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FIBONACCI-MANN ITERATION FOR MONOTONE ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN MODULAR SPACES

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ABSTRACT. In this work, we extend the fundamental results of Schu to the class of monotone asymptotically nonexpansive mappings in modular function spaces. In particular, we study the behavior of the Fibonacci-Mann iteration process defined by

$$x_{n+1} = t_n T^{\phi(n)}(x_n) + (1 - t_n)x_n,$$

for $n \in \mathbb{N}$, when T is a monotone asymptotically nonexpansive self-mapping.

1. INTRODUCTION

Modular function spaces (MFS) find their roots in the study of the classical function spaces $L^p(\Omega)$ and their extensions by Orlicz and others. For more details on MFS, we recommend the book by Kozlowski [11]. Another interesting use of the modular structure, for whoever is looking for more applications, is the excellent book by Diening et al. [4] about Lebesgue and Sobolev spaces with variable exponents. Fixed point theory in MFS was initiated in 1990 in the original paper [9]. Since then this theory has seen an explosion which culminated in the publication of the recent book by Khamsi and Kozlowski [8]. In this work, we continue investigating the fixed point problem in MFS. To be precise, we investigate the case of monotone mappings. This area of metric fixed point theory is new and attracted some attention after the publication of Ran and Reuring's paper [15]. An interesting reference with many applications of the fixed point theory of monotone mappings is the excellent book by Carl and Heikkilä [3].

Since this work deal with the metric fixed point theory, we recommend the book by Khamsi and Kirk [6].

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2. Preliminaries

For the basic definitions and properties of MFS, we refer the readers to the books [8, 11].

Throughout this work, Ω stands for a nonempty set, Σ a nontrivial σ-algebra of subsets of Ω, $\mathcal{P}$ a δ-ring of subsets of Δ such that $P \cap S \in \mathcal{P}$ for any $P \in \mathcal{P}$ and $S \in \Sigma$. We will assume that there exists an increasing sequence $\{\Delta_n\} \subset \mathcal{P}$ such that $\Omega = \bigcup \Omega_n$. $\mathcal{M}_\infty$ will stand for the space of all extended measurable functions $f : \Omega \to [-\infty, \infty]$ for which there exists $\{g_n\} \subset \mathcal{E}$, with $|g_n| \leq |f|$ and $g_n(t) \to f(t)$, for all $t \in \Omega$, where $\mathcal{E}$ stands for the vector space of simple functions whose supports is in $\mathcal{P}$.

Definition 2.1. [8, 11] *A convex and even function $\rho : \mathcal{M}_\infty \to [0, \infty]$ is called a regular modular if:*

(i) $\rho(f) = 0$ implies $f = 0$ $\rho - a.e.$;
(ii) $|f(t)| \leq |g(t)|$ for all $t \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$ (we will say that ρ is monotone);
(iii) $|f_n(t)| \uparrow |f(t)|$ for all $t \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$ (ρ has the Fatou property).

Recall that a subset $A \in \Sigma$ is said to be ρ-null if $\rho(g\mathbb{1}_A) = 0$, for any $g \in \mathcal{E}$, and a property holds ρ-almost everywhere (ρ-a.e.) if the exceptional set is ρ-null. The notation $\mathbb{1}_A$ denotes the characteristic function of the set A. Consider the set

$$\mathcal{M} = \{f \in \mathcal{M}_\infty; |f(t)| < \infty \ \rho - a.e\}.$$

The MFS L_ρ is defined as:

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \ as \ \lambda \to 0\}.$$

The following theorem is essential throughout this work.

Theorem 2.1. [8, 11] *Let ρ be a convex regular modular.*

(1) If $\rho(\beta f_n) \to 0$, for some $\beta > 0$, then there exists a subsequence $\{f_{\psi(n)}\}$ such that $f_{\psi(n)} \to 0$ $\rho - a.e.$
(2) If $f_n \to f$ $\rho - a.e.$, then $\rho(g) \leq \liminf_{n \to \infty} \rho(g_n)$.

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The following definition is needed since it connects the metric properties with its modular version.

Definition 2.2. [8, 11] *Let ρ be a convex regular modular.*

(1) $\{g_n\}$ is said to ρ-converge to g if $\lim\limits_{n\to\infty} \rho(g_n - g) = 0$.

(2) A sequence $\{g_n\}$ is called ρ-Cauchy if $\lim\limits_{n,m\to\infty} \rho(g_n - g_m) = 0$.

(3) A subset C of L_ρ is said to be ρ-closed if for any sequence $\{g_n\}$ in C ρ-convergent to g implies that $g \in C$.

(4) A subset C of L_ρ is called ρ-bounded if its ρ-diameter

$$\delta_\rho(C) = \sup\{\rho(g - h);\ g,\ h \in C\}$$

is finite.

Note that despite the fact that ρ does not satisfy the triangle inequality in general, the ρ limit is unique. However, the ρ-convergence may not imply ρ-Cauchy behavior. But it is interesting to know that ρ-balls $B_\rho(x, r) = \{y \in L_\rho; \rho(x-y) \leq r\}$ are ρ-closed, and any ρ-Cauchy sequence in L_ρ is ρ-convergent, i.e. L_ρ is ρ-complete [8, 11].

Using Theorem 2.1 Part (1), we get the following result:

Theorem 2.2. *Let ρ be a convex regular modular. Let $\{g_n\} \subset L_\rho$ be a sequence which ρ-converges to g. The following hold:*

(i) if $\{g_n\}$ is monotone increasing, i.e., $g_n \leq g_{n+1}$ ρ-a.e., for any $n \geq 1$, then $g_n \leq g$ ρ-a.e., for any $n \geq 1$.;

(ii) if $\{g_n\}$ is monotone decreasing, i.e., $g_{n+1} \leq g_n$ ρ-a.e., for any $n \geq 1$, then $g \leq g_n$ ρ-a.e., for any $n \geq 1$.

Next we discuss a property called uniform convexity which plays an important part in metric fixed point theory.

Definition 2.3. [8] *Let ρ be a convex regular modular.*

(i) Let $r > 0$ and $\varepsilon > 0$. Define

$$\delta_\rho(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}\,\rho\left(\frac{f+g}{2}\right);\ (f, g) \in D(r, \varepsilon)\right\},$$

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where

$$D(r,\varepsilon)=\{(f,g); f,g\in L_\rho,\ \rho(f)\le r,\ \rho(g)\le r, \rho(f-g)\ge \varepsilon r\}.$$

ρ is said to be uniformly convex (UC) if for every $R>0$ and $\varepsilon>0$, we have $\delta_\rho(R,\varepsilon)>0$.

(ii) ρ is said to be (UUC) if for every $s\ge 0, \varepsilon>0$ there exists $\eta(s,\varepsilon)>0$ such that $\delta_\rho(R,\varepsilon)>\eta(s,\varepsilon)>0$, for $R>s$.

Remark 2.1. *It is known that, under suitable assumptions, the uniform convexity of the modular in Orlicz spaces is satisfied provided the Orlicz function is uniformly convex* [10, 17]*. As an example of Orlicz functions that are uniformly convex, we may take $\varphi_1(t)=e^{|t|}-|t|-1$ and $\varphi_2(t)=e^{t^2}-1$* [13, 12]*. Note that these two functions fail the important property known as the the Δ_2 condition.*

Modular functions which are uniformly convex enjoys a property similar to reflexivity in Banach spaces.

Theorem 2.3. [8, 10] *Let ρ be a (UUC) convex regular modular. Then L_ρ has property (R), i.e. every nonincreasing sequence $\{C_n\}$ of nonempty, ρ-bounded, ρ-closed, convex subsets of L_ρ has nonempty intersection.*

Remark 2.2. *Let ρ be a (UUC) convex regular modular. Let K be a ρ-bounded convex ρ-closed nonempty subset of L_ρ. Let $\{f_n\}\subset K$ be a monotone increasing sequence. Since order intervals in L_ρ are convex and ρ-closed, then the property (R) implies*

$$\bigcap_{n\ge 1}\Big\{f\in K;\ f_n\le f\ \ \rho-a.e.\Big\}\neq\emptyset.$$

In other words, there exists $f\in K$ such that $f_n\le f$ ρ-a.e., for any $n\ge 1$. A similar conclusion holds for decreasing sequences.

The following lemma is useful throughout this work.

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Lemma 2.1. [7] *Let ρ be a (UUC) convex regular modular. Let $R > 0$ and $\{\alpha_n\} \subset [a, b]$ with $0 < a \leq b < 1$. Assume that*

$$\begin{cases} \limsup_{n\to\infty} \rho(f_n) \leq R, \\ \limsup_{n\to\infty} \rho(g_n) \leq R, \\ \lim_{n\to\infty} \rho(\alpha_n\, f_n + (1-\alpha_n)\, g_n) = R. \end{cases}$$

Then $\lim_{n\to\infty} \rho(f_n - g_n) = 0$ holds.

The concept of ρ-type functions will prove to be an important tool dealing with the existence of fixed points.

Definition 2.4. *Let ρ be a convex regular modular. Let K be a nonempty subset of L_ρ. A function $\tau : K \to [0, \infty]$ is called a ρ-type if there exists a sequence $\{g_m\}$ of elements of L_ρ such that*

$$\tau(f) = \limsup_{m\to\infty} \rho(g_m - f),$$

for any $f \in K$. Let τ be a type. A sequence $\{f_n\}$ is called a minimizing sequence of τ in K if $\lim_{n\to\infty} \tau(f_n) = \inf\{\tau(f);\ f \in K\}$.

We have the following amazing result about type functions in MFS.

Lemma 2.2. [7] *Let ρ be a (UUC) convex regular modular. Let K be a ρ-bounded ρ-closed convex nonempty subset of L_ρ. Then any minimizing sequence of any ρ-type defined on K is ρ-convergent. Its limit is independent of the minimizing sequence.*

Before we finish this section, we give the modular definitions of monotone Lipschitzian mappings.

Definition 2.5. *Let ρ be a convex regular modular. Let C be nonempty subset of L_ρ. A mapping $T : C \to C$ is said to be monotone if $T(f) \leq T(g)$ ρ-a.e. whenever $f \leq g$ ρ-a.e., for any $f, g \in C$. Moreover T is called monotone asymptotically* *nonexpansive (in short M-A-N) if T is monotone and there exists $\{k_n\} \subset [1, +\infty)$ such that $\lim\limits_{n\to\infty} k_n = 1$ and*

$$\rho(T^n(g) - T^n(h)) \leq k_n \ \rho(g - h),$$

for any g and h in C such that $g \leq h$ ρ-a.e., and $n \geq 1$. $g \in C$ is called a fixed point of T if and only if $T(g) = g$.

The fixed point theory for asymptotically nonexpansie mappings finds its root in the work of Goebel and Kirk [5]. Following the success of the fixed point theory of monotone mappings, an existence fixed point theorem for M-A-N mappings was proved in [1] and its modular version in [2]. Before we state the main result of [2], recall that a map T is said to be ρ-continuous if $\{g_n\}$ ρ-converges to g implies $\{T(g_n)\}$ ρ-converges to $T(g)$.

Theorem 2.4. [2] *Let ρ be a (UUC) convex regular modular. Let C be a ρ-bounded ρ-closed convex nonempty subset of L_ρ. Let $T : C \to C$ be a ρ-continuous M-A-N mapping. Assume there exists $f_0 \in K$ such that $f_0 \leq T(f_0)$ (resp. $T(f_0) \leq f_0$) ρ-a.e. Then T has a fixed point f such that $f_0 \leq f$ (resp. $f \leq f_0$) ρ-a.e.*

The original proof of the existence of a fixed point of asymptotically nonexpansive mappings was not constructive. It was Shu [16] who considered a modified Mann iteration to generate an approximate fixed point sequence for such mappings. While studying asymptotically nonexpansive mappings, Schu modified the Mann iteration by

$$x_{n+1} = t_n T^n(x_n) + (1 - t_n) x_n, \tag{SMI}$$

for $t_n \in [0, 1]$ and $n \in \mathbb{N}$. His main motivation resides in the fact that the iterates of T are becoming almost nonexpansive. In the investigation of monotone mappings, the iteration sequence (SMI) fails to generate a sequence which is monotone. A very important fact when investigating the existence of fixed points of such mappings. This problem forced the authors in [1] to modify the iteration (SMI) by using the Fibonacci sequence $\{\phi(n)\}$ defined by

$$\phi(0) = \phi(1) = 1, \ \ \textit{and} \ \ \phi(n + 1) = \phi(n) + \phi(n - 1),$$

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for any $n \geq 1$. The new iteration scheme, called Fibonacci-Mann iteration, is defined by

$$\text{(FMI)} \qquad h_{n+1} = \alpha_n \, T^{\phi(n)}(h_n) + (1 - \alpha_n) \, h_n,$$

for $\alpha_n \in [0,1]$ and $n \in \mathbb{N}$. This new iteration scheme allowed the authors of [1] to prove the main results of Schu [16] for M-A-N mappings defined in uniformly convex Banach spaces. A surprising fact since this class of mappings may fail to be continuous.

Next we discuss the behavior of the iteration (FMI) which will generate an approximate fixed point of M-A-N mapping in MFS.

The proof of the following lemma is order theoretical and is similar to the original proof done in [1] in the context of Banach spaces.

Lemma 2.3. [1] *Let ρ be a convex regular modular. Let C be a convex nonempty subset of L_ρ. Let $T : C \to C$ be a monotone mapping. Let $h_0 \in C$ be such that $h_0 \leq T(h_0)$ (resp. $T(h_0) \leq h_0$) ρ-a.e. Let $\{\alpha_n\} \subset [0,1]$. Consider the (FMI) sequence $\{h_n\}$ generated by h_0 and $\{\alpha_n\}$. Let f be a fixed point of T such that $h_0 \leq f$ (resp. $f \leq h_0$) ρ-a.e. Then*

(i) $h_0 \leq h_n \leq h_{n+1} \leq T^{\phi(n)}(h_n) \leq f$ *(resp.* $f \leq T^{\phi(n)}(h_n) \leq h_{n+1} \leq h_n \leq h_0$*)* ρ*-a.e.,*

(ii) $h_0 \leq T^{\phi(n)}(h_0) \leq T^{\phi(n)}(h_n) \leq f$ *(resp.* $f \leq T^{\phi(n)}(h_n) \leq T^{\phi(n)}(h_0) \leq h_0$*)* ρ*-a.e.,*

for any $n \in \mathbb{N}$.

The next lemma is crucial in the proof of the main results of this work.

Lemma 2.4. *Let ρ be a convex regular modular. Let C be a ρ-bounded and convex nonempty subset of L_ρ. Assume $T : C \to C$ is M-A-N mapping with the Lipschitza constants $\{k_n\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $h_0 \in C$ be such that $h_0 \leq T(h_0)$ (resp. $T(h_0) \leq h_0$) ρ-a.e. Let $\{\alpha_n\} \subset [0,1]$. Consider the (FMI) sequence $\{h_n\}$ generated by h_0 and $\{\alpha_n\}$. Let f be a fixed point of T such that $h_0 \leq f$ (resp. $f \leq h_0$) ρ-a.e. Then $\lim_{n\to\infty} \rho(h_n - f)$ exists.*

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Proof. Without loss of generality, assume that $h_0 \leq T(h_0)$ ρ-a.e. From the definition of $\{h_n\}$, we have

$$\begin{array}{rcl}\rho(h_{n+1}-f) & \leq & \alpha_n\ \rho(T^{\phi(n)}(h_n)-f)+(1-\alpha_n)\ \rho(h_n-f)\\ & = & \alpha_n\ \rho(T^{\phi(n)}(h_n)-T^{\phi(n)}(f))+(1-\alpha_n)\ \rho(h_n-f)\\ & \leq & \alpha_n\ (k_{\phi(n)}-1)\ \rho(h_n-f)+\rho(h_n-f)\\ & = & (k_{\phi(n)}-1)\ \rho(h_n-f)+\rho(h_n-f)\end{array}$$

for any $n \geq 1$, since T is M-A-N. Hence

$$\rho(h_{n+1}-f)-\rho(h_n-f) \leq (k_{\phi(n)}-1)\ \delta_\rho(C),$$

for any $n \in \mathbb{N}$. Hence

$$\rho(h_{n+m}-f)-\rho(h_n-f) \leq \delta_\rho(C)\sum_{i=0}^{m}(k_{\phi(n+i)}-1),$$

for any $n, m \geq 1$. If we let $m \to \infty$, we get

$$\limsup_{m\to\infty}\rho(h_m-f) \leq \rho(h_n-f)+\delta_\rho(C)\sum_{i=n}^{\infty}(k_{\phi(i)}-1) \leq \rho(h_n-f)+\delta_\rho(C)\sum_{i=n}^{\infty}(k_i-1),$$

for any $n \geq 1$. Next let $n \to \infty$, we get

$$\limsup_{m\to\infty}\rho(h_m-f) \leq \liminf_{n\to\infty}\rho(h_n-f)+\delta_\rho(C)\liminf_{n\to\infty}\sum_{i=n}^{\infty}(k_i-1) = \liminf_{n\to\infty}\rho(h_n-f).$$

Therefore, we have $\limsup_{m\to\infty}\rho(h_m-f) = \liminf_{n\to\infty}\rho(h_n-f)$, which implies the desired conclusion. □

3. MAIN RESULTS

The next result shows that the sequence generated by (FMI) has an approximate fixed point behavior which is crucial throughout.

Proposition 3.1. *Let ρ be a (UUC) convex regular modular. Let $C \subset L_\rho$ be a ρ-bounded ρ-closed convex nonempty subset. Let $T : C \to C$ be a M-A-N mapping with the associated constants $\{k_n\}$ satisfying $\sum_{n=1}^{\infty}(k_n-1) < \infty$. Let $h_0 \in C$ be such that $h_0 \leq T(h_0)$ (resp. $T(h_0) \leq h_0$) ρ-a.e. Let f be a fixed point of T*

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such that $h_0 \leq f$ *(resp.* $f \leq h_0$*)* ρ*-a.e. Let* $\{\alpha_n\} \subset [a, b]$*, with* $0 < a \leq b < 1$*. Consider the (FMI) sequence* $\{h_n\}$ *generated by* h_0 *and* $\{\alpha_n\}$*. Then*

$$\lim_{n\to\infty} \rho(h_n - T^{\phi(n)}(h_n)) = 0.$$

Proof. Without loss of generality, we assume $h_0 \leq T(h_0)$ ρ-a.e. From Lemma 2.4, we know that $h_n \leq h_{n+1} \leq f$ ρ-a.e., which implies $\rho(f - h_{n+1}) \leq \rho(f - h_n)$, for any $n \in \mathbb{N}$, i.e., $\{\rho(f - h_n)\}$ is a decreasing sequence of positive numbers. Hence $\lim\limits_{n\to\infty} \rho(h_n - f)$ exists. Set $R = \lim\limits_{n\to\infty} \rho(h_n - f)$. Assume that $R = 0$, i.e., $\{h_n\}$ ρ-converges to f. From Lemma 2.3, we get $h_n \leq T^{\phi(n)}(h_n) \leq f$ ρ-a.e., which implies

$$\rho(T^{\phi(n)}(h_n) - h_n) \leq \rho(f - h_n),$$

for any $n \in \mathbb{N}$. Hence we have $\lim\limits_{n\to\infty} \rho(T^{\phi(n)}(h_n) - h_n) = 0$. Next, we assume $R > 0$. We have

$$\begin{aligned} \limsup_{n\to\infty} \rho(T^{\phi(n)}(h_n) - f) &= \limsup_{n\to\infty} \rho(T^{\phi(n)}(h_n) - T^{\phi(n)}(f)) \\ &\leq \limsup_{n\to\infty} k_{\phi(n)}\ \rho(h_n - f) \\ &= R. \end{aligned}$$

On the other hand, we have $\rho(h_{n+1} - f) \leq \alpha_n\ \rho(T^{\phi(n)}(h_n) - f) + (1 - \alpha_n)\ \rho(h_n - f)$, for any $n \geq 1$. Let $\mathcal{U}$ be a non-trivial ultrafilter over $\mathbb{N}$. We have

$$R = \lim_{\mathcal{U}} \rho(h_{n+1} - f) \leq \alpha\ \lim_{\mathcal{U}} \rho(T^{\phi(n)}(h_n) - f) + (1 - \alpha)\ R,$$

with $\lim\limits_{\mathcal{U}}\ \alpha_n = \alpha \in [a, b]$. Since $\alpha \neq 0$, we get $\lim\limits_{\mathcal{U}} \rho(T^{\phi(n)}(h_n) - f) \geq R$. Since $\mathcal{U}$ was an arbitrary ultrafilter, we get

$$R \leq \liminf_{n\to\infty} \rho(T^{\phi(n)}(h_n) - f) \leq \limsup_{n\to\infty} \rho(T^{\phi(n)}(h_n) - f) \leq R.$$

So $\lim\limits_{n\to\infty} \rho(T^{\phi(n)}(h_n) - f) = R$. Since

$$\lim_{n\to\infty}\ \rho\Big(\alpha_n\ T^{\phi(n)}(h_n) + (1 - \alpha_n)\ h_n - f\Big) = \lim_{n\to\infty} \rho(h_{n+1} - f) = R,$$

Lemma 2.1 implies that $\lim\limits_{n\to\infty} \rho(h_n - T^{\phi(n)}(h_n)) = 0$, which completes the proof of our claim. □

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Recall that the map $T : C \to C$ is said to be ρ-compact if $\{T(f_n)\}$ has a ρ-convergent subsequence for any sequence $\{f_n\}$ in C. The following result is the monotone version of Theorem 2.2 of [16].

Theorem 3.1. *Let ρ be a (UUC) convex regular modular. Let $C \subset L_\rho$ be a ρ-bounded and ρ-closed convex nonempty subset of L_ρ. Let $T : C \to C$ be a M-A-N mapping with the Lipschitz constants $\{k_n\}$. Assume that T^m is ρ-compact for some $m \geq 1$. Let $h_0 \in C$ be such that $h_0 \leq T(h_0)$ (resp. $T(h_0) \leq h_0$) ρ-a.e. Let $\{\alpha_n\} \subset [a, 1]$ with $0 < a \leq 1$. Consider the (FMI) sequence $\{h_n\}$ generated by h_0 and $\{\alpha_n\}$. Then $\{h_n\}$ ρ-converges to a fixed point f of T such that $h_0 \leq f$ (resp. $f \leq h_0$) ρ-a.e.*

Proof. Without loss of generality, we assume $h_0 \leq T(h_0)$ ρ-a.e. Since T is monotone, the sequence $\{T^n(h_0)\}$ is monotone increasing. Since T^m is ρ-compact, there exists a subsequence $\{T^{\phi(n)}(h_0)\}$ which ρ-converges to $f \in C$. Let us show that $\{T^n(h_0)\}$ ρ-converges to f and is a fixed point of T. Using the properties of partial order, we have $T^n(h_0) \leq f$ ρ-a.e., for any $n \in \mathbb{N}$. In particular, we have

$$T^{\phi(n)}(h_0) \leq T^{\phi(n)+1}(h_0) \leq f \quad \rho - a.e.$$

for any $n \in \mathbb{N}$. Hence

$$\rho(f - T^{\phi(n)+1}(h_0)) \leq \rho(f - T^{\phi(n)}(h_0)),$$

for any $n \in \mathbb{N}$. This will imply $\{T^{\phi(n)+1}(h_0)\}$ ρ-converges to f. But

$$\rho(T(f) - T^{\phi(n)+1}(h_0)) \leq k_1 \ \rho(f - T^{\phi(n)}(h_0)),$$

for any $n \in \mathbb{N}$, which implies that $\{T^{\phi(n)+1}(h_0)\}$ ρ-converges to $T(f)$ as well, which implies $T(f) = f$ from the uniqueness of the ρ-limit. It is clear from the properties of the modular ρ, that $\{\rho(f - T^n(h_0))\}$ is a decreasing sequence of positive real numbers. Hence

$$\lim_{n\to\infty} \ \rho(f - T^n(h_0)) = \lim_{n\to\infty} \ \rho(f - T^{\phi(n)}(h_0)) = 0,$$

i.e., $\{T^n(h_0)\}$ ρ-converges to f. Let us finish the proof of Theorem 3.1 by showing that $\{h_n\}$ ρ-converges to f. Since f is a fixed point of T which satisfies $h_0 \leq f$, then Lemma 2.3 implies $T^{\phi(n)}(h_0) \leq T^{\phi(n)}(h_n) \leq f$ ρ-a.e., which implies

$$\rho(f - T^{\phi(n)}(h_n)) \leq \rho(f - T^{\phi(n)}(h_0)),$$

for any $n \in \mathbb{N}$. Hence $\{T^{\phi(n)}(h_n)\}$ ρ-converges to f. Since $\{h_n\}$ is monotone increasing and bounded above by f, we know that $\{\rho(f - h_n)\}$ is a decreasing sequence of positive real numbers. Hence $\lim_{n\to\infty} \rho(f - h_n) = R$ exists. Let us prove that $R = 0$. Let $\mathcal{U}$ be a non-trivial ultrafilter over $\mathbb{N}$. We have

$$\lim_{\mathcal{U}} \rho(h_{n+1} - f) \leq \alpha \lim_{\mathcal{U}} \rho(T^{\phi(n)}(h_n) - f) + (1 - \alpha) \lim_{\mathcal{U}} \rho(h_n - f),$$

with $\lim_{\mathcal{U}} \alpha_n = \alpha \in [a, 1]$. Since $\lim_{\mathcal{U}} \rho(h_{n+1} - f) = \lim_{\mathcal{U}} \rho(h_n - f) = R$, and $\lim_{\mathcal{U}} \rho(T^n(h_n) - f) = 0$, we get $R \leq (1 - \alpha)\, R$ which implies $R = 0$ since $\alpha \neq 0$. □

Before investigate a weaker convergence of the (FMI) sequence, we will need the following result which may be seen as similar to the classical Opial condition [14].

Proposition 3.2. *Let $C \subset L_\rho$ be a ρ-a.e. compact and ρ-bounded convex nonempty subset of L_ρ. Let $\{f_n\}$ be a monotone increasing (resp. decreasing) bounded sequence in C. Consider the ρ-type function $\varphi : C_\infty \to [0, +\infty)$ defined by*

$$\varphi(h) = \lim_{n\to\infty} \rho(f_n - h),$$

where $C_\infty = \{h \in C;\ f_n \preceq h$ (resp. $h \preceq f_n$) for any $n \in \mathbb{N}\}$. Then $\{f_n\}$ is ρ-a.e. convergent to $f \in C_\infty$ and

$$\varphi(f) = \inf\{\varphi(h);\ h \in C_\infty\}.$$

Moreover, if ρ is (UUC), then any minimizing sequence $\{h_n\}$ of φ in C_∞ ρ-converges to f. In particular, φ has a unique minimum point in C_∞.

Proof. Without loss of generality, assume that $\{f_n\}$ be a monotone increasing. Since C is ρ-a.e. compact, there exists a subsequence $\{f_{\phi(n)}\}$ which is ρ-a.e. convergent to some $f \in C$. Using Theorem 2.2, we conclude that $f_n \leq f$ ρ-a.e., for any $n \in \mathbb{N}$, which implies $f \in C_\infty$ and proves it is nonempty. Let $h \in C_\infty$. Then the sequence $\{\rho(h - f_n)\}$ is a decreasing sequence of finite positive numbers since C is ρ-bounded. Hence $\varphi(h) = \lim\limits_{n\to\infty} \rho(f_n - h)$ exists. As we saw before, there exists a subsequence $\{f_{\phi(n)}\}$ of $\{f_n\}$ which ρ-a.e.-converges to $f \in C_\infty$. Let us prove that $\{f_n\}$ ρ-a.e.-converges to f. Indeed, for any $n \in \mathbb{N}$, there exists a unique $k_n \in \mathbb{N}$ such that $\phi(k_n) \leq n < \phi(k_n + 1)$. Clearly we have $k_n \to \infty$ when $n \to \infty$. Moreover, we have $f_{\phi(k_n)} \leq f_n \leq f$, for any $n \in \mathbb{N}$. Since $\{f_{\phi(k_n)}\}$ ρ-a.e. converges to f, we conclude that $\{f_n\}$ also ρ-a.e. converges to f. Next let $h \in C_\infty$. Then we must have $f_n \leq f \leq h$ ρ-a.e., which implies

$$\rho(f - f_n) \leq \rho(h - f_n),$$

for any $n \in \mathbb{N}$. Hence $\varphi(f) \leq \varphi(h)$, i.e.,

$$\varphi(f) = \inf\{\varphi(h);\ h \in C_\infty\}.$$

The last part of Proposition 3.2 is a classical result which may be found in [7]. □

Now we are ready to state a modular monotone version of Theorem 2.1 of [16].

Theorem 3.2. *Let ρ be a (UUC) convex regular modular. Let $C \subset L_\rho$ be a ρ-a.e. compact and ρ-bounded convex nonempty subset of L_ρ. Let $T : C \to C$ be a M-A-N mapping with the Lipschitz constants $\{k_n\}$. Assume that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Let $h_0 \in C$ be such that h_0 and $T(h_0)$ are comparable. Let $\{\alpha_n\} \subset [a, b]$ with $0 < a \leq b < 1$. Consider the (FMI) sequence $\{h_n\}$ generated by h_0 and $\{\alpha_n\}$. Then $\{h_n\}$ is ρ-a.e.-convergent. Its ρ-limit is a fixed point of T comparable to h_0.*

Proof. Without loss of generality, assume that $h_0 \preceq T(h_0)$. In this case, we know that $\{T^n(h_0)\}$ is monotone increasing. Proposition 3.2 implies that $\{T^n(h_0)\}$ is ρ-a.e. convergent to $f \in C_\infty = \{h \in C;\ T^n(h_0) \leq h\ \ \rho - a.e.$ for any $n \in \mathbb{N}\}$, the minimum point of $\varphi : C_\infty \to [0, +\infty)$ defined by

$$\varphi(h) = \lim_{n\to\infty} \rho(T^n(h_0) - h).$$

From the definition of M-A-N, we get

$$\varphi(f) \leq \varphi(T^m(f)) \leq k_m\ \varphi(f),$$

for any $m \geq 1$. Hence $\{T^m(f)\}$ is a minimizing sequence of φ since $\lim\limits_{m\to\infty} k_m = 1$. Using Proposition 3.2, we conclude that $\{T^m(f)\}$ ρ-converges to f. Note that since $T^n(h_0) \leq f$ ρ-a.e., we get $T^{n+1}(h_0) \leq T(f)$ ρ-a.e., for any $n \in \mathbb{N}$, which implies $f \leq T(f)$ ρ-a.e., for any $n \in \mathbb{N}$. Hence $\{T^m(f)\}$ is monotone increasing and ρ-converges to f, which implies $T^m(f) \leq f$ ρ-a.e. Hence $T(f) = f$ holds, i.e., f is a fixed point of T. Using Lemma 2.3, we have

$$T^{\phi(n)}(h_0) \leq T^{\phi(n)}(h_n) \leq f \quad \rho - a.e.,$$

for any $n \in \mathbb{N}$, which implies that $\{T^{\phi(n)}(h_n)\}$ also ρ-a.e.-converges to f. Proposition 3.1 implies

$$\lim_{n\to\infty} \rho(h_n - T^{\phi(n)}(h_n)) = 0.$$

Using the properties of ρ-convergence and ρ-a.e.-convergence [8], there exists a sequence of increasing integers $\{k_n\}$ such that $\{h_{k_n} - T^{\phi(k_n)}(h_{k_n})\}$ ρ-a.e.-converges to 0. Therefore, we must have $\{h_{k_n}\}$ ρ-a.e.-converges to f. Since $\{h_n\}$ is monotone increasing, Proposition 3.2 completes the proof of Theorem 3.2 by implying that $\{h_n\}$ ρ-a.e.-converges to f, a fixed point of T. □

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