RANDOM FIXED POINT THEOREMS FOR GENERALIZED RANDOM $\alpha - \psi$--CONTRACTIVE MAPPINGS WITH APPLICATIONS TO STOCHASTIC DIFFERENTIAL EQUATION

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Abstract. In this paper, we prove some random fixed point theorems for generalized random $\alpha - \psi$--contractive mappings in a Polish space and, as some applications, we show the existence of random solutions of second order random differential equation.

Keywords: random fixed point, random $\alpha$--admissible with respect to $\eta$, generalized random $\alpha - \psi$--contractive mapping.

Mathematics Subject Classification: 47H10; 47H40.

1. INTRODUCTION

Random fixed point theorems are stochastic generalization of a classical fixed point theorems. Random fixed point theorems for contraction mapping in a Polish space, i.e., a separable complete metric space, were proved by Špaček [22], Hanš [5,6]. Some random fixed point theorems play a main role in developing theory of random differential and random integral equations (see, [2, 8, 15]). In 1996, Mukhejea [16] proved the random fixed point theorem of Schauder’s type in atomic probability measure space. In 1984, Sehgal and Waters [20] proved the random fixed point theorem of the classical Rothe’s fixed point theorem. The random fixed point theory and applications developed very rapidly (see, Bharucha-Reid [3], Itoh [7], Beg and Shahzad [1], Li [14], Kumam et al. [10–13], Nieto [17]).

In 2012, Samet et al. [19] introduced a new concept of $\alpha - \psi$--contractive type and $\alpha$--admissible mappings and establish fixed point theorems for such mappings in complete metric spaces. Afterwards Karapinar and Samet [8] introduced the concepts of

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generalized $\alpha-\psi$-contractive type mapping. In 2013, Salimi et al. [18] modified the notion of $\alpha$-admissible and $\alpha-\psi$-contractive mappings and established certain fixed point theorems. Our results are proper generalizations of the recent results in [9,19].

Recently, Tchier and Vetro [21] introduced the concepts of random $\alpha$-admissible and random $\alpha-\psi$-contractive mappings and established random fixed point theorems. The purpose of this paper is to prove some random fixed point theorems for generalized random $\alpha-\psi$-contractive mappings in a Polish space and, by using our main results, we show the existence of random solutions of second order random differential equation.

2. PRELIMINARIES

We denote the Borel $\sigma$-algebra on a metric space $M$ by $B(M)$. Let $(\Omega, \Sigma)$ be a measurable space with $\Sigma$ a $\sigma$-algebra of subsets of $\Omega$. So that by $\Sigma \times B(M)$ we mean the smallest $\sigma$-algebra on $\Omega \times M$ containing all the sets $A \times B$ (with $A \in \Sigma$ and $B \in B(M)$).

**Definition 2.1.** Let $(\Omega, \Sigma)$ be a measurable space, $M$ and $N$ be two metric spaces. A mapping $f : \Omega \times M \to N$ is called Carathéodory if, for all $m \in M$, the mapping $\omega \to f(\omega, m)$ is $(\Sigma, B(N))$-measurable $(\Sigma$-measurable for short) and, for all $\omega \in \Omega$, the mapping $m \to f(\omega, m)$ is continuous.

**Theorem 2.2.** [4] If $(\Omega, \Sigma)$ is a measurable space, $M$ is a separable metric space, $N$ is a metric space, and $f : \Omega \times M \to N$ is a Carathéodory mapping, then $f$ is $\Sigma \times B(M)$-measurable.

**Corollary 2.3.** [4] If $(\Omega, \Sigma)$ is a measurable space, $M$ is a separable metric space, $N$ is a metric space, and $f : \Omega \times M \to N$ is a Carathéodory mapping, and $u : \Omega \to M$ is $\Sigma$-measurable, then mapping $\omega \to f(\omega, u(\omega))$ is a $\Sigma$-measurable mapping from $\Omega$ into $N$.

**Definition 2.4.** [4] Let $(\Omega, \Sigma)$ be a measurable space, $M$ a separable metric space and $N$ a metric space. A function $f : \Omega \times M \to N$ is said to be superpositionally measurable (sup-measurable for short), if for all $u : \Omega \to M$ is $\Sigma$-measurable, the function $\omega \to f(\omega, u(\omega))$ is $\Sigma$-measurable from $\Omega$ into $N$.

**Remark 2.5.** [4] Corollary 2.3 says that a Carathéodory function is sup-measurable. Also, every $\Sigma \times B(M)$-measurable functions $f : \Omega \times M \to N$ is sup-measurable.

**Definition 2.6.** A mapping $f : \Omega \times M \to M$ is called random operator whenever, for any $x \in M$, $f(\cdot, x)$ is $\Sigma$-measurable. So, a random fixed point of $f$ is $\Sigma$-measurable mapping $z : \Omega \times M$ such that $z(\omega) = f(\omega, z(\omega))$ for all $\omega \in \Omega$.

**Lemma 2.7.** Let $M, N$ be two locally compact metric spaces. A mapping $f : \Omega \times M \to N$ is Carathéodory if and only if the mapping $\omega \to r(\omega)(\cdot) = f(\omega, \cdot)$ is $\Sigma$-measurable from $\Omega$ to $C(M, N)$ (i.e., the space of all continuous functions from $M$ into $N$ endowed with the compact-open topology).

Let $\Psi$ be the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where $\psi^n$ denote the $n$th iterate of $\psi$.

**Lemma 2.8.** For every nondecreasing function $\psi : [0, +\infty) \to [0, +\infty)$, the following implication holds:

$$\forall t > 0, \lim_{n \to +\infty} \psi^n(t) = 0 \implies \psi(t) < t.$$
Definition 2.9. Let $T : \Omega \times M \to M$ and $\alpha : \Omega \times M \times M \to [0, +\infty)$. We say that $T$ is a random $\alpha-$admissible if $u, v \in M, \omega \in \Omega, \alpha(\omega, u, v) \geq 1 \implies \alpha(\omega, T(\omega, u), T(\omega, v)) \geq 1$.

Definition 2.10. Let $(\Omega, \Sigma)$ be a measurable space, $(M, d)$ be a separable metric space, and $T : \Omega \times M \to M$ be a given mapping. We say that $T$ is a random $\alpha - \psi-$contractive mapping if there exist functions $\alpha : \Omega \times M \times M \to [0, +\infty)$ and $\psi_\omega \in \Psi, \omega \in \Omega$, such that

$$\alpha(\omega, u, v)d(T(\omega, u), T(\omega, v)) \leq \psi_\omega(d(u, v)),$$

for all $u, v \in M$ and $\omega \in \Omega$ such that $\alpha(\omega, u, v) \geq 1$.

3. Main Results

Definition 3.1. Let $T : \Omega \times M \to M$ and $\alpha, \eta : \Omega \times M \times M \to [0, +\infty)$. We say that $T$ is a random $\alpha-$admissible with respect to $\eta$ if $u, v \in M, \omega \in \Omega, \alpha(\omega, u, v) \geq \eta(\omega, u, v) \implies \alpha(\omega, T(\omega, u), T(\omega, v)) \geq \eta(\omega, T(\omega, u), T(\omega, v))$.

Note that if we take $\eta(\omega, u, v) = 1$, then this definition reduces to Definition 2.9.

Definition 3.2. Let $(\Omega, \Sigma)$ be a measurable space, $(M, d)$ be a separable space, and $T : \Omega \times M \to M$ be a given mapping. We say that $T$ is a generalized random $\alpha - \psi-$contractive mapping if there exist functions $\alpha, \eta : \Omega \times M \times M \to [0, +\infty)$ and $\psi_\omega \in \Psi, \omega \in \Omega$, such that

$$\alpha(\omega, u, v) \geq \eta(\omega, u, v) \implies d(T(\omega, u), T(\omega, v)) \leq \psi_\omega(O(\omega, (u, v))),$$

(3.1)

where

$$O(\omega, (u, v)) = \max \left\{ d(u, v), \frac{d(u, T(\omega, u)) + d(v, T(\omega, v))}{2}, \frac{d(u, T(\omega, v)) + d(v, T(\omega, u))}{2} \right\}$$

for all $u, v \in M$ and $\omega \in \Omega$.

Theorem 3.3. Let $(\Omega, \Sigma)$ be a measurable space, $(M, d)$ be a Polish space, $T : \Omega \times M \to M$ and $\alpha, \eta : \Omega \times M \times M \to [0, +\infty)$. The hypotheses are the following:

(H1) $T$ is a random $\alpha-$admissible with respect to $\eta$.

(H2) there exists a measurable mapping $u_0 : \Omega \to M$ such that, for all $\omega \in \Omega$. $\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq \eta(\omega, u_0(\omega), T(\omega, u_0(\omega)))$.

(H3) $T$ is a Carathéodory mapping.

(H4) $T$ is a generalized random $\alpha - \psi-$contractive mapping.

Then $T$ has a random fixed point, that is, there exists $\zeta : \Omega \to M$ measurable such that $T(\omega, \zeta(\omega)) = \zeta(\omega)$ for all $\omega \in \Omega$.

Proof. Hypothese (H2) ensures that there exists a measurable mapping $u_0 : \Omega \to M$ such that

$$\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq \eta(\omega, u_0(\omega), T(\omega, u_0(\omega)),$$

for all $\omega \in \Omega$. Define the sequence $\{u_n(\omega)\}$ in $M$ by $u_n(\omega) = T^n(\omega, u_0(\omega)) = T(\omega, u_{n-1}(\omega))$ for all $n \in \mathbb{N} \cup \{0\}, \omega \in \Omega$.

If $u_n(\omega) = u_{n+1}(\omega)$ for all $n \in \mathbb{N} \cup \{0\}$, for all $\omega \in \Omega$, then $\zeta(\omega) = u_n(\omega)$ is a random fixed point of $T$. 
Assume that \( u_n(\omega) \neq u_{n+1}(\omega) \) for all \( n \in \mathbb{N} \cup \{0\} \), for one \( \omega \in \Omega \). Since \( T \) is a random \( \alpha \)-admissible with respect to \( \eta \) (H1) and \( \alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) = \eta(\omega, u_0(\omega), T(\omega, u_0(\omega))) \) we have
\[
\alpha(\omega, u_1(\omega), u_2(\omega)) = \alpha(\omega, T(\omega, u_0(\omega)), T^2(\omega, u_0(\omega))) \\
\geq \eta(\omega, T(\omega, u_0(\omega)), T^2(\omega, u_0(\omega))) = \eta(\omega, u_1(\omega), u_2(\omega)).
\]
Continuing this process, we get
\[
\alpha(\omega, u_n(\omega), u_{n+1}(\omega)) \geq \eta(\omega, u_n(\omega), u_{n+1}(\omega)) \text{ for all } n \in \mathbb{N} \cup \{0\}, \omega \in \Omega. \quad (3.2)
\]
So, by (3.2) and hypothesis (H4) with \( u = u_{n-1}(\omega) \), \( v = u_n(\omega) \), we get
\[
d(T(\omega, u_{n-1}(\omega)), T(\omega, u_n(\omega))) \leq \psi(\omega, (u_{n-1}(\omega), u_n(\omega))).
\]
On the other hand,
\[
O(\omega, (u_{n-1}(\omega), u_n(\omega))) = \max \left\{ d(u_{n-1}(\omega), u_n(\omega)), \right. \\
\frac{d(u_{n-1}(\omega), T(\omega, u_{n-1}(\omega))) + d(u_n(\omega), T(\omega, u_n(\omega)))}{2}, \right. \\
\left. \frac{d(u_{n-1}(\omega), T(\omega, u_n(\omega))) + d(u_n(\omega), T(\omega, u_{n-1}(\omega)))}{2} \right\}
\]
\[
= \max \left\{ d(u_{n-1}(\omega), u_n(\omega)), \right. \\
\frac{d(u_{n-1}(\omega), u_n(\omega)) + d(u_n(\omega), u_{n+1}(\omega))}{2}, \right. \\
\frac{d(u_{n-1}(\omega), u_{n+1}(\omega))}{2} \right\}
\]
\[
\leq \max \left\{ d(u_{n-1}(\omega), u_n(\omega)), \right. \\
\frac{d(u_{n-1}(\omega), u_n(\omega)) + d(u_n(\omega), u_{n+1}(\omega))}{2} \right\}
\]
\[
\leq \max \{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\},
\]
which implies
\[
d(u_n(\omega), u_{n+1}(\omega)) \leq \psi(\omega, \max \{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\}).
\]
Now, if \( \max \{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\} = d(u_n(\omega), u_{n+1}(\omega)) \) for all \( n \in \mathbb{N} \), then
\[
d(u_n(\omega), u_{n+1}(\omega)) \leq \psi(\omega, \max \{d(u_{n-1}(\omega), u_n(\omega)), d(u_n(\omega), u_{n+1}(\omega))\})
\]
\[
= \psi(\omega, d(u_n(\omega), u_{n+1}(\omega)))
\]
\[
< d(u_n(\omega), u_{n+1}(\omega)),
\]
which is a contradiction. Hence, for all \( n \in \mathbb{N} \), we have
\[
d(u_n(\omega), u_{n+1}(\omega)) \leq \psi(\omega, d(u_{n-1}(\omega), u_n(\omega))).
\]
By induction, we have
\[
d(u_n(\omega), u_{n+1}(\omega)) \leq \psi^p(\omega, d(u_0(\omega), u_1(\omega))).
\]
Fix $\epsilon > 0$, and let $N \in \mathbb{N}$ such that
\[
\sum_{n \geq N} \psi_\omega d(u_n(\omega), u_{n+1}(\omega)) < \epsilon \quad \text{for all } n \in \mathbb{N}.
\]

Also, let $n, m \in \mathbb{N}$ with $m > n \geq N$. Then, by the triangular inequality, we get
\[
d(u_n(\omega), u_m(\omega)) \leq \sum_{k=n}^{m-1} d(u_k(\omega), u_{k+1}(\omega))
\leq \sum_{k=n}^{m-1} \psi_k^\omega(d(u_0(\omega), u_1(\omega)))
\leq \sum_{n \geq n(\epsilon)} \psi_n^\omega(d(u_0(\omega), u_1(\omega)))
< \epsilon.
\]

The argument show that the sequence $\{u_n(\omega)\}$ is a Cauchy sequence. Since $(M, d)$ is complete, there exists $\zeta : \Omega \to M$ such that $u_n(\omega) \to \zeta(\omega)$ as $n \to +\infty$ for all $\omega \in \Omega$. Since $T$ is a Carathéodory mapping (hypothesis (H3)), it follows that $u_n$ is measurable for all $n \in \mathbb{N}$ and that $u_{n+1}(\omega) = T(\omega, u_n(\omega)) \to T(\omega, \zeta(\omega))$ as $n \to +\infty$ for all $\omega \in \Omega$. By the uniqueness of the limit, we get $\zeta(\omega) = T(\omega, \zeta(\omega))$, that is, $\zeta(\omega)$ is a random fixed point of $T$. Note that $\zeta$ is a measurable since it is a limit of a sequence of measurable.

By taking $\eta(\omega, u, v) = 1$, $\forall \omega \in \Omega$, $u, v \in M$ in Theorem 3.3, we have the following result.

**Corollary 3.4.** Let $(\Omega, \Sigma)$ be a measurable space, $(M, d)$ be a Polish space, $T : \Omega \times M \to M$ and $\alpha : \Omega \times M \times M \to [0, +\infty)$. The hypotheses are the following:

(H1) $T$ is a random $\alpha$–admissible.
(H2) there exists a measurable mapping $u_0 : \Omega \to M$ such that, for all $\omega \in \Omega$.
\[
\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq 1.
\]
(H3) $T$ is a Carathéodory mapping.
(H4) $T$ is a generalized random $\alpha$ – $\psi$–contractive mapping.

Then $T$ has a random fixed point, that is, there exists $\zeta : \Omega \to M$ is measurable such that $T(\omega, \zeta(\omega)) = \zeta(\omega)$ for all $\omega \in \Omega$.

**Theorem 3.5.** Let $(\Omega, \Sigma)$ be a measurable space, $(M, d)$ be a Polish space, $T : \Omega \times M \to M$ and $\alpha : \Omega \times M \times M \to [0, +\infty)$. The hypotheses are the following:

(G1) $T$ is a random $\alpha$–admissible with respect to $\eta$.
(G2) there exists a measurable mapping $u_0 : \Omega \to M$ such that, for all $\omega \in \Omega$.
\[
\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geq \eta(\omega, u_0(\omega), T(\omega, u_0(\omega))).
\]
(G3) $T$ is a sup-measurable.
(G4) $T$ is a generalized random $\alpha$ – $\psi$–contractive mapping.
(G5) If $\{u_n(\omega)\}$ is a sequence in $M$ such that
\[
\alpha(\omega, u_n(\omega), u_{n+1}(\omega)) \geq \eta(\omega, u_n(\omega), u_{n+1}(\omega))
\]
for all $\omega \in \Omega$, for all $n \in \mathbb{N} \cup \{0\}$ and $u_n(\omega) \to u(\omega)$ as $n \to +\infty$, then
\[
\alpha(\omega, u_n(\omega), u(\omega)) \geq \eta(\omega, u_n(\omega), u(\omega)),
\]
for all $\omega \in \Omega$, for all $n \in \mathbb{N} \cup \{0\}$.
Then \( T \) has a random fixed point, that is, there exists \( \zeta : \Omega \to M \) is measurable such that \( T(\omega, \zeta(\omega)) = \zeta(\omega) \) for all \( \omega \in \Omega \).

Proof. A similar reasoning as in the proof of Theorem 3.3 gives us that the sequence \( \{u_n(\omega)\} \) is a Cauchy sequence for all \( \omega \in \Omega \). This means that there exists \( \zeta : \Omega \to M \) such that \( u_n(\omega) \to \zeta(\omega) \) as \( n \to +\infty \) for all \( \omega \in \Omega \). On the other hand, from (3.2) and hypothesis (G5), we have
\[
\alpha(\omega, u_n(\omega), \zeta(\omega)) \geq \eta(\omega, u_n(\omega), \zeta(\omega)) \text{ for all } n \in \mathbb{N} \cup \{0\}, \omega \in \Omega.
\] (3.3)
Now, using the triangle inequality (3.3) and (G4), we get
\[
d(T(\omega, \zeta(\omega)), \zeta(\omega)) \leq d(T(\omega, \zeta(\omega)), T(\omega, u_n(\omega))) + d(u_{n+1}(\omega), \zeta(\omega))
\]
\[
\leq \psi_\omega(d(\zeta(\omega), u_n(\omega))) + d(u_{n+1}(\omega), \zeta(\omega)).
\]
Taking the limit as \( n \to +\infty \) and since \( \psi_\omega \) is continuous at \( t = 0 \), we have
\[
d(T(\omega, \zeta(\omega)), \zeta(\omega)) = 0,
\]
that is, \( T(\omega, \zeta(\omega)) = \zeta(\omega) \) for all \( \omega \in \Omega \). The hypothesis that \( T \) is sup-measurable implies that \( u_n \) is measurable for all \( n \in \mathbb{N} \) and hence \( \zeta \) is measurable. Thus \( \zeta \) is a random fixed point of \( T \).

4. Application to ordinary random differential equations

We consider the following two-point boundary value problem of second order random differential equation:
\[
\begin{align*}
-\frac{d^2 u}{dt^2}(\omega, t) &= f(\omega, t, u(\omega, t)), \quad t \in [0, 1], \\
u(\omega, u) &= u(\omega, 1) = 0
\end{align*}
\] (4.1)
for all \( \omega \in \Omega \), we have \( f : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R} \) has certain regularities and \( \Omega \) is nonempty.

By a random solution of system (4.1), we mean a measurable mapping \( u : \Omega \to C([0, 1], \mathbb{R}) \) satisfying (4.1), where \( C([0, 1], \mathbb{R}) \) denote the space of all continuous functions defined on \([0, 1]\). The space \( C([0, 1], \mathbb{R}) \) endowed with the metric
\[
d_\infty(x, y) = \|x - y\|_{\infty}.
\]

In this section, we prove a theorem producing the existence of random soution of system (4.1).

Let \((\Omega, \Sigma)\) be a measurable space. Let \( f : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function, which means that \( \omega \mapsto f(\omega, t, u) \) is measurable for all \((t, u) \in [0, 1] \times \mathbb{R}\) and \((t, u) \mapsto f(\omega, t, u) \) is continuous for all \( \omega \in \Omega \).

Then consider the integral operator \( F : \Omega \times C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R}) \) defined by
\[
F(\omega, u)(t) = \int_0^1 G(t, s)f(\omega, s, u(s))ds,
\] (4.2)
for all \( u \in C([0, 1], \mathbb{R}) \) and \( \omega \in \Omega \), where \( G : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is continuous function, and \( g : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function.

Remark 4.1. \( F \) is a random operator from \( \Omega \times C([0, 1], \mathbb{R}) \) into \( C([0, 1], \mathbb{R}) \). In fact, given \( u \in C([0, 1], \mathbb{R}) \) since \( f \) is a Carathéodory function for \( s \in [0, 1] \) fixed, the function \( h : \Omega \times [0, 1] \to \mathbb{R} \), defined by \( h(\omega, t) = G(t, s)f(\omega, s, u(s)) \), is Carathéodory. By Lemma 2.7, the integral in (4.2) is limit of a finite sum of measurable functions. So, the mapping \( \omega \to F(\omega, u) \) is measurable, and hence \( F \) is a random operator.
**Remark 4.2.** Let \( h : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function, \( u \in C([0, 1], \mathbb{R}) \), and let \( \{u_n\} \subseteq C([0, 1], \mathbb{R}) \) be a sequence convergent to \( u \). Then there exists an interval \([a, b] \subseteq \mathbb{R}\) such that \( u_n(s), u(s) \in [a, b] \) for all \( s \in [0, 1] \). The continuity of the function \( h(\omega, \cdot, \cdot) \) in \([0, 1] \times \mathbb{R}\) for fixed \( \omega \in \Omega \) ensures that the function \( h(\omega, \cdot, \cdot) \) is uniformly continuous in \([0, 1] \times [a, b]\).

The hypotheses are the following:

(i) For each \( \omega \in \Omega \), there exist \( \psi_\omega \in \Psi \) and \( \theta : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) such that

\[
\theta(\omega, a, b) \geq 0 \text{ for all } a, b \in \mathbb{R}, \text{ then for every } t \in [0, 1], \text{ we have }
\]

\[
|f(\omega, t, a) - f(\omega, t, b)| \\
\leq \psi_\omega \left( \max \left\{ |a(t) - b(t)|, \frac{1}{2}|a(t) - F(\omega, a(t))| + |b(t) - F(\omega, b(t))| \right. \right) \left. \right) \right).
\]

(ii) There exists a measurable mapping \( u_0 : \Omega \to C([0, 1], \mathbb{R}) \) such that, for all \( \omega \in \Omega \), we have

\[
\theta(\omega, u_0(\omega)(t), F(\omega, u_0(\omega))(t)) \geq 0 \text{ for all } t \in [0, 1]
\]

(iii) For each \( \omega \in \Omega \) and for all \( t \in [0, 1], u, v \in C([0, 1], \mathbb{R}) \), we have

\[
\theta(\omega, u(t), v(t)) \geq 0 \Rightarrow \theta(\omega, F(\omega, u(t)), F(\omega, v(t))) \geq 0.
\]

(iv) \( \int_0^1 G(t, s) ds \leq 1 \) for all \( t \in [0, 1] \) and \( s \in [0, 1] \).

**Theorem 4.3.** If hypotheses (i) – (iv) hold, then the random integral operator \( F \) has a random fixed point.

**Proof.** For fixed \( \omega \in \Omega \) we show that \( F(\omega, \cdot) \) is continuous. Indeed, consider a sequence \( \{u_n\} \subseteq C([0, 1], \mathbb{R}) \) with \( u_n \to u \in C([0, 1], \mathbb{R}) \) as \( n \to +\infty \). By Remark 4.2, there exists \([a, b] \subseteq \mathbb{R}\) such that \( u_n(s), u(s) \in [a, b] \) for all \( s \in [0, 1] \). In addition, the functions \( f(\omega, \cdot, \cdot) \) is uniformly continuous in \([0, 1] \times [a, b]\). Thus, for fixed \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
|f(\omega, s_1, u_1) - f(\omega, s_2, u_2)| < \epsilon,
\]

for all \( s_1, s_2 \in [0, 1] \) and \( u_1, u_2 \in [a, b] \) such that \( |s_1 - s_2| + |u_1 - u_2| < \delta \).

Now, let \( n(\delta) \in \mathbb{N} \) such that \( \|u_n - u\|_\infty < \delta \) whenever \( n \geq n(\delta) \). Then, for every \( n \geq n(\delta) \), we have

\[
|f(\omega, s, u_n(s)) - f(\omega, s, u(s))| < \epsilon.
\]

Consequently, for \( t \in [0, 1] \) and \( n \geq n(\delta) \), we have

\[
|F(\omega, u_n)(t) - F(\omega, u)(t)| \leq \int_0^1 |G(t, s)||f(\omega, s, u_n(s)) - f(\omega, s, u(s))| ds
\]

\[
\leq \epsilon
\]

\[
\Rightarrow \|F(\omega, u_n) - F(\omega, u)\|_\infty \leq \epsilon.
\]

So, \( d_\infty(F(\omega, u_n), F(\omega, u)) \to 0 \) as \( n \to +\infty \) \( \Rightarrow F(\omega, \cdot) \) is a continuous operator for each fixed \( \omega \in \Omega \).

Thus, by Remark 4.2, \( F : \Omega \times C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R}) \) is a Carathéodory function.
Next step is to show that the integral operator $F$ satisfies a generalized random $\alpha-\psi-$contractive type condition as in (H4). So, for each $\omega \in \Omega$ and all $u, v \in C([0, 1], \mathbb{R})$ such that $\theta(\omega, u(t), v(t)) \geq 0$ for all $t \in [0, 1]$, we prove that
\[
d_\infty(F(\omega, u), F(\omega, v)) \leq \psi_\omega(O(\omega, (u, v)))
\]
where
\[
O(\omega, (u, v)) = \max \left\{ \frac{d(u, v)}{2}, \frac{d(u, F(\omega, u)) + d(v, F(\omega, v))}{2}, \frac{d(u, F(\omega, v)) + d(v, F(\omega, u))}{2} \right\}.
\]
Indeed, let $\omega \in \Omega$ be fixed, and $u, v \in C([0, 1], \mathbb{R})$ be such that $\theta(\omega, u(t), v(t)) \geq 0$ for all $t \in [0, 1]$, then
\[
|F(\omega, u)(t) - F(\omega, v)(t)|
= \left| \int_0^1 G(t, s)[f(\omega, s, u(s)) - f(\omega, s, v(s))]ds \right|
\leq \int_0^1 G(t, s)|f(\omega, s, u(s)) - f(\omega, s, v(s))|ds
\leq \int_0^1 G(t, s) \left[ \psi_\omega \left( \max \left\{ |u(s) - v(s)|, \frac{1}{2} [|u(s) - F(\omega, u(s))| + |v(s) - F(\omega, v(s))]| \right\} \right) \right] ds
\leq \int_0^1 G(t, s) \left[ \psi_\omega \left( \max \left\{ |u(s) - v(s)|, \frac{1}{2} [|u(s) - F(\omega, u(s))| + |v(s) - F(\omega, v(s))]| \right\} \right) \right] ds
= \left( \int_0^1 G(t, s)ds \right) \psi_\omega \left( \max \left\{ |u(s) - v(s)|, \frac{1}{2} [|u(s) - F(\omega, u(s))| + |v(s) - F(\omega, v(s))]| \right\} \right)
\leq \psi_\omega \left( \max \left\{ |u(s) - v(s)|, \frac{1}{2} [|u(s) - F(\omega, u(s))| + |v(s) - F(\omega, v(s))]| \right\} \right).
\]
Then
\[
\|F(\omega, u) - F(\omega, v)\|
\leq \psi_\omega \left( \max \left\{ |u(s) - v(s)|, \frac{1}{2} [|u(s) - F(\omega, u(s))| + |v(s) - F(\omega, v(s))]| \right\} \right).
\]
Let $\alpha : \Omega \times C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R}) \to [0, +\infty)$ be function given as
\[
\alpha(\omega, u, v) = \begin{cases} 
1 & \text{if } \theta(\omega, u(t), v(t)) \geq 0 \text{ for all } t \in [0, 1], \\
0 & \text{otherwise}
\end{cases}
\]
GENERALIZED RANDOM $\alpha - \psi$-CONTRACTIVE MAPPINGS

for all $\omega \in \Omega$. So, for all $u, v \in C([0, 1], \mathbb{R})$ with $\alpha(\omega, u, v) \geq 1$, we get

$$
\|F(\omega, u) - F(\omega, v)\|_{\infty} \\
\leq \psi_{\omega}\left(\max\left\{\|u(s) - v(s)\|_{\infty}, \frac{1}{2}\left[\|u(s) - F(\omega, u(s))\|_{\infty} + \|v(s) - F(\omega, v(s))\|_{\infty}\right],\right.\right.
\left.\frac{1}{2}\left[\|u(s) - F(\omega, v(s))\|_{\infty} + \|v(s) - F(\omega, u(s))\|_{\infty}\right]\right\},
$$

which means that $F$ is a generalized random $\alpha - \psi$-contractive integral operator.

Note that, for each $\omega \in \Omega$ and all $t \in [0, 1]$, $u, v \in C([0, 1], \mathbb{R})$, we have

$$
\alpha(\omega, u, v) \geq 1 \\
\Rightarrow \theta(\omega, u(t), v(t)) \geq 0 \; \text{ for all } \; t \in [0, 1] \\
\Rightarrow \theta(\omega, F(\omega, u)(t), F(\omega, v)(t)) \geq 0 \\
\alpha(\omega, F(\omega, u), F(\omega, v)) \geq 1,
$$

which means that $F$ is a random $\alpha$–admissible integral operator.

All of the hypotheses of Corollary 3.4 are satisfied, and hence the mapping $F$ has a random fixed point. □

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