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Generalized Preinvex Functions and Their Applications

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Abstract : This study focuses on a new class of functions called sub-b-s-preinvex, that is a generalization of sub-b-s-convex and preinvex functions, and discusses some of their properties. A new sub-b-s-preinvex programming is introduced and the sufficient conditions of optimality under this type of function is established.

Keywords: Geodesic E-convex sets;Geodesic E-convex functions; Riemannian manifolds

1 Introduction

Convex functions play an important role in optimation theory, convex analysis, Minkowski space and fractal mathematics[1, 4, 6, 7, 9, 12, 13, 14] and the generalized convex functions is one of the main topics that researchers worked a lot on it. A class of b-vex functions were introduced in[2]. The definitions of two kinds of s-convex functions and some of their properties were given in [8]. Also, a new generalized functions called sub-b-convex was studied in [5]. Sub-b-s-convex functions that extended of the concept of sub-b-convexity were introduced in [10]. Semi-b-preinvex functions were given in [11] as a generalization of the semi preinvex functions. The main aim of this paper is given a new calss of function called sub-b-s-preinvex function, that can be reduced in to sub-b-preinvex when s=1, and studied some of their properties. Furthermore, a new class of sets called sub-b-s-preinvex sets is defined. A new sub-b-s-preinvex programming is introduced and the sufficient conditions of optimality under this type of function is established.

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2 Main results

Let us given some definitions of sub-b-convexity and preinvexity functions before give our main results .

Definition 2.1. *[10] A function $h : K \longrightarrow \mathbb{R}$ is called a sub-b-s-convex function on a non-empty convex set $K \subset \mathbb{R}^n$ w.r.t.$b : K \times K \times [0, 1] \longrightarrow \mathbb{R}$, if*

$$h\left(\delta u_1 + (1-\delta)u_2\right) \leq \delta^s h(u_1) + (1-\delta)^s h(u_2) + b(u_1, u_2, \delta),$$

$\forall u_1, u_2 \in K, \delta \in [0, 1], s \in (0, 1]$.

Ben-Israel and Mond [3] defined a class of functions called preinvex in the non-empty invex set $K \subset \mathbb{R}^n$ $w.r.t.\eta$ such as

Definition 2.2. *A function $h : K \longrightarrow \mathbb{R}$ is preinvex on K w.r.t.η, if there exists an n-dimensional vector function $\eta : K \times K \longrightarrow \mathbb{R}^n$ such that*

$$h\left(u_2 + \delta\eta(u_1, u_2)\right) \leq \delta h(u_1) + (1-\delta)h(u_2),$$

$\forall u_1, u_2 \in K, \delta \in [0, 1]$.

Now, the concepts of sub-b-s-preinvex function and sub-b-s-preinvex set are given. Futhermore, some of their properties are studied.

Definition 2.3. *A function $h : K \longrightarrow \mathbb{R}$ is called a sub-b-s-preinvex function on K w.r.t.η, b, if*

$$h\left(u_2 + \delta\eta(u_1, u_2)\right) \leq \delta^s h(u_1) + (1-\delta)^s h(u_2) + b(u_1, u_2, \delta), \tag{2.1}$$

where $b : K \times K \times [0, 1] \longrightarrow \mathbb{R}, \forall u_1, u_2 \in K\delta \in [0, 1], s \in (0, 1]$.

Remark 2.4.
1. *If $\eta(u_1, u_2) = u_1 - u_2$ in (2.1), then sub-b-s-preinvex w.r.t.η, b becomes sub-b-s-convex function. Also, if we put s=1, then (2.1) becomes sub-b-convex function.*

2. *When $\eta(u_1, u_2) = u_1 - u_2$ and $b(u, u, \delta) \leq 0in$ (2.1), then the sub-b-s-preinvex function becoms convex function.*

Theorem 2.5. *If $h_1, h_2 : K \longrightarrow \mathbb{R}$ are sub-b-s-preinvex functions w.r.t.η,b, then $h_1 + h_2$ and $\beta h_1 (\beta \geqslant 0)$ are also sub-b-s-preinvex functions w.r.t.η,b.*

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Corollary 2.6. *If $h_k : K \longrightarrow \mathbb{R}$, where $k = 1, 2, \cdots, n$ are sub-b-s-preinvex functions w.r.t.η,b_k, then the function which is $H = \sum_{k=1}^{n} a_k h_k$,$a_k \geqslant 0$ $(k = 1, 2, \cdots, n)$ is also sub-b-s-preinvex function w.r.t.η,b where $b = \sum_{k=1}^{n} a_k b_k$.*

Proposition 2.7. *If $h_k : K \longrightarrow \mathbb{R}$, where $k = 1, 2, \cdots, n$ are sub-b-s-preinvex functions w.r.t.η,b_k, then the function which is $H = \max h_k,$, $k = 1, 2, \cdots, n$ is also sub-b-s-preinvex function w.r.t.η,b where $b = \max b_k$.*

Theorem 2.8. *Let $h_1 : K \longrightarrow \mathbb{R}$ be a sub-b-s-preinvex function w.r.t.η,b_1 and $h_2 : \mathbb{R} \longrightarrow \mathbb{R}$ be an increasing function, then $h_1 o h_2$ is a sub-b-s-preinvex functions w.r.t.η, b where $b = h_2 o b_1$, if h_2 satisfies the following conditions*

1. $h_2 (\beta u_1) = \beta h_2(u_1), \forall u_1 \in \mathbb{R}, \beta \geqslant 0.$

2. $h_2 (u_1 + u_2) = h_2(u_1) + h_2(u_2), , \forall u_1, u_2 \in \mathbb{R}, \beta \geqslant 0.$

Proof.

$$\begin{aligned}
(h_2 o h_1)(u_2 + \delta\eta(u_1, u_2)) &= h_2 (h_1 (u_2 + \delta\eta(u_1, u_2))) \\
&\leq h_2 (\delta^s h_1(u_1) + (1 - \delta)^s h_1(u_2) + b_1(u_1, u_2, \delta)) \\
&= \delta^s h_2 (h_1(u_1)) + (1 - \delta)^s h_2 (h_1(u_2)) + h_2 (b_1(u_1, u_2, \delta)) \\
&= \delta^s (h_2 o h_1)(u_1) + (1 - \delta)^s (h_2 o h_1)(u_2) + b(u_1, u_2, \delta)
\end{aligned}$$

which means that $h_2 o h_1$ is sub-b-s-preinvex functions $w.r.t.\eta, b$. □

Now, we give a definition of sub-b-s-preinvex set $w.r.t.\eta, b$.

Definition 2.9. *A set $K \subseteq \mathbb{R}^{n+1}$ is called a sub-b-s-preinvex set $w.r.t.\eta, b$, if*

$$(u_2 + \delta\eta(u_1, u_2), \delta^s \beta_1 + (1 - \delta)^s \beta_2 + b(u_1, u_2, \delta)) \in K,$$

$\forall (u_1, \beta_1), (u_2, \beta_2) \in K$, $u_1, u_2 \in \mathbb{R}^n$, $\delta \in [0, 1]$, $s \in (0, 1]$ and $b : \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \longrightarrow \mathbb{R}$.

The epigraph of the sub-b-s-preinvex function $h : k \longrightarrow \mathbb{R}$ can be given as

$$G(h) = \{(u, \beta) : u \in K, \beta \in \mathbb{R}, h(u) \leq \beta\}.$$

From the above defintion, the following theorem can be stated and proven

Theorem 2.10. *$h : k \longrightarrow \mathbb{R}$ is a sub-b-s-preinvex function $w.r.t.\eta, b$ iff its epigraph is also a sub-b-s-preinvex set $w.r.t.\eta, b$.*

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Proof. Let that $(u_1, \beta_1), (u_2, \beta_2) \in G(h)$, then by using the hypothesis, we find $h(u_1) \leq \beta_1$ and $h(u_2) \leq \beta_2$.

$$\begin{aligned} h\left(u_2 + \delta\eta(u_1, u_2)\right) &\leq \delta^s h(u_1) + (1-\delta)^s h(u_2) + b(u_1, u_2, \delta) \\ &\leq \delta^s \beta_1 + (1-\delta)^s \beta_2 + b(u_1, u_2, \delta). \end{aligned} \tag{2.2}$$

Hence,

$$(u_2 + \delta\eta(u_1, u_2), \beta_1 + (1-\delta)^s \beta_2 + b(u_1, u_2, \delta)) \in G(h).$$

Therefore, $G(h)$ is sub-b-s-preinvex set $w.r.t.\eta, b$.
Now, assume that $G(h)$ is sub-b-s-preinvex set $w.r.t.\eta, b$, then

$$(u_1, h(u_1)), (u_2, h(u_2)) \in G(h),$$

where $u_1, u_2 \in K$.
$(u_2 + \delta\eta(u_1, u_2), \delta^s h(u_1) + (1-\delta)^s h(u_2) + b(u_1, u_2, \delta)) \in G(h)$ which means that

$$h\left(u_2 + \delta\eta(u_1, u_2)\right) \leq \delta^s h(u_1) + (1-\delta)^s h(u_2) + b(u_1, u_2, \delta).$$

Then h is sub-b-s-preinvex function $w.r.t.\eta, b$. □

Proposition 2.11. *Assume that K_i is a family of is sub-b-s-preinvex sets $w.r.t.\eta, b$. Then $\cap_{i \in I} K_i$ is also a sub-b-s-preinvex set $w.r.t.\eta, b$.*

Proof. Consider $(u_1, \beta_1), (u_2, \beta_2) \in \cap_{i \in I} K_i$, then $(u_1, \beta_1), (u_2, \beta_2) \in K_i, \forall i \in I$ and since K_i is a sub-b-s-preinvex sets $w.r.t.\eta, b$ hence, we can obtain

$$(u_2 + \delta\eta(u_1, u_2), \beta_1 + (1-\delta)^s \beta_2 + b(u_1, u_2, \delta)) \in K_i, \forall i \in I.$$

$\Rightarrow$

$$(u_2 + \delta\eta(u_1, u_2), \beta_1 + (1-\delta)^s \beta_2 + b(u_1, u_2, \delta)) \in \cap_{i \in I} K_i.$$

□

According to Theorem2.10 and Proposition2.11, the following proposition is got:

Proposition 2.12. *Let h_i be sub-b-s-preinvex functions $w.r.t.\eta, b$, then a function $H = \sup_{i \in I} h_i$ is also sub-b-s-preinvex function $w.r.t.\eta, b$.*

Theorem 2.13. *Let $h : k \longrightarrow \mathbb{R}$ be a non-negative differentiable sub-b-s-preinvex function $w.r.t.\eta, b$. Then*

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1. $dh_{u_2}\eta(u_1,u_2) \leq \delta^{s-1}\left(h(u_1)+h(u_2)\right)+\lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)}{\delta}$,
2. $dh_{u_2}\eta(u_1,u_2) \leq \delta^{s-1}\left(h(u_1)-h(u_2)\right)+\frac{h(u_2)}{\delta}+\lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)}{\delta}$.

Proof. 1. By using the hypothesis, we can write

$$h\left(u_2+\delta\eta(u_1,u_2)\right)=h(u_2)+\delta dh_{u_2}\eta(u_1,u_2)+O(\delta),$$

Also,

$$h\left(u_2+\delta\eta(u_1,u_2)\right)\leq \delta^s h(u_1)+(1-\delta)^s h(u_2)+b(u_1,u_2,\delta).$$

Furthermore,

$$\begin{aligned} h\left(u_2+\delta\eta(u_1,u_2)\right) &\leq \delta^s h(u_1)+(1-\delta)^s h(u_2)+b(u_1,u_2,\delta)\\ &\leq \delta^s h(u_1)+(1+\delta^s)h(u_2)+b(u_1,u_2,\delta). \end{aligned}$$

Then,

$$h(u_2)+\delta dh_{u_2}\eta(u_1,u_2)+O(\delta)\leq \delta^s h(u_1)+(1+\delta^s)h(u_2)+b(u_1,u_2,\delta)$$

by taking $\lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)}{\delta}$ which is the maximum of $\frac{b(u_1,u_2,\delta)}{\delta}-\frac{O(\delta)}{\delta}$, we find the first result.

2. Similarly,

$$\begin{aligned} &h(u_2)+\delta dh_{u_2}\eta(u_1,u_2)+O(\delta)\\ &\quad\leq \delta^s h(u_1)+(1+\delta^s)h(u_2)+b(u_1,u_2,\delta)\\ &\quad= \delta^s h(u_1)+(1+\delta^s)h(u_2)-\delta^s h(u_2)+\delta^s h(u_2)+b(u_1,u_2,\delta)\\ &\quad= \delta^s\left(h(u_1)-h(u_2)\right)+b(u_1,u_2,\delta)+\left((1-\delta)^s+\delta^s\right)h(u_2). \end{aligned}$$

However, we know that $(1-\delta)^s+\delta^s, \forall\delta\in[0,1]$ and $s\in(0,1]$ and since h is non-negative function, hence

$$h(u_2)+\delta h_{u_2}\eta(u_1,u_2)+O(\delta)\leq \delta^s\left(h(u_1)-h(u_2)\right)+2h(u_2)+b(u_1,u_2,\delta).$$

Then, by dividing the last inequality byδ and taking $\lim_{\delta\longrightarrow 0_+}$, we get the second part of theorem.

□

Theorem 2.14. *Let $h:k\longrightarrow\mathbb{R}$ be a negative differentiable sub-b-s-preinvex function w.r.t.η, b. Then*

$$dh_{u_2}\eta(u_1,u_2)\leq \delta^{s-1}\left(h(u_1)-h(u_2)\right)+\lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)}{\delta}.$$

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Proof. we get the result by using the hypotheses, since

$$dh_{u_2}\eta(u_1,u_2) \leq \delta^{s-1}\left(h(u_1)-h(u_2)\right)+\frac{b(u_1,u_2,\delta)}{\delta}-\frac{O(\delta)}{\delta}$$

and then by taking $\lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)}{\delta}$ which is the maximum of $\frac{b(u_1,u_2,\delta)}{\delta}-\frac{O(\delta)}{\delta}$. □

Corollary 2.15. *Assume that $h : k \longrightarrow \mathbb{R}$ is a differentiable sub-b-s-preinvex function w.r.t.η, b, and*

1. *h is a non-negative function, then*

$$d\left(h_{u_2}-h_{u_1}\right)\eta(u_1,u_2) \leq \frac{h(u_1)+h(u_2)}{\delta}+\lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)+b(u_2,u_1,\delta)}{\delta},$$

2. *h is a negative function, then*

$$d\left(h_{u_2}-h_{u_1}\right)\eta(u_1,u_2) \leq \lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)-b(u_2,u_1,\delta)}{\delta}.$$

Proof. 1. Let h be non-negative function and by using Theorem 2.13, then

$$dh_{u_2}\eta(u_1,u_2) \leq \delta^{s-1}\left(h(u_1)-h(u_2)\right)+\frac{h(u_2)}{\delta}+\lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)}{\delta}.$$

Also,

$$dh_{u_1}\eta(u_1,u_2) \leq \delta^{s-1}\left(h(u_2)-h(u_1)\right)+\frac{h(u_1)}{\delta}+\lim_{\delta\longrightarrow 0_+}\frac{b(u_2,u_1,\delta)}{\delta}.$$

Thus,

$$d\left(h_{u_2}-h_{u_1}\right)\eta(u_1,u_2) \leq \frac{h(u_1)+h(u_2)}{\delta}+\lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)+b(u_2,u_1,\delta)}{\delta}.$$

2. Since h is negative function and according to Theorem 2.14, the second result can be obtained directly.

□

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3 Application

Next, some application to our results are given:
Let us consider the unconstraint proplem (P)

$$(P): \min \{h(u), u \in K\} \tag{3.1}$$

Theorem 3.1. *Consider that $h : k \longrightarrow \mathbb{R}$ is a non-negative differentiable sub-b-s-preinvex function w.r.t.η, b. If $u^* \in K$ and*

$$dh_{u^*}\eta(u,u^*) \geqslant \frac{h(u^*)}{\delta} + \lim_{\delta \longrightarrow 0_+} \frac{b(u,u^*,\delta)}{\delta}, \ \forall u \in K, \delta \in [0,1], s \in (0,1] \tag{3.2}$$

then u^* is the optimal solution to (P) respect to h on K.

Proof. By using the hypothesis and the second pair of Theorem 2.13, we obtain

$$dh_{u^*}\eta(u,u^*) - \frac{h(u^*)}{\delta} - \lim_{\delta \longrightarrow 0_+} \frac{b(u,u^*,\delta)}{\delta} \leq \delta^{s-1}\left(h(u) - h(u^*)\right),$$

$\forall \delta \in [0,1], s \in (0,1]$, and since

$$dh_{u^*}\eta(u,u^*) \geqslant \frac{h(u^*)}{\delta} + \lim_{\delta \longrightarrow 0_+} \frac{b(u,u^*,\delta)}{\delta}.$$

That is $h(u) - h(u^*) \geqslant 0$, which means that u^* is the optimal solution. □

Example 3.2. *Let us take the following function $h : \mathbb{R}^+ \longrightarrow \mathbb{R}$ such that $h(u) = 2u^s$, where $s \in (0,1]$ and let $b(u_1,u_2,\delta) = \delta u_1^2 + 4\delta u_2^2$ and*

$$\eta(u_1,u_2) \quad = \quad \begin{cases} -u_2; u_1 = u_2, \\ 1 - u_2; u_1 \neq_2 . \end{cases}$$

Since $b(u_1,u_2,\delta) \geqslant 0, \forall \delta \in (0,1]$, then it is easy to say that h is a sub-b-s-preinvex function. Not that only, but also $h(u)$ is a non-negative differentiable and $\lim_{\delta \longrightarrow 0_+} \frac{b(u_1,u_2,\delta)}{\delta}$ exists for every $u_1, u_2 \in \mathbb{R}^+$ and $\delta \in (0,1]$. From this information the following unconstraint sub-b-s-preinvex programming can be given

$$(P): \min \left\{h(u), u \in \mathbb{R}^+\right\}$$

$dh_{u^*}\eta(u,u^*) = 2s(u^*)^{s-1}\eta(u,u^*)$, $\frac{h(u^*)}{\delta} = \frac{2(u^*)^s}{\delta}$ *and* $\lim_{\delta \longrightarrow 0_+} \frac{b(u,u^*,\delta)}{\delta} = u^2 + 4(u^*)^2$.
Then we can say that $u^* = 0$ and

$$dh_{u^*}\eta(u,u^*) \geqslant \frac{h(u^*)}{\delta} + \lim_{\delta \longrightarrow 0_+} \frac{b(u,u^*,\delta)}{\delta}$$

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holds $\forall u \in K, \delta \in (0,1], s \in (0,1)$. *Hence, then minimum value of* $h(u)$ *at zero.*

Corollary 3.3. *Assume that* $h : k \longrightarrow \mathbb{R}$ *is a strictly non-negative differentiable sub-b-s-preinvex functionw.r.t.*η, b *and let* $u^* \in K$ *which satisfies condition(3.2), then* u^* *is the unique optimal solution of* h *on* K.

Proof. Since h is strictly non-negative differentiable sub-b-s-preinvex function $w.r.t.\eta, b$ and by using Theorem2.13, we get

$$dh_{u_2}\eta(u_1,u_2) < \delta^{s-1}\left(h(u_1)-h(u_2)\right) + \frac{h(u_2)}{\delta} + \lim_{\delta\longrightarrow 0_+}\frac{b(u_1,u_2,\delta)}{\delta}.$$

Let $v_1, v_2 \in K$ where $v_1 \neq v_2$ be optimal solutions og (P). Then $h(v_1) = h(v_2)$ and

$$dh_{v_2}\eta(v_1,v_2) - \frac{h(v_2)}{\delta} - \lim_{\delta\longrightarrow 0_+}\frac{b(v_1,v_2,\delta)}{\delta} < \delta^{s-1}\left(h(v_1)-h(v_2)\right).$$

By using (3.2), we find $\delta^{s-1}\left(h(v_1)-h(v_2)\right) > 0$, but $h(v_1) = h(v_2)$, then $v_1 = v_2 = u^*$, the proof is completed. □

Now, let nonlinear programming :

$$(P_*) : \min\left\{h(u) : u \in \mathbb{R}^n, f_i(u) \leq 0, i \in I, where I = 1,2,\cdots,m\right\}$$

and let F_e is the feasible set of (P_*) which is given as

$$F_e = \left\{u \in \mathbb{R}^n : f_i(u) \leq 0, i \in I\right\}.$$

In addition, for $u^* \in F_e$, we define $N(u^*) = \{i : f_i(u^*) = 0, i \in I\}$.

Theorem 3.4. *Assume that* $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a non-negative differentiable sub-b-s-preinvex function w.r.t.*η, b *and* $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ *are differentiable sub-b-s-preinvex functions w.r.t.*$\eta, b_i, i \in I$. *Also, let*

$$dh_{u^*}\eta(u,u^*) + \sum_{i\in I} v_i df_{iu^*}\eta(u,u^*) = 0,\ u^* \in F_s,\ v_i \geqslant 0,\ i \in I. \qquad (3.3)$$

If

$$\frac{h(u^*)}{\delta} + \lim_{\delta\longrightarrow 0_+}\frac{b(u,u^*,\delta)}{\delta} \leq -\sum_{i=1} v_i lim_{\delta\longrightarrow 0_+}\frac{b(u,u^*,\delta)}{\delta}, \qquad (3.4)$$

then u^* *is an optimal solution of* (P_*).

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Proof. $f_i(u) \leq 0 = f_i(u^*), \forall u \in F_s$, also because each f_i is a sub-b-s-preinvex function and from Theorem 2.14, we get

$$df_{iu^*}\eta(u,u^*) - \lim_{\delta \longrightarrow 0_+} \frac{b(u,u^*,\delta)}{\delta} \leq \delta^{s-1}\left(f_i(u) - f_i(u^*)\right) \leq 0. \tag{3.5}$$

Moreover, we obtain that

$$\begin{aligned} dh_{u^*}\eta(u,u^*) &= -\sum_{i\in I} v_i df_{iu^*}\eta(u,u^*) \\ &= -\sum_{i\in N(u^*)} v_i df_{iu^*}\eta(u,u^*). \end{aligned} \tag{3.6}$$

From (3.4) and (3.6), it results

$$\begin{aligned} & dh_{u^*}\eta(u,u^*) - \frac{h(u^*)}{\delta} - \lim_{\delta \longrightarrow 0_+} \frac{b(u,u^*,\delta)}{\delta} \\ & \qquad \geqslant -\sum_{i\in N(u^*)} v_i \left(df_{iu^*}\eta(u,u^*) - \lim_{\delta \longrightarrow 0_+} \frac{b(u,u^*,\delta)}{\delta} \right). \end{aligned} \tag{3.7}$$

Here, we use (3.5) and (3.7) to get

$$dh_{u^*}\eta(u,u^*) \geqslant \frac{h(u^*)}{\delta} + \lim_{\delta \longrightarrow 0_+} \frac{b(u,u^*,\delta)}{\delta}$$

and according to Theorem 3.1, which implies that

$$h(u) \geqslant h(u^*), \forall u \in F_s.$$

Hence, u^* is an optimal solution of (P_*). □

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