Generalized Preinvex Functions and Their Applications

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Abstract: This study focuses on a new class of functions called sub-b-s-preinvex, that is a generalization of sub-b-s-convex and preinvex functions, and discusses some of their properties. A new sub-b-s-preinvex programming is introduced and the sufficient conditions of optimality under this type of function is established.

Keywords: Geodesic E-convex sets; Geodesic E-convex functions; Riemannian manifolds

1 Introduction

Convex functions play an important role in optimization theory, convex analysis, Minkowski space and fractal mathematics [1, 4, 6, 7, 9, 12, 13, 14]. And the generalized convex functions is one of the main topics that researchers worked a lot on it. A class of b-vex functions were introduced in [2]. The definitions of two kinds of s-convex functions and some of their properties were given in [8]. Also, a new generalized functions called sub-b-convex was studied in [5]. Sub-b-s-convex functions that extended of the concept of sub-b-convexity were introduced in [10]. Semi-b-preinvex functions were given in [11] as a generalization of the semi preinvex functions. The main aim of this paper is given a new class of function called sub-b-s-preinvex function, that can be reduced in to sub-b-preinvex when s=1, and studied some of their properties. Furthermore, a new class of sets called sub-b-s-preinvex sets is defined. A new sub-b-s-preinvex programming is introduced and the sufficient conditions of optimality under this type of function is established.
2 Main results

Let us given some definitions of sub-b-convexity and preinvexity functions before give our main results.

Definition 2.1. [10] A function \( h : K \rightarrow \mathbb{R} \) is called a sub-b-s-convex function on a non-empty convex set \( K \subset \mathbb{R}^n \) w.r.t. \( b \) if
\[
h(\delta u_1 + (1 - \delta) u_2) \leq \delta h(u_1) + (1 - \delta) h(u_2) + b(u_1, u_2, \delta),
\]
\( \forall u_1, u_2 \in K, \delta \in [0, 1], s \in (0, 1] \).

Ben-Israel and Mond [3] defined a class of functions called preinvex in the non-empty invex set \( K \subset \mathbb{R}^n \) w.r.t. \( \eta \) such as

Definition 2.2. A function \( h : K \rightarrow \mathbb{R} \) is preinvex on \( K \) w.r.t. \( \eta \), if there exists an \( n \)-dimensional vector function \( \eta : K \times K \rightarrow \mathbb{R}^n \) such that
\[
h(u_2 + \delta \eta(u_1, u_2)) \leq \delta h(u_1) + (1 - \delta) h(u_2),
\]
\( \forall u_1, u_2 \in K, \delta \in [0, 1] \).

Now, the concepts of sub-b-s-preinvex function and sub-b-s-preinvex set are given. Furthermore, some of their properties are studied.

Definition 2.3. A function \( h : K \rightarrow \mathbb{R} \) is called a sub-b-s-preinvex function on \( K \) w.r.t. \( \eta, b \), if
\[
h(u_2 + \delta \eta(u_1, u_2)) \leq \delta h(u_1) + (1 - \delta) h(u_2) + b(u_1, u_2, \delta),
\]
\( (2.1) \)
\( \forall u_1, u_2 \in K, \delta \in [0, 1], s \in (0, 1] \).

Remark 2.4. 1. If \( \eta(u_1, u_2) = u_1 - u_2 \) in \( (2.1) \), then sub-b-s-preinvex w.r.t. \( \eta, b \) becomes sub-b-s-convex function. Also, if we put \( s=1 \), then \( (2.1) \) becomes sub-b-convex function.

2. When \( \eta(u_1, u_2) = u_1 - u_2 \) and \( b(u, u, \delta) \leq 0 \) in \( (2.1) \), then the sub-b-s-preinvex function becomes convex function.

Theorem 2.5. If \( h_1, h_2 : K \rightarrow \mathbb{R} \) are sub-b-s-preinvex functions w.r.t. \( \eta, b \), then \( h_1 + h_2 \) and \( \beta h_1 (\beta \geq 0) \) are also sub-b-s-preinvex functions w.r.t. \( \eta, b \).
Theorem 2.10. From the above definition, the following theorem can be stated and proven as

Proof.

Proposition 2.7. If \( h_k : K \to \mathbb{R} \), where \( k = 1, 2, \ldots, n \) are sub-b-s-preinvex functions w.r.t.\( \eta, b_k \), then the function which is \( H = \sum_{k=1}^{n} a_k h_k, a_k \geq 0 \) \((k = 1, 2, \ldots, n)\) is also sub-b-s-preinvex function w.r.t.\( \eta, b \) where \( b = \sum_{k=1}^{n} a_k b_k \).

Corollary 2.6. If \( h_k : K \to \mathbb{R} \), where \( k = 1, 2, \ldots, n \) are sub-b-s-preinvex functions w.r.t.\( \eta, b_k \), then the function which is \( H = \max h_k, k = 1, 2, \ldots, n \) is also sub-b-s-preinvex function w.r.t.\( \eta, b \) where \( b = \max b_k \).

Theorem 2.8. Let \( h_1 : K \to \mathbb{R} \) be a sub-b-s-preinvex function w.r.t.\( \eta, b_1 \) and \( h_2 : \mathbb{R} \to \mathbb{R} \) be an increasing function, then \( h_1 oh_2 \) is a sub-b-s-preinvex functions w.r.t.\( \eta, b \) where \( b = h_2 oh_1 \), if \( h_2 \) satisfies the following conditions

1. \( h_2 (\beta u_1) = \beta h_2 (u_1), \forall u_1 \in \mathbb{R}, \beta \geq 0. \)
2. \( h_2 (u_1 + u_2) = h_2 (u_1) + h_2 (u_2), \forall u_1, u_2 \in \mathbb{R}, \beta \geq 0. \)

Proof.

\[
(h_2 oh_1)(u_2 + \delta \eta(u_1, u_2)) = h_2 (h_1 (u_2 + \delta \eta(u_1, u_2))) \leq h_2 (\delta^s h_1 (u_1) + (1 - \delta)^s h_1 (u_2) + b_1 (u_1, u_2, \delta)) = \delta^s h_2 (h_1 (u_1)) + (1 - \delta)^s h_2 (h_1 (u_2)) + h_2 (b_1 (u_1, u_2, \delta)) = \delta^s (h_2 oh_1)(u_1) + (1 - \delta)^s (h_2 oh_1)(u_2) + b(u_1, u_2, \delta)
\]

which means that \( h_2 oh_1 \) is sub-b-s-preinvex functions w.r.t.\( \eta, b \). □

Now, we give a definition of sub-b-s-preinvex set w.r.t.\( \eta, b \).

Definition 2.9. A set \( K \subseteq \mathbb{R}^{n+1} \) is called a sub-b-s-preinvex set w.r.t.\( \eta, b \), if

\[
(u_2 + \delta \eta(u_1, u_2), \delta^s \beta_1 + (1 - \delta)^s \beta_2 + b(u_1, u_2, \delta)) \in K,
\]

\( \forall(u_1, \beta_1), (u_2, \beta_2) \in K, u_1, u_2 \in \mathbb{R}^n, \delta \in [0, 1], s \in (0, 1) \) and \( b : \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \to \mathbb{R} \).

The epigraph of the sub-b-s-preinvex function \( h : k \to \mathbb{R} \) can be given as

\[
G(h) = \{(u, \beta) : u \in K, \beta \in \mathbb{R}, h(u) \leq \beta\}.
\]

From the above definition, the following theorem can be stated and proven

Theorem 2.10. \( h : k \to \mathbb{R} \) is a sub-b-s-preinvex function w.r.t.\( \eta, b \) iff its epigraph is also a sub-b-s-preinvex set w.r.t.\( \eta, b \).
Proof. Let that \((u_1, \beta_1), (u_2, \beta_2) \in G(h)\), then by using the hypothesis, we find \(h(u_1) \leq \beta_1\) and \(h(u_2) \leq \beta_2\).

\[
\begin{align*}
    h(u_2 + \delta \eta(u_1, u_2)) & \leq \delta^* h(u_1) + (1 - \delta)^* h(u_2) + b(u_1, u_2, \delta) \\
    & \leq \delta^* \beta_1 + (1 - \delta)^* \beta_2 + b(u_1, u_2, \delta).
\end{align*}
\]

Hence,

\[
(u_2 + \delta \eta(u_1, u_2), \beta_1 + (1 - \delta)^* \beta_2 + b(u_1, u_2, \delta)) \in G(h).
\]

Therefore, \(G(h)\) is sub-b-s-preinvex set w.r.t.\(\eta, b\).

Now, assume that \(G(h)\) is sub-b-s-preinvex set w.r.t.\(\eta, b\), then

\[
(u_1, h(u_1)), (u_2, h(u_2)) \in G(h),
\]

where \(u_1, u_2 \in K\).

\[
(u_2 + \delta \eta(u_1, u_2), \delta^* h(u_1) + (1 - \delta)^* h(u_2) + b(u_1, u_2, \delta)) \in G(h)
\]

which means that

\[
h(u_2 + \delta \eta(u_1, u_2)) \leq \delta^* h(u_1) + (1 - \delta)^* h(u_2) + b(u_1, u_2, \delta).
\]

Then \(h\) is sub-b-s-preinvex function w.r.t.\(\eta, b\).

\[\square\]

**Proposition 2.11.** Assume that \(K_i\) is a family of is sub-b-s-preinvex sets w.r.t.\(\eta, b\). Then \(\cap_{i \in I} K_i\) is also a sub-b-s-preinvex set w.r.t.\(\eta, b\).

**Proof.** Consider \((u_1, \beta_1), (u_2, \beta_2) \in \cap_{i \in I} K_i\), then \((u_1, \beta_1), (u_2, \beta_2) \in K_i, \forall i \in I\) and since \(K_i\) is a sub-b-s-preinvex sets w.r.t.\(\eta, b\) hence, we can obtain

\[
(u_2 + \delta \eta(u_1, u_2), \beta_1 + (1 - \delta)^* \beta_2 + b(u_1, u_2, \delta)) \in K_i, \forall i \in I.
\]

\[
\Rightarrow (u_2 + \delta \eta(u_1, u_2), \beta_1 + (1 - \delta)^* \beta_2 + b(u_1, u_2, \delta)) \in \cap_{i \in I} K_i.
\]

\[\square\]

According to Theorem 2.10 and Proposition 2.11, the following proposition is got:

**Proposition 2.12.** Let \(h_i\) be sub-b-s-preinvex functions w.r.t.\(\eta, b\), then a function \(H = \sup_{i \in I} h_i\) is also sub-b-s-preinvex function w.r.t.\(\eta, b\).

**Theorem 2.13.** Let \(h : k \rightarrow \mathbb{R}\) be a non-negative differentiable sub-b-s-preinvex function w.r.t.\(\eta, b\). Then
1. \( dh_{u_2}η(u_1, u_2) \leq \delta^{s-1} (h(u_1) + h(u_2)) + \lim_{\delta \rightarrow 0^+} \frac{b(u_1, u_2, δ)}{δ} \),

2. \( dh_{u_2}η(u_1, u_2) \leq \delta^{s-1} (h(u_1) - h(u_2)) + \frac{h(u_2)}{δ} + \lim_{\delta \rightarrow 0^+} \frac{b(u_1, u_2, δ)}{δ} \).

**Proof.** 1. By using the hypothesis, we can write

\[
h (u_2 + δη(u_1, u_2)) = h(u_2) + δ dh_{u_2}η(u_1, u_2) + O(δ),
\]

Also,

\[
h (u_2 + δη(u_1, u_2)) \leq \delta^s h(u_1) + (1 - \delta)^s h(u_2) + b(u_1, u_2, δ).
\]

Furthermore,

\[
h (u_2 + δη(u_1, u_2)) \leq \delta^s h(u_1) + (1 - \delta)^s h(u_2) + b(u_1, u_2, δ) \\
\leq \delta^s h(u_1) + (1 + \delta^s) h(u_2) + b(u_1, u_2, δ).
\]

Then,

\[
h(u_2) + δ dh_{u_2}η(u_1, u_2) + O(δ) \leq \delta^s h(u_1) + (1 + \delta^s) h(u_2) + b(u_1, u_2, δ)
\]

by taking \( \lim_{\delta \rightarrow 0^+} \frac{b(u_1, u_2, δ)}{δ} \) which is the maximum of \( \frac{b(u_1, u_2, δ)}{δ} - \frac{O(δ)}{δ} \), we find the first result.

2. Similarly,

\[
h(u_2) + δ dh_{u_2}η(u_1, u_2) + O(δ) \\
\leq \delta^s h(u_1) + (1 + \delta^s) h(u_2) + b(u_1, u_2, δ) \\
= \delta^s h(u_1) + (1 + \delta^s) h(u_2) - \delta^s h(u_2) + b(u_1, u_2, δ) \\
= \delta^s (h(u_1) - h(u_2)) + b(u_1, u_2, δ) + ((1 - \delta)^s + \delta^s) h(u_2).
\]

However, we know that \((1 - \delta)^s + \delta^s, \forall \delta \in [0,1]\) and \(s \in (0,1]\) and since \( h \) is non-negative function, hence

\[
h(u_2) + δ dh_{u_2}η(u_1, u_2) + O(δ) \leq \delta^s (h(u_1) - h(u_2)) + 2h(u_2) + b(u_1, u_2, δ).
\]

Then, by dividing the last inequality by \( δ \) and taking \( \lim_{\delta \rightarrow 0^+} \), we get the second part of theorem. 

\[\square\]

**Theorem 2.14.** Let \( h : k \rightarrow \mathbb{R} \) be a negative differentiable sub-b-s-preinvex function w.r.t.\( η, b \). Then

\[
dh_{u_2}η(u_1, u_2) \leq \delta^{s-1} (h(u_1) - h(u_2)) + \lim_{\delta \rightarrow 0^+} \frac{b(u_1, u_2, δ)}{δ}.
\]

5
Proof. we get the result by using the hypotheses, since
\[ dh_{u_2} \eta(u_1, u_2) \leq \delta^{s-1} (h(u_1) - h(u_2)) + \frac{b(u_1, u_2, \delta)}{\delta} - \frac{O(\delta)}{\delta} \]
and then by taking \( \lim_{\delta \to 0^+} \frac{b(u_1, u_2, \delta)}{\delta} \) which is the maximum of \( \frac{b(u_1, u_2, \delta)}{\delta} - \frac{O(\delta)}{\delta} \).

\[ \square \]

Corollary 2.15. Assume that \( h : k \to \mathbb{R} \) is a differentiable sub-b-s-preinvex function w.r.t. \( \eta, b \), and

1. \( h \) is a non-negative function, then
\[
d (h_{u_2} - h_{u_1}) \eta(u_1, u_2) \leq \frac{h(u_1) + h(u_2)}{\delta} + \lim_{\delta \to 0^+} \frac{b(u_1, u_2, \delta) + b(u_2, u_1, \delta)}{\delta},
\]

2. \( h \) is a negative function, then
\[
d (h_{u_2} - h_{u_1}) \eta(u_1, u_2) \leq \lim_{\delta \to 0^+} \frac{b(u_1, u_2, \delta) - b(u_2, u_1, \delta)}{\delta}.
\]

Proof. 1. Let \( h \) be non-negative function and by using Theorem 2.13, then
\[
dh_{u_2} \eta(u_1, u_2) \leq \delta^{s-1} (h(u_1) - h(u_2)) + \frac{h(u_2)}{\delta} + \lim_{\delta \to 0^+} \frac{b(u_1, u_2, \delta)}{\delta}.
\]
Also,
\[
dh_{u_1} \eta(u_1, u_2) \leq \delta^{s-1} (h(u_2) - h(u_1)) + \frac{h(u_1)}{\delta} + \lim_{\delta \to 0^+} \frac{b(u_2, u_1, \delta)}{\delta}.
\]
Thus,
\[
d (h_{u_2} - h_{u_1}) \eta(u_1, u_2) \leq \frac{h(u_1) + h(u_2)}{\delta} + \lim_{\delta \to 0^+} \frac{b(u_1, u_2, \delta) + b(u_2, u_1, \delta)}{\delta}.
\]

2. Since \( h \) is negative function and according to Theorem 2.14 the second result can be obtained directly.
3 Application

Next, some application to our results are given:

Let us consider the unconstraint problem \((P)\):

\[
(P) : \min \{ h(u), u \in K \}
\] (3.1)

**Theorem 3.1.** Consider that \(h : k \rightarrow \mathbb{R}\) is a non-negative differentiable sub-b-s-preinvex function w.r.t. \(\eta, b\). If \(u^* \in K\) and

\[
\frac{dh_{u^*}}{\delta} \eta(u, u^*) \geq \frac{h(u^*)}{\delta} + \lim_{\delta \rightarrow 0^+} \frac{b(u, u^*, \delta)}{\delta}, \quad \forall u \in K, \delta \in [0, 1], s \in (0, 1) \tag{3.2}
\]

then \(u^*\) is the optimal solution to \((P)\) respect to \(h\) on \(K\).

**Proof.** By using the hypothesis and the second pair of Theorem 2.13, we obtain

\[
\frac{dh_{u^*}}{\delta} \eta(u, u^*) - \frac{h(u^*)}{\delta} - \lim_{\delta \rightarrow 0^+} \frac{b(u, u^*, \delta)}{\delta} \leq \delta^{s-1} (h(u) - h(u^*)),
\]

\(\forall \delta \in [0, 1], s \in (0, 1)\), and since

\[
\frac{dh_{u^*}}{\delta} \eta(u, u^*) \geq \frac{h(u^*)}{\delta} + \lim_{\delta \rightarrow 0^+} \frac{b(u, u^*, \delta)}{\delta}.
\]

That is \(h(u) - h(u^*) \geq 0\), which means that \(u^*\) is the optimal solution. \(\square\)

**Example 3.2.** Let us take the following function \(h : \mathbb{R}^+ \rightarrow \mathbb{R}\) such that \(h(u) = 2u^s\), where \(s \in (0, 1]\) and let \(b(u_1, u_2, \delta) = \delta u_1^2 + 4\delta u_2^2\) and

\[
\eta(u_1, u_2) = \begin{cases} 
-u_2; & u_1 = u_2, \\
1 - u_2; & u_1 \neq u_2.
\end{cases}
\]

Since \(b(u_1, u_2, \delta) \geq 0, \forall \delta \in (0, 1]\), then it is easy to say that \(h\) is a sub-b-s-preinvex function. Not that only, but also \(h(u)\) is a non-negative differentiable and \(\lim_{\delta \rightarrow 0^+} \frac{b(u_1, u_2, \delta)}{\delta}\) exists for every \(u_1, u_2 \in \mathbb{R}^+\) and \(\delta \in (0, 1]\).

From this information the following unconstraint sub-b-s-preinvex programming can be given

\[
(P) : \min \{ h(u), u \in \mathbb{R}^+ \}
\]

\[
dh_{u^*} \eta(u, u^*) = 2s(u^*)^{s-1} \eta(u, u^*), \quad h(u^*) = \frac{2(u^*)^s}{s} \quad \text{and} \quad \lim_{\delta \rightarrow 0^+} \frac{b(u, u^*, \delta)}{\delta} = u^2 + 4(u^*)^2.
\]

Then we can say that \(u^* = 0\) and

\[
dh_{u^*} \eta(u, u^*) \geq \frac{h(u^*)}{\delta} + \lim_{\delta \rightarrow 0^+} \frac{b(u, u^*, \delta)}{\delta}
\]
holds \( \forall u \in K, \delta \in (0,1], s \in (0,1) \). Hence, then minimum value of \( h(u) \) at zero.

**Corollary 3.3.** Assume that \( h : k \longrightarrow \mathbb{R} \) is a strictly non-negative differentiable sub-b-s-preinvex function w.r.t. \( \eta, b \) and let \( u^* \in K \) which satisfies condition (3.2), then \( u^* \) is the unique optimal solution of \( h \) on \( K \).

**Proof.** Since \( h \) is strictly non-negative differentiable sub-b-s-preinvex function w.r.t. \( \eta, b \) and by using Theorem 2.13, we get

\[
dh_{u^* \eta}(u_1, u_2) < s^{-1} (h(u_1) - h(u_2)) + \frac{h(u_2)}{\delta} + \lim_{\delta \rightarrow 0^+} b(u_1, u_2, \delta).
\]

Let \( v_1, v_2 \in K \) where \( v_1 \neq v_2 \) be optimal solutions og \( (P) \). Then \( h(v_1) = h(v_2) \) and

\[
dh_{v_2 \eta}(v_1, v_2) - \frac{h(v_2)}{\delta} - \lim_{\delta \rightarrow 0^+} b(v_1, v_2, \delta) < s^{-1} (h(v_1) - h(v_2)).
\]

By using (3.2), we find \( s^{-1} (h(v_1) - h(v_2)) > 0 \), but \( h(v_1) = h(v_2) \), then \( v_1 = v_2 = u^* \), the proof is completed. \( \square \)

Now, let nonlinear programming :

\[
(P_\ast) : \min \{h(u) : u \in \mathbb{R}^n, f_i(u) \leq 0, i \in I, \text{where } I = 1, 2, \cdots, m\}
\]

and let \( F_e \) is the feasible set of \( (P_\ast) \) which is given as

\[
F_e = \{u \in \mathbb{R}^n : f_i(u) \leq 0, i \in I\}.
\]

In addition, for \( u^* \in F_e \), we define \( N(u^*) = \{i : f_i(u^*) = 0, i \in I\} \).

**Theorem 3.4.** Assume that \( h : \mathbb{R}^n \longrightarrow \mathbb{R} \) is a non-negative differentiable sub-b-s-preinvex function w.r.t. \( \eta, b \) and \( f_i : \mathbb{R}^n \longrightarrow \mathbb{R} \) are differentiable sub-b-s-preinvex functions w.r.t. \( \eta, b_i, i \in I \). Also, let

\[
dh_{u^* \eta}(u, u^*) + \sum_{i \in I} v_i df_i u^* = 0, \quad u^* \in F_\ast, \quad v_i \geq 0, \quad i \in I. \quad (3.3)
\]

If

\[
\frac{h(u^*)}{\delta} + \lim_{\delta \rightarrow 0^+} b(u, u^*, \delta) \leq - \sum_{i=1} v_i \lim_{\delta \rightarrow 0^+} b(u, u^*, \delta), \quad (3.4)
\]

then \( u^* \) is an optimal solution of \( (P_\ast) \).
Proof. $f_i(u) \leq 0 = f_i(u^*), \forall u \in F_s$, also because each $f_i$ is a sub-b-s-preinvex function and from Theorem 2.14, we get

$$df_{iu^*}\eta(u, u^*) - \lim_{\delta \to 0^+} \frac{b(u, u^*, \delta)}{\delta} \leq \delta^{s-1}(f_i(u) - f_i(u^*)) \leq 0. \quad (3.5)$$

Moreover, we obtain that

$$dh_{u^*}\eta(u, u^*) = - \sum_{i \in I} v_i df_{iu^*}\eta(u, u^*)$$
$$= - \sum_{i \in N(u^*)} v_i df_{iu^*}\eta(u, u^*). \quad (3.6)$$

From (3.4) and (3.6), it results

$$dh_{u^*}\eta(u, u^*) - \frac{h(u^*)}{\delta} - \lim_{\delta \to 0^+} \frac{b(u, u^*, \delta)}{\delta}$$
$$\geq - \sum_{i \in N(u^*)} v_i \left( df_{iu^*}\eta(u, u^*) - \lim_{\delta \to 0^+} \frac{b(u, u^*, \delta)}{\delta} \right). \quad (3.7)$$

Here, we use (3.5) and (3.7) to get

$$dh_{u^*}\eta(u, u^*) \geq \frac{h(u^*)}{\delta} + \lim_{\delta \to 0^+} \frac{b(u, u^*, \delta)}{\delta}$$

and according to Theorem 3.1 which implies that

$$h(u) \geq h(u^*), \forall u \in F_s.$$

Hence, $u^*$ is an optimal solution of $(P_s)$. \hfill $\square$

References


