In 1996, Mukhejea [16] proved the random fixed point theorem of Schauder's type in otomic probability measure space. In 1984, Sehgal and Waters [20] proved the random fixed point theorem of the classical Rothe's fixed point theorem. The random fixed point theory and applications developed very rapidly (see, Bharucha-Reid [3], Itoh [7], Beg and Shahzad [1], Li [14], Kumam et al. [10–13], Nieto [17]).

In 2012, Samet et al. [19] introduced a new concept of $\alpha-\psi-$contractive type and $\alpha-$admissible mappings and establish fixed point theorems for such mappings in complete metric spaces. Afterwards Karapinar and Samet [9] introduced the concepts of

§ *Corresponding author: poom.kum@kmutt.ac.th (P. Kumam).*

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generalized $\alpha-\psi-$contractive type mapping. In 2013, Salimi et al. [18] modified the notion of $\alpha-$admissible and $\alpha-\psi-$contractive mappings and established certain fixed point theorems. Our results are proper generalizations of the recent results in [9, 19].

Rencently, Tchier and Vetro [21] introduced the concepts of random $\alpha-$admissible and random $\alpha-\psi-$contractive mappings and established random fixed point theorems.

The purpose of this paper is to prove some random fixed point theorems for generalized random $\alpha-\psi-$contractive mappings in a Polish space and, by using our main results, we show the existence of random solutions of second order random differential equation.

2. PRELIMINARIES

We denote the Borel $\sigma-$algebra on a metric space M by $B(M)$. Let (Ω,Σ) be a measurable space with Σ a $\sigma-$algebra of subsets of Ω. So that by $\Sigma\times B(M)$ we mean the smallest $\sigma-$algebra on $\Omega\times M$ containing all the sets $A\times B$ (with $A\in\Sigma$ and $B\in B(M)$).

Definition 2.1. Let (Ω,Σ) be a measurable space, M and N be two metric spaces. A mapping $f:\Omega\times M\to N$ is called Carathéodory if, for all $m\in M$, the mapping $\omega\to f(\omega,m)$ is $(\Sigma,B(N))-$measurable ($\Sigma-$measurable for short) and, for all $\omega\in\Omega$, the mapping $m\to f(\omega,m)$ is continuous.

Theorem 2.2. *[4] If (Ω,Σ) is a measurable space, M is a separable metric space, N is a metric space, and $f:\Omega\times M\to N$ is a Carathéodory mapping, then f is $\Sigma\times B(M)-$measurable.*

Corollary 2.3. *[4] If (Ω,Σ) is a measurable space, M is a separable metric space, N is a metric space, and $f:\Omega\times M\to N$ is a Carathéodory mapping, and $u:\Omega\to M$ is $\Sigma-$measurable, then mapping $\omega\to f(\omega,u(\omega))$ is a $\Sigma-$measurable mapping from Ω into N.*

Definition 2.4. [4] Let (Ω,Σ) be a measurable space, M a separable metric space and N a metric space. A function $f:\Omega\times M\to N$ is said to be superpositionally measurable (sup-measurable for short), if for all $u:\Omega\to M$ is $\Sigma-$measurable, the function $\omega\to f(\omega,u(\omega))$ is $\Sigma-$measurable from Ω into N.

Remark 2.5. [4] Corollary 2.3 says that a Carathéodory function is sup-measurable. Also, every $\Sigma\times B(M)-$measurable functions $f:\Omega\times M\to N$ is sup-measurable.

Definition 2.6. A mapping $f:\Omega\times M\to M$ ys called random operator whenever, for any $x\in M$, $f(\cdot,x)$ is $\Sigma-$measurable. So, a random fixed point of f is $\Sigma-$measurable mapping $z:\Omega\times M$ such that $z(\omega)=f(\omega,z(\omega))$ for all $\omega\in\Omega$.

Lemma 2.7. *Let M,N be two locally compact metric spaces. A mapping $f:\Omega\times M\to N$ is Carathéodory if and only if the mapping $\omega\to r(\omega)(\cdot)=f(\omega,\cdot)$ is $\Sigma-$measurable from Ω to $C(M,N)$ (i.e., the space of all continuous functions from M into N endowed with the compact-open topology).*

Let Ψ be the family of all nondecreasing functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\sum_{n=1}^{+\infty}\psi^n(t)<+\infty$ for each $t>0$, where ψ^n denote the nth iterate of ψ.

Lemma 2.8. *For every nondecreasing function $\psi:[0,+\infty)\to[0,+\infty)$, the following implication holds:*

$$\forall t>0,\quad \lim_{n\to+\infty}\psi^n(t)=0 \Longrightarrow \psi(t)<t.$$

Definition 2.9. Let $T:\Omega\times M\to M$ and $\alpha:\Omega\times M\times M\to[0,+\infty)$. We say that T is a random $\alpha-$admissible if

$$u,v\in M,\ \omega\in\Omega,\ \ \alpha(\omega,u,v)\geq 1\Longrightarrow \alpha(\omega,T(\omega,u),T(\omega,v))\geq 1.$$

Definition 2.10. Let (Ω,Σ) be a measurable space, (M,d) be a separable metric space, and $T:\Omega\times M\to M$ be a given mapping. We say that T is a random $\alpha-\psi-$contractive mapping if there exist functions $\alpha:\Omega\times M\times M\to[0,+\infty)$ and $\psi_\omega\in\Psi$, $\omega\in\Omega$, such that

$$\alpha(\omega,u,v)d(T(\omega,u),T(\omega,v))\leq\psi_\omega(d(u,v)),$$

for all $u,v\in M$ and $\omega\in\Omega$ such that $\alpha(\omega,u,v)\geq 1$.

3. Main Results

Definition 3.1. Let $T:\Omega\times M\to M$ and $\alpha,\eta:\Omega\times M\times M\to[0,+\infty)$. We say that T is a random $\alpha-$admissible with respect to η if

$$u,v\in M,\ \omega\in\Omega,\ \ \alpha(\omega,u,v)\geq\eta(\omega,u,v)\Rightarrow\alpha(\omega,T(\omega,u),T(\omega,v))\geq\eta(\omega,T(\omega,u),T(\omega,v)).$$

Note that if we take $\eta(\omega,u,v)=1$, then this definition reduces to Definition 2.9.

Definition 3.2. Let (Ω,Σ) be a measurable space, (M,d) be a separable space, and $T:\Omega\times M\to M$ be a given mapping. We say that T is a generalized random $\alpha-\psi-$contractive mapping if there exist functions $\alpha,\eta:\Omega\times M\times M\to[0,+\infty)$ and $\psi_\omega\in\Psi$, $\omega\in\Omega$, such that

$$\alpha(\omega,u,v)\geq\eta(\omega,u,v)\Rightarrow d(T(\omega,u),T(\omega,v))\leq\psi_\omega(O(\omega,(u,v))),\tag{3.1}$$

where

$$O(\omega,(u,v))=\max\left\{d(u,v),\frac{d(u,T(\omega,u))+d(v,T(\omega,v))}{2},\frac{d(u,T(\omega,v))+d(v,T(\omega,u))}{2}\right\}$$

for all $u,v\in M$ and $\omega\in\Omega$.

Theorem 3.3. *Let (Ω,Σ) be a measurable space, (M,d) be a Polish space, $T:\Omega\times M\to M$ and $\alpha,\eta:\Omega\times M\times M\to[0,+\infty)$. The hypotheses are the following:*

(H1) *T is a random $\alpha-$admissible with respect to η.*
(H2) *there exists a measurable mapping $u_0:\Omega\to M$ such that, for all $\omega\in\Omega$.*

$$\alpha(\omega,u_0(\omega),T(\omega,u_0(\omega)))\geq\eta(\omega,u_0(\omega),T(\omega,u_0(\omega))).$$

(H3) *T is a Carathéodory mapping.*
(H4) *T is a generalized random $\alpha-\psi-$contractive mapping.*

Then T has a random fixed point, that is, there exists $\zeta:\Omega\to M$ is measurable such that $T(\omega,\zeta(\omega))=\zeta(\omega)$ for all $\omega\in\Omega$.

Proof. Hypothese (H2) ensures that there exists a measurable mapping $u_0:\Omega\to M$ such that

$$\alpha(\omega,u_0(\omega),T(\omega,u_0(\omega)))\geq\eta(\omega,u_0(\omega),T(\omega,u_0(\omega))),$$

for all $\omega\in\Omega$. Define the sequence $\{u_n(\omega)\}$ in M by

$$u_n(\omega)=T^n(\omega,u_0(\omega))=T(\omega,u_{n-1}(\omega))\ \ for\ \ all\ \ n\in\mathbb{N}\cup\{0\},\omega\in\Omega.$$

If $u_n(\omega)=u_{n+1}(\omega)$ for all $n\in\mathbb{N}\cup\{0\}$, for all $\omega\in\Omega$, then $\zeta(\omega)=u_n(\omega)$ is a random fixed point of T.

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Assume that $u_n(\omega) \neq u_{n+1}(\omega)$ for all $n \in \mathbb{N} \cup \{0\}$, for one $\omega \in \Omega$. Since T is a random $\alpha-$admissible with respect to η (H1) and $\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) = \eta(\omega, u_0(\omega), T(\omega, u_0(\omega)))$ we have

$$\begin{aligned}\alpha(\omega, u_1(\omega), u_2(\omega)) &= \alpha(\omega, T(\omega, u_0(\omega)), T^2(\omega, u_0(\omega))) \\ &\geq \eta(\omega, T(\omega, u_0(\omega)), T^2(\omega, u_0(\omega))) = \eta(\omega, u_1(\omega), u_2(\omega)).\end{aligned}$$

Continuing this process, we get

$$\alpha(\omega, u_n(\omega), u_{n+1}(\omega)) \geq \eta(\omega, u_n(\omega), u_{n+1}(\omega)) \;\; for \;\; all \;\; n \in \mathbb{N} \cup \{0\}, \omega \in \Omega. \tag{3.2}$$

So, by (3.2) and hypothesis (H4) with $u = u_{n-1}(\omega)$, $v = u_n(\omega)$, we get

$$d(T(\omega, u_{n-1}(\omega)), T(\omega, u_n(\omega))) \leq \psi_\omega(O(\omega, (u_{n-1}(\omega), u_n(\omega)))).$$

On the other hand,

$$\begin{aligned}O(\omega, (u_{n-1}(\omega), u_n(\omega))) &= \max\Big\{ d(u_{n-1}(\omega), u_n(\omega)), \\ &\quad \frac{d(u_{n-1}(\omega), T(\omega, u_{n-1}(\omega))) + d(u_n(\omega), T(\omega, u_n(\omega)))}{2}, \\ &\quad \frac{d(u_{n-1}(\omega), T(\omega, u_n(\omega))) + d(u_n(\omega), T(\omega, u_{n-1}(\omega)))}{2} \Big\} \\ &= \max\Big\{ d(u_{n-1}(\omega), u_n(\omega)), \\ &\quad \frac{d(u_{n-1}(\omega), u_n(\omega)) + d(u_n(\omega), u_{n+1}(\omega))}{2}, \\ &\quad \frac{d(u_{n-1}(\omega), u_{n+1}(\omega))}{2} \Big\} \\ &\leq \max\Big\{ d(u_{n-1}(\omega), u_n(\omega)), \\ &\quad \frac{d(u_{n-1}(\omega), u_n(\omega)) + d(u_n(\omega), u_{n+1}(\omega))}{2} \Big\} \\ &\leq \max\{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\},\end{aligned}$$

which implies

$$d(u_n(\omega), u_{n+1}(\omega)) \leq \psi_\omega(\max\{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\}).$$

Now, if $\max\{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\} = d(u_n(\omega), u_{n+1}(\omega))$ for all $n \in \mathbb{N}$, then

$$\begin{aligned}d(u_n(\omega), u_{n+1}(\omega)) &\leq \psi_\omega(\max\{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\}) \\ &= \psi_\omega(d(u_n(\omega), u_{n+1}(\omega))) \\ &< d(u_n(\omega), u_{n+1}(\omega)),\end{aligned}$$

which is a contradiction. Hence, for all $n \in \mathbb{N}$, we have

$$d(u_n(\omega), u_{n+1}(\omega)) \leq \psi_\omega d(u_{n-1}(\omega), u_n(\omega)).$$

By induction, we have

$$d(u_n(\omega), u_{n+1}(\omega)) \leq \psi_\omega^n d(u_0(\omega), u_1(\omega)).$$

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Fix $\epsilon > 0$, and let $N \in \mathbb{N}$ such that

$$\sum_{n\geq N} \psi_\omega d(u_n(\omega), u_{n+1}(\omega)) < \epsilon \ \ for \ \ all \ \ n \in \mathbb{N}.$$

Also, let $n, m \in \mathbb{N}$ with $m > n \geq N$. Then, by the triangular inequality, we get

$$\begin{aligned} d(u_n(\omega), u_m(\omega)) &\leq \sum_{k=n}^{m-1} d(u_k(\omega), u_{k+1}(\omega)) \\ &\leq \sum_{k=n}^{m-1} \psi_\omega^k(d(u_0(\omega), u_1(\omega))) \\ &\leq \sum_{n\geq n(\epsilon)} \psi_\omega^n(d(u_0(\omega), u_1(\omega))) \\ &< \epsilon. \end{aligned}$$

The argument show that the sequence $\{u_n(\omega)\}$ is a Cauchy sequence. Since (M, d) is complete, there exists $\zeta : \Omega \to M$ such that $u_n(\omega) \to \zeta(\omega)$ as $n \to +\infty$ for all $\omega \in \Omega$. Since T is a Carathéodory mapping (hypothesis(H3)), it follows that u_n is measurable for all $n \in \mathbb{N}$ and that $u_{n+1}(\omega) = T(\omega, u_n(\omega)) \to T(\omega, \zeta(\omega))$ as $n \to +\infty$ for all $\omega \in \Omega$. By the uniqueness of the limit, we get $\zeta(\omega) = T(\omega, \zeta(\omega))$, that is, $\zeta(\omega)$ is a random fixed point of T. Note that ζ is a measurable since it is a limit of a sequence of measurable. □

By taking $\eta(\omega, u, v) = 1$, $\forall \omega \in \Omega$, $u, v \in M$ in Theorem 3.3, we have the following result.

Corollary 3.4. *Let (Ω, Σ) be a measurable space, (M, d) be a Polish space, $T : \Omega \times M \to M$ and $\alpha : \Omega \times M \times M \to [0, +\infty)$. The hypotheses are the following:*

(H1) *T is a random $\alpha-$admissible.*

(H2) *there exists a measurable mapping $u_0 : \Omega \to M$ such that, for all $\omega \in \Omega$.*

$$\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq 1.$$

(H3) *T is a Carathéodory mapping.*

(H4) *T is a generalized random $\alpha - \psi-$contractive mapping.*

Then T has a random fixed point, that is, there exists $\zeta : \Omega \to M$ is measurable such that $T(\omega, \zeta(\omega)) = \zeta(\omega)$ for all $\omega \in \Omega$.

Theorem 3.5. *Let (Ω, Σ) be a measurable space, (M, d) be a Polish space, $T : \Omega \times M \to M$ and $\alpha : \Omega \times M \times M \to [0, +\infty)$. The hypotheses are the following:*

(G1) *T is a random $\alpha-$admissible with respect to η.*

(G2) *there exists a measurable mapping $u_0 : \Omega \to M$ such that, for all $\omega \in \Omega$.*

$$\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq \eta(\omega, u_0(\omega), T(\omega, u_0(\omega))).$$

(G3) *T is a sup-measurable.*

(G4) *T is a generalized random $\alpha - \psi-$contractive mapping.*

(G5) *If $\{u_n(\omega)\}$ is a sequence in M such that*

$$\alpha(\omega, u_n(\omega), u_{n+1}(\omega)) \geq \eta(\omega, u_n(\omega), u_{n+1}(\omega))$$

for all $\omega \in \Omega$, for all $n \in \mathbb{N} \cup \{0\}$ and $u_n(\omega) \to u(\omega)$ as $n \to +\infty$, then

$$\alpha(\omega, u_n(\omega), u(\omega)) \geq \eta(\omega, u_n(\omega), u(\omega)),$$

for all $\omega \in \Omega$, for all $n \in \mathbb{N} \cup \{0\}$.

Then T has a random fixed point, that is, there exists $\zeta : \Omega \to M$ is measurable such that $T(\omega, \zeta(\omega)) = \zeta(\omega)$ for all $\omega \in \Omega$.

Proof. A similar reasoning as in the proof of Theorem 3.3 gives us that the sequence $\{u_n(\omega)\}$ is a Cauchy sequence for all $\omega \in \Omega$. This means that there exists $\zeta : \Omega \to M$ such that $u_n(\omega) \to \zeta(\omega)$ as $n \to +\infty$ for all $\omega \in \Omega$. On the other hand, from (3.2) and hypothesis (G5), we have

$$\alpha(\omega, u_n(\omega), \zeta(\omega)) \geq \eta(\omega, u_n(\omega), \zeta(\omega)) \ \ for \ \ all \ \ n \in \mathbb{N} \cup \{0\}, \omega \in \Omega. \tag{3.3}$$

Now, using the triangle inequality (3.3) and (G4), we get

$$\begin{aligned} d(T(\omega, \zeta(\omega)), \zeta(\omega)) &\leq d(T(\omega, \zeta(\omega)), T(\omega, u_n(\omega))) + d(u_{n+1}(\omega), \zeta(\omega)) \\ &\leq \psi_\omega(d(\zeta(\omega), u_n(\omega))) + d(u_{n+1}(\omega), \zeta(\omega)). \end{aligned}$$

Taking the limit as $n \to +\infty$ and since ψ_ω is continuous at $t = 0$, we have

$$d(T(\omega, \zeta(\omega)), \zeta(\omega)) = 0,$$

that is, $T(\omega, \zeta(\omega)) = \zeta(\omega)$ for all $\omega \in \Omega$. The hypothesis that T is sup-measurable implies that u_n is measurable for all $n \in \mathbb{N}$ and hence ζ is measurable. Thus ζ is a random fixed point of T. □

4. Application to ordinary random differential equations

We consider the following two-point boundary value problem of second order random differential equation:

$$\begin{cases} -\dfrac{d^2u}{dt^2}(\omega, t) = f(\omega, t, u(\omega, t)), \quad t \in [0, 1], \\ u(\omega, u) = u(\omega, 1) = 0 \end{cases} \tag{4.1}$$

for all $\omega \in \Omega$, we have $f : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ has certain regularities and Ω is nonempty.

By a random solution of system (4.1), we mean a measurable mapping $u : \Omega \to C([0, 1], \mathbb{R})$ satisfying (4.1), where $C([0, 1], \mathbb{R})$ denote the space of all continuous functions defined on $[0, 1]$. The space $C([0, 1], \mathbb{R})$ endowed with the metric

$$d_\infty(x, y) = \|x - y\|_\infty.$$

In this section, we prove a theorem producing the existence of random soution of system (4.1).

Let (Ω, Σ) be a measurable space. Let $f : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which means that $\omega \mapsto f(\omega, t, u)$ is measurable for all $(t, u) \in [0, 1] \times \mathbb{R}$ and $(t, u) \mapsto f(\omega, t, u)$ is continuous for all $\omega \in \Omega$.

Then consider the integral operator $F : \Omega \times C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})$ defined by

$$F(\omega, u)(t) = \int_0^1 G(t, s) f(\omega, s, u(s)) ds, \tag{4.2}$$

for all $u \in C([0, 1], \mathbb{R})$ and $\omega \in \Omega$, where $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function, and $g : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Remark 4.1. F is a random operator from $\Omega \times C([0, 1], \mathbb{R})$ into $C([0, 1], \mathbb{R}$. In fact, given $u \in C([0, 1], \mathbb{R})$ since f is a Carathéodory function for $s \in [0, 1]$ fixed, the function $h : \Omega \times [0, 1] \to \mathbb{R}$, defined by $h(\omega, t) = G(t, s) f(\omega, s, u(s))$, is Carathéodory. By Lemma 2.7, the integral in (4.2) is limit of a finite sum of measurable functions. So, the mapping $\omega \to F(\omega, u)$ is measurable, and hence F is a random operator.

Remark 4.2. Let $h:\Omega\times[0,1]\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function, $u\in C([0,1],\mathbb{R})$, and let $\{u_n\}\subset C([0,1],\mathbb{R})$ be a sequence convergent to u. Then there exists an interval $[a,b]\subset\mathbb{R}$ such that $u_n(s),u(s)\in[a,b]$ for all $s\in[0,1]$. The continuity of the function $h(\omega,\cdot,\cdot)$ in $[0,1]\times\mathbb{R}$ for fixed $\omega\in\Omega$ ensures that the function $h(\omega,\cdot,\cdot)$ is uniformly continuous in $[0,1]\times[a,b]$.

The hypotheses are the following:

(i) For each $\omega\in\Omega$, there exist $\psi_\omega\in\Psi$ and $\theta:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that if $\theta(\omega,a,b)\geq 0$ for all $a,b\in\mathbb{R}$, then for every $t\in[0,1]$, we have

$$\begin{aligned}&|f(\omega,t,a)-f(\omega,t,b)|\\&\leq\psi_\omega\Bigg(\max\Bigg\{|a(t)-b(t)|,\frac{1}{2}[|a(t)-F(\omega,a(t))|+|b(t)-F(\omega,b(t))|],\\&\frac{1}{2}[|a(t)-F(\omega,b(t))|+|b(t)-F(\omega,a(t))|]\Bigg\}\Bigg).\end{aligned}$$

(ii) There exists a measurable mapping $u_0:\Omega\to C([0,1],\mathbb{R})$ such that, for all $\omega\in\Omega$, we have

$$\theta(\omega,u_0(\omega)(t),F(\omega,u_0(\omega))(t))\geq 0\quad for\ \ all\quad t\in[0,1]$$

(iii) For each $\omega\in\Omega$ and for all $t\in[0,1]$, $u,v\in C([0,1],\mathbb{R})$, we have

$$\theta(\omega,u(t),v(t))\geq 0\Rightarrow\theta(\omega,F(\omega,u)(t),F(\omega,v)(t))\geq 0.$$

(iv) $\int_0^1 G(t,s)ds\leq 1$ for all $t\in[0,1]$ and $s\in[0,1]$.

Theorem 4.3. *If hypotheses $(i)-(iv)$ hold, then the random integral operator F has a random fixed point.*

Proof. For fixed $\omega\in\Omega$ we show that $F(\omega,\cdot)$ is continuous. Indeed, consider a sequence $\{u_n\}\in C([0,1],\mathbb{R})$ with $u_n\to u\in C([0,1],\mathbb{R})$ as $n\to+\infty$. By Remark 4.2, there exists $[a,b]\subset\mathbb{R}$ such that $u_n(s),u(s)\in[a,b]$ for all $s\in[0,1]$. In addition, the functions $f(\omega,\cdot,\cdot)$ is uniformly continuous in $[0,1]\times[a,b]$. Thus, for fixed $\epsilon>0$, there exists $\delta>0$ such that

$$|f(\omega,s_1,u_1)-f(\omega,s_2,u_2)|<\epsilon,$$

for all $s_1,s_2\in[0,1]$ and $u_1,u_2\in[a,b]$ such that $|s_1-s_2|+|u_1-u_2|<\delta$.

Now, let $n(\delta)\in\mathbb{N}$ such that $\|u_n-u\|_\infty<\delta$ whenever $n\geq n(\delta)$. Then, for every $n\geq n(\delta)$, we have

$$|f(\omega,s,u_n(s))-f(\omega,s,u(s))|<\epsilon.$$

Consequently, for $t\in[0,1]$ and $n\geq n(\delta)$, we have

$$\begin{aligned}|F(\omega,u_n)(t)-F(\omega,u)(t)|&\leq\int_0^1|G(t,s)||f(\omega,s,u_n(s))-f(\omega,s,u(s))|ds\\&\leq\epsilon\end{aligned}$$

$$\Rightarrow\|F(\omega,u_n)-F(\omega,u)\|_\infty\leq\epsilon.$$

So, $d_\infty(F(\omega,u_n),F(\omega,u))\to 0$ as $n\to+\infty\Rightarrow F(\omega,\cdot)$ is a continuous operator for each fixed $\omega\in\Omega$.

Thus, by Remark 4.2, $F:\Omega\times C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is a Carathéodory function.

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Next step is to show that the integral operator F satisfies a generalized random $\alpha-\psi-$contractive type condition as in (H4). So, for each $\omega \in \Omega$ and all $u, v \in C([0,1],\mathbb{R})$ such that $\theta(\omega, u(t), v(t)) \geq 0$ for all $t \in [0,1]$, we prove that

$$d_\infty(F(\omega,u), F(\omega,v)) \leq \psi_\omega(O(\omega,(u,v)))$$

where

$$O(\omega,(u,v)) = \max\left\{d(u,v), \frac{d(u,F(\omega,u)) + d(v,F(\omega,v))}{2}, \frac{d(u,F(\omega,v)) + d(v,F(\omega,u))}{2}\right\}.$$

Indeed, let $\omega \in \Omega$ be fixed, and $u, v \in C([0,1],\mathbb{R})$ be such that $\theta(\omega,u(t),v(t)) \geq 0$ for all $t \in [0,1]$, then

$$\begin{aligned}
&|F(\omega,u)(t) - F(\omega,v)(t)| \\
&= \left|\int_0^1 G(t,s)[f(\omega,s,u(s)) - f(\omega,s,v(s))]ds\right| \\
&\leq \int_0^1 G(t,s)|f(\omega,s,u(s)) - f(\omega,s,v(s))|ds \\
&\leq \int_0^1 G(t,s)\Bigg[\psi_\omega\Bigg(\max\Bigg\{|u(s)-v(s)|, \frac{1}{2}\big[|u(s) - F(\omega,u(s))| + |v(s) - F(\omega,v(s))|\big], \\
&\frac{1}{2}\big[|u(s) - F(\omega,v(s))| + |v(s) - F(\omega,u(s))|\big]\Bigg\}\Bigg)\Bigg]ds \\
&\leq \int_0^1 G(t,s)\Bigg[\psi_\omega\Bigg(\max\Bigg\{|u(s)-v(s)\|, \frac{1}{2}\big[\|u(s) - F(\omega,u(s))\| + \|v(s) - F(\omega,v(s))\|\big], \\
&\frac{1}{2}\big[\|u(s) - F(\omega,v(s))\| + \|v(s) - F(\omega,u(s))\|\big]\Bigg\}\Bigg)\Bigg]ds \\
&= \left(\int_0^1 G(t,s)ds\right)\psi_\omega\Bigg(\max\Bigg\{|u(s)-v(s)\|, \frac{1}{2}\big[\|u(s) - F(\omega,u(s))\| + \|v(s) - F(\omega,v(s))\|\big], \\
&\frac{1}{2}\big[\|u(s) - F(\omega,v(s))\| + \|v(s) - F(\omega,u(s))\|\big]\Bigg\}\Bigg) \\
&\leq \psi_\omega\Bigg(\max\Bigg\{|u(s)-v(s)\|, \frac{1}{2}\big[\|u(s) - F(\omega,u(s))\| + \|v(s) - F(\omega,v(s))\|\big], \\
&\frac{1}{2}\big[\|u(s) - F(\omega,v(s))\| + \|v(s) - F(\omega,u(s))\|\big]\Bigg\}\Bigg).
\end{aligned}$$

Then

$$\begin{aligned}
&\|F(\omega,u) - F(\omega,v)\| \\
&\leq \psi_\omega\Bigg(\max\Bigg\{|u(s)-v(s)\|, \frac{1}{2}\big[\|u(s) - F(\omega,u(s))\| + \|v(s) - F(\omega,v(s))\|\big], \\
&\frac{1}{2}\big[\|u(s) - F(\omega,v(s))\| + \|v(s) - F(\omega,u(s))\|\big]\Bigg\}\Bigg).
\end{aligned}$$

Let $\alpha : \Omega \times C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}) \to [0,+\infty)$ be function given as

$$\alpha(\omega,u,v) = \begin{cases} 1 & if \quad \theta(\omega,u(t),v(t)) \geq 0 \quad for\ \ all \quad t \in [0,1], \\ 0 & otherwise \end{cases}$$

for all $\omega \in \Omega$. So, for all $u, v \in C([0,1], \mathbb{R})$ with $\alpha(\omega, u, v) \geq 1$, we get

$$\begin{aligned}
&\|F(\omega, u) - F(\omega, v)\|_\infty \\
&\leq \psi_\omega\Bigg(\max\Big\{ |u(s) - v(s)\|_\infty, \frac{1}{2}\big[\|u(s) - F(\omega, u(s))\|_\infty + \|v(s) - F(\omega, v(s))\|_\infty\big], \\
&\frac{1}{2}\big[\|u(s) - F(\omega, v(s))\|_\infty + \|v(s) - F(\omega, u(s))\|_\infty\big]\Big\}\Bigg),
\end{aligned}$$

which means that F is a generalized random $\alpha - \psi-$contractive integral operator.

Note thar, for each $\omega \in \Omega$ and all $t \in [0,1]$, $u, v \in C([0,1], \mathbb{R})$, we have

$$\begin{aligned}
&\alpha(\omega, u, v) \geq 1 \\
&\Rightarrow \theta(\omega, u(t), v(t)) \geq 0 \quad for\ \ all \quad t \in [0,1] \\
&\Rightarrow \theta(\omega, F(\omega, u)(t), F(\omega, v)(t)) \geq 0 \\
&\alpha(\omega, F(\omega, u), F(\omega, v)) \geq 1,
\end{aligned}$$

which means that F is a random $\alpha-$admissible integral oprator.

All of the hypotheses of Corollary 3.4 are satisfied, and hence the mapping F has a random fixed point. □

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