Mean-field type games between two players driven by backward stochastic differential equations *

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Abstract: In this paper, mean-field type games between two players with backward stochastic dynamics are defined and studied. They make up a class of non-zero-sum differential games where the players' state dynamics solve backward stochastic differential equations (BSDEs) that depend on the marginal distributions of player states. Players try to minimize their individual cost functionals, also depending on the marginal state distributions. Under some regularity conditions, we derive necessary and sufficient conditions for existence of Nash equilibria. Player behavior is illustrated by numerical examples, and is compared to a centrally planned solution where the social cost, the sum of player costs, is minimized. The inefficiency of a Nash equilibrium, compared to socially optimal behavior, is quantified by the so-called price of anarchy. Numerical simulations of the price of anarchy indicate how the improvement in social cost achievable by a central planner depends on problem parameters.

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1. Introduction

A mean-field type game (MFTG) is a game in which payoffs and dynamics depend not only on the state and control profiles of the players, but also on the distribution of the state-control processes. MFTG has a plethora of applications in the engineering sciences, see [15] and the references therein.

The theory of MFTG and mean-field type control (MFTC), initiated in [1], is well developed for forward stochastic dynamics, i.e. given initial conditions [5, 8, 12]. In the deterministic case, initial and terminal conditions are equivalent. However, in the stochastic case they are not, and there are applications where stochastic dynamics with terminal conditions are of interest; in [3], we propose a model for pedestrians groups moving towards *targets they must reach*, such as deliveries and emergency personnel. The hard terminal condition leads to the formulation of a dynamic model for crowd motion where the state dynamics satisfies a BSDE. A game between such groups is of interest since it could be a tool for decentralized decision making

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under conflicting interests. Mean-field effects appear in pedestrian crowd models as approximations of aggregate human interaction, so the game would in fact be a MFTG [2]. Other areas of application include strategies for financial investments, where often future conditions are specified [14, 17, 18, 19, 21] and lead to dynamic models using BSDEs. On financial markets, price is determined by the aggregates (mean-field effects) such as supply and demand, and players on that market naturally compete.

BSDEs were introduced in [6] and the general theory is by now fully developed [28, 31]. Optimal control of BSDEs with coefficients involving the marginal state distribution, mean-field BSDEs, has recently gained attention. A key tool in optimal control of BSDEs (and SDEs) is the analysis of forward-backward systems of stochastic differential equations, arising both in dynamic programming and from the Pontryagin type stochastic maximum principle. Forward-backward SDEs are thoroughly treated in [30, 31], and the mean-field version in [12]. Mean-field BSDEs were derived as limits of particle systems in [7]. Existence and uniqueness results for mean-field BSDEs, as well as a comparison theorem, are provided in [9]. In [25] the linear-quadratic BSDE control problem with deterministic coefficients is studied. Recent work on the control of BSDEs includes [24, 29].

The natural next step is to extend control of mean-field BSDEs to games where the state processes are mean-field BSDEs. In this paper we analyze a game between two players following mean-field BSDE dynamics. This is in fact a MFTG, since the distribution of each player is effected by both players' choice of strategy. We also look at the cooperative situation, which is a control problem, where a central planner optimizes the social cost. The social cost is the the sum of player costs. The fraction between the worst case social cost in the game and the optimal social cost quantifies the efficiency of game equilibria and was first studied in [22] for traffic coordination on networks under the name *coordination ratio*. Later, [26] coined the term *price of anarchy*.

Following the path laid-out in [1], we derive necessary and sufficient conditions for equilibria and social optima, establishing a Pontryagin type maximum principle under Lipschitz and differentiability assumptions on the involved cost and dynamic coefficient functions. As a consequence, we have existence of a Nash equilibrium and a verification theorem. We solve a linear-quadratic (LQ) case explicitly up to a system of ordinary differential equations (ODEs) and provide numerical examples that pinpoint differences in player behavior between the game and the centrally planned case. We study the price of anarchy, the price the players as a society have to pay for having independent choice, by simulation in the LQ case. By varying parameters, we observe how a central planner's improvement depends on the players' preferences.

The paper is organized as follows. The MFTG is defined in Section 2. Section 3 and 4 deal with necessary and sufficient conditions for Nash equilibria and social optima; maximum principles for the MFTG and the MFTC are derived. An LQ problem is solved explicitly in Section 5, and numerical results are presented. The paper concludes with some remarks on possible extensions in Section 6, followed by an appendix containing proofs.

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2. Problem formulation

Let $T > 0$ be a finite real number representing the time horizon of the game. Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ on which two independent standard Brownian motions $W^1_\cdot, W^2_\cdot$ are defined, d_1- and d_2-dimensional respectively. Additionally, $y^1_T, y^2_T \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^d)$ and ξ, $\mathcal{F}_0$-measurable, are defined on the space. We assume that these five random objects are independent and that they generate the filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$. Notice that ξ makes $\mathcal{F}_0$ non-trivial. Let $\mathcal{G}$ be the σ-algebra on $[0,T]\times\Omega$ of $\mathcal{F}_t$-progressively measurable sets. For $k \geq 1$, let $\mathbb{S}^{2,k}$ be the set of $\mathbb{R}^k$-valued and continuous $\mathcal{G}$-measurable processes $X_\cdot := \{X_t : t \in [0,T]\}$ such that $\mathbb{E}[\sup_{t\in[0,T]} |X_t|^2] < \infty$, and let $\mathbb{H}^{2,k}$ be the set of $\mathbb{R}^k$-valued $\mathcal{G}$-measurable processes $X_\cdot$ such that $\mathbb{E}[\int_0^T |X_s|^2 ds] < \infty$.

Let (U^i, d_{U^i}) be a separable metric space, $i = 1, 2$. Player i picks her control $u^i_\cdot$ from the set

$$\mathcal{U}^i := \left\{ u : [0,T]\times\Omega \to U^i \mid u_\cdot \text{ is } \mathbb{F}\text{-adapted, } \mathbb{E}\left[\int_0^T d_{U^i}(u_s)^2 ds\right] < \infty \right\}. \tag{2.1}$$

The distribution of any random variable $\chi \in \mathcal{X}$ will be denoted by $\mathcal{L}(\xi) \in \mathcal{P}(\mathcal{X})$, and $-i$ will denote the index $\{1,2\}\backslash i$. Given a pair of controls $(u^1_\cdot, u^2_\cdot) \in \mathcal{U}^1 \times \mathcal{U}^2$, consider the system of controlled BSDEs

$$dY^i_t = b^i(t, \Theta^i_t, \Theta^{-i}_t, Z_t)dt + Z^{i,1}_t dW^1_t + Z^{i,2}_t dW^2_t, \quad Y^i_T = y^i_T, \; i = 1, 2, \tag{2.2}$$

where $\Theta^i_t = (Y^i_t, \mathcal{L}(Y^i_t), u^i_t)$ and $Z_t = [Z^{1,1}_t Z^{1,2}_t Z^{2,1}_t Z^{2,2}_t]$. Furthermore,

$$b^i : \Omega \times [0,T] \times S \times U^i \times S \times U^{-i} \times \mathbb{R}^{d\times(2d_1+2d_2)} \to \mathbb{R}^d, \tag{2.3}$$

where $S := \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)$ is equipped with the norm $\|(y,\mu)\|_S := |y| + d_2(\mu)$, d_2 being the 2-Wasserstein metric on $\mathcal{P}(\mathbb{R}^d)$. $\mathbb{R}^{d\times(2d_1+2d_2)}$ is equipped with the trace norm $\|Z\|_F = \mathrm{tr}(ZZ^*)^{1/2}$. Note that if X is a square integrable random variable in $\mathbb{R}^d$, then $d_2(\mathcal{L}(X)) < \infty$ and $\mathcal{L}(X) \in \mathcal{P}_2(\mathbb{R}^d)$, the space of measures with finite d_2-norm.

Given $(u^1_\cdot, u^2_\cdot) \in \mathcal{U}^1\times\mathcal{U}^2$, a pair of $\mathbb{R}^d\times\mathbb{R}^{d\times(d_1+d_2)}$-valued $\mathcal{G}$-measurable processes $(Y^i_\cdot, [Z^{i,1}_\cdot Z^{i,2}_\cdot])$, $i = 1, 2$, is a solution to (2.2) if

$$Y^i_t = y^i_T - \int_t^T b^i(s, \Theta^i_s, \Theta^{-i}_s, Z_s)ds - \sum_{j=1}^2 \int_t^T Z^{i,j}_s dW^j_s, \quad \forall t \in [0,T], \; a.s., \tag{2.4}$$

and $(Y^i_\cdot, [Z^{i,1}_\cdot Z^{i,2}_\cdot]) \in \mathbb{S}^{2,d} \times \mathbb{H}^{2,d\times(d_1+d_2)}$.

Remark 2.1. Any terminal condition $y^i_T \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^d)$ naturally induces a $\mathbb{F}$-martingale $Y^i_t := \mathbb{E}[y^i_T \mid \mathcal{F}_t]$. The martingale representation theorem then gives existence of a unique process $[Z^{i,1}_\cdot, Z^{i,2}_\cdot] \in \mathbb{H}^{2,d\times(d_1+d_2)}$ such that $Y^i_t = y^i_T + Z^{i,1}_t dW^1_t + Z^{i,2}_t dW^2_t$, i.e. $[Z^{i,1}_\cdot, Z^{i,2}_\cdot]$ plays the role of the projection and without it, $Y^i_\cdot$ would not be $\mathcal{G}$-measurable. Hence the noise $(W^1_\cdot, W^2_\cdot)$ generating the filtration is *common to both players*, and $[Z^{i,1}_\cdot, Z^{i,2}_\cdot]$, $i = 1, 2$ is their respective reaction to it. Player i may actually be effected by all the noise in the filtration even if only some components of $(W^1_\cdot, W^2_\cdot)$ appear in b^i. An interpretation of $[Z^{i,1}_\cdot, Z^{i,2}_\cdot]$ is that

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it is a *second control* of player i: first she plays $u^i_\cdot$ to heed preferences on energy use, initial position etc., then she picks $[Z^{i,1}_\cdot, Z^{i,2}_\cdot]$ so that her path to y^i_T is the optimal prediction based on available information in the filtration at any given time. The component b^i in (2.2) acts as a velocity in.

Existence and uniqueness of (2.2) is given by a slight variation of [9, Theorem 3.1], where the one-dimensional case is treated. For the d-dimensional mean-field free case, see [27].

Assumption 1. *The process $b^i(\omega,\cdot,0,\dots,0)$, $i=1,2$, belongs to $\mathbb{H}^{2,d}$ and for any $v^i = (y^i,\mu^i,u^i,y^{-i},\mu^{-i},u^{-i},z) \in S\times U^1\times S\times U^2\times\mathbb{R}^{d\times(2d_1+2d_2)}$, $b^i(\omega,\cdot,v^i)$, $i=1,2$, is $\mathcal{G}$-measurable.*

Assumption 2. *Given a pair of control values $(u^1,u^2)\in U^1\times U^2$, there exists a constant $L>0$ such that for all $t\in[0,T]$ and tuples $(y^1,\mu^1,y^2,\mu^2,z),(\bar{y}^1,\bar{\mu}^1,\bar{y}^2,\bar{\mu}^2,\bar{z})\in S\times S\times\mathbb{R}^{d(2d_1+2d_2)}$*

$$
\begin{aligned}
&|b^i(t,y^i,\mu^i,u^i,y^{-i},\mu^{-i},u^{-i},z) - b^i(t,\bar{y}^i,\bar{\mu}^i,u^i,\bar{y}^{-i},\bar{\mu}^{-i},u^{-i},\bar{z})| \\
&\quad\le L\left(\sum_{j=1}^{2}\|(y^j,\mu^j)-(\bar{y}^j,\bar{\mu}^j)\|_S + \|z-\bar{z}\|_F\right), \quad \mathbb{P}\text{-a.s.},\ i=1,2.
\end{aligned}
\tag{2.5}
$$

Theorem 2.1. *Let Assumptions 1 and 2 hold. Then, for any terminal conditions $y^1_T, y^2_T \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^d)$ and $(u^1_\cdot,u^2_\cdot)\in\mathcal{U}^1\times\mathcal{U}^2$, the system of mean-field BSDEs (2.2) has a unique solution $(Y^i_\cdot,[Z^{i,1}_\cdot,Z^{i,2}_\cdot])\in\mathbb{S}^{2,d}\times\mathbb{H}^{2,d\times(d_1+d_2)}$, $i=1,2$.*

Next, we introduce the best reply of player i as follows:

$$
J^i(u^i_\cdot;u^{-i}_\cdot) := \mathbb{E}\left[\int_0^T f^i(t,\Theta^i_t,\Theta^{-i}_t)dt + h^i(Y^i_0,\mathcal{L}(Y^i_0),Y^{-i}_0,\mathcal{L}(Y^{-i}_0))\right] \tag{2.6}
$$

for given maps $f^i:[0,T]\times S\times U^i\times S\times U^{-i}\to\mathbb{R}$ and $h^i:\Omega\times S\times S\to\mathbb{R}$.

Assumption 3. *For any pair of controls $(u^1_\cdot,u^2_\cdot)\in\mathcal{U}^1\times\mathcal{U}^2$, $f^i(\cdot,\Theta^i_\cdot,\Theta^{-i}_\cdot)\in L^1_{\mathcal{F}}(0,T;\mathbb{R})$ and $h(Y^i_0,\mathcal{L}(Y^i_0),Y^{-i}_0,\mathcal{L}(Y^{-i}_0))\in L^1_{\mathcal{F}_0}(\Omega;\mathbb{R})$.*

The problems we consider next are

1. The Mean-field Type Game (MFTG): find the Nash equilibrium controls of

$$
\begin{cases}
\inf\limits_{u^i_\cdot\in\mathcal{U}^i} & J^i(u^i_\cdot;u^{-i}_\cdot), \quad i=1,2, \\
\text{s.t.} & dY^i_t = b^i(t,\Theta^i_t,\Theta^{-i}_t,Z_t)dt + Z^{i,1}dW^1_t + Z^{i,2}_t dW^2_t,\ Y^i_T = y^i_T.
\end{cases}
\tag{2.7}
$$

2. The Mean-field Type Control Problem (MFTC): find the optimal control pair of

$$
\begin{cases}
\inf\limits_{(u^1_\cdot,u^2_\cdot)\in\mathcal{U}^1\times\mathcal{U}^2} & J(u^1_\cdot,u^2_\cdot) := J^1(u^1_\cdot;u^2_\cdot)+J^2(u^2_\cdot;u^1_\cdot), \\
\text{s.t.} & dY^i_t = b^i(t,\Theta^i_t,\Theta^{-i}_t,Z_t)dt + Z^{i,1}dW^1_t + Z^{i,2}_t dW^2_t,\ Y^i_T = y^i_T, \\
& i=1,2.
\end{cases}
\tag{2.8}
$$

In the game each player assumes that the other player acts rationally, i.e. minimizes cost, and picks her control as the best response to that. This leads to a set of two inequalities,

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characterizing control pairs $(u^1_\cdot, u^2_\cdot)$ that constitute Nash equilibria. In this paper, each player is aware of the other player's control set, best response function and state dynamics. Therefore, even though the decision process is decentralized, both players solve the same set of inequalities. When there are multiple Nash equilibria, there is an ambiguity around which one to play if the players do not communicate. In the control problem, a central planner decides what strategies are played by both players. The central planner might just be the two players cooperating towards a common goal, or some superior decision maker. A control pair that minimizes the *social cost* J is chosen. This cost can be though of as the total cost for a society made up by the two players in the game. Logically, we expect the optimal social cost to be lower than the social cost in a Nash equilibrium. The ratio between the worst case social cost in the game and the optimal social cost is called the price of anarchy, and we will highlight it in the numerical simulations in Section 5 where we also observe behavioral differences between MFTG and MFTC given identical data.

3. Problem 1: MFTG

This section is the derivation of necessary and sufficient equilibrium conditions of (2.7). Given the existence of such a pair of controls, we derive the conditions by the means of a Pontryagin type stochastic maximum principle.

Assume that $(\hat{u}^1_\cdot, \hat{u}^2_\cdot)$ is a Nash equilibrium for the MFTG, i.e. satisfies the following system of inequalities,

$$\begin{cases} J^1(\hat{u}^1_\cdot; \hat{u}^2_\cdot) \leq J^1(u^1_\cdot; \hat{u}^2_\cdot), \quad u^1_\cdot \in \mathcal{U}^1, \\ J^2(\hat{u}^2_\cdot; \hat{u}^1_\cdot) \leq J^2(u^2_\cdot; \hat{u}^2_\cdot), \quad u^2_\cdot \in \mathcal{U}^2. \end{cases} \tag{3.1}$$

Consider the first inequality, with $\bar{u}^{\varepsilon,1}$ chosen as a spike-perturbation of $\hat{u}^1$. That is, for $u_\cdot \in \mathcal{U}^1$,

$$\bar{u}^{\varepsilon,1}_t := \begin{cases} \hat{u}^1_t, & t \in [0,T] \backslash E_\varepsilon, \\ u_t, & t \in E_\varepsilon. \end{cases} \tag{3.2}$$

Here, E_ε is any subset of $[0,T]$ of Lebesgue measure ε. Clearly, $\bar{u}^{\varepsilon,1}_\cdot \in \mathcal{U}^1$. When player 1 plays the spike-perturbed control $\bar{u}^{\varepsilon,1}_\cdot$ and player 2 plays the equilibrium control $\hat{u}^2_\cdot$, we denote the dynamics by

$$\begin{cases} d\bar{Y}^{\varepsilon,1}_t = b^1(t, \bar{\Theta}^{\varepsilon,1}_t, \bar{Y}^{\varepsilon,2}_t, \mathcal{L}(\bar{Y}^{\varepsilon,2}_t), \hat{u}^2_t, \bar{Z}^\varepsilon_t)dt + \bar{Z}^{\varepsilon,1,1}_t dW^1_t + \bar{Z}^{\varepsilon,1,2}_t dW^2_t, \; \bar{Y}^1_T = y^1, \\ d\bar{Y}^{\varepsilon,2}_t = b^2(t, \bar{Y}^{\varepsilon,2}_t, \mathcal{L}(\bar{Y}^{\varepsilon,2}_t), \hat{u}^2_t, \bar{\Theta}^{\varepsilon,1}_t, \bar{Z}^\varepsilon_t)dt + \bar{Z}^{\varepsilon,2,1}_t dW^1_t + \bar{Z}^{\varepsilon,2,2}_t dW^2_t, \; \bar{Y}^2_T = y^2. \end{cases} \tag{3.3}$$

The performance of the perturbed dynamics (3.3) will be compared with that of the equilibrium dynamics

$$\begin{cases} d\hat{Y}^1_t = b^1(t, \hat{\Theta}^1_t, \hat{\Theta}^2_t, \hat{Z}_t)dt + \hat{Z}^{1,1}_t dW^1_t + \hat{Z}^{1,2}_t dW^2_t, \; \hat{Y}^1_T = y^1, \\ d\hat{Y}^2_t = b^2(t, \hat{\Theta}^2_t, \hat{\Theta}^1_t, \hat{Z}_t)dt + \hat{Z}^{2,1}_t dW^1_t + \hat{Z}^{2,2}_t dW^2_t, \; \hat{Y}^2_T = y^2. \end{cases} \tag{3.4}$$

For simplicity, we write for $\varphi \in \{b^1, f^1, h^1\}$, $\psi \in \{b^2, f^2, h^2\}$, $\vartheta \in \{b^i, f^i, h^i, i = 1, 2\}$,

$$\begin{aligned}\bar{\varphi}^{\varepsilon}_t &:= \varphi(t, \bar{\Theta}^{\varepsilon,1}_t, \bar{Y}^{\varepsilon,2}_t, \mathcal{L}(\bar{Y}^{\varepsilon,2}_t), \hat{u}^2_t, \bar{Z}^{\varepsilon}_t),\\ \bar{\psi}^{\varepsilon}_t &:= \psi(t, \bar{Y}^{\varepsilon,2}_t, \mathcal{L}(\bar{Y}^{\varepsilon,2}_t), \hat{u}^2_t, \bar{\Theta}^{\varepsilon,1}_t, \bar{Z}^{\varepsilon}_t),\\ \hat{\vartheta}_t &:= \vartheta(t, \hat{\Theta}^i_t, \hat{\Theta}^{-i}_t, \hat{Z}_t).\end{aligned} \tag{3.5}$$

In this shorthand notation, which will be used from now on, the difference in performance is

$$J^1(\bar{u}^{\varepsilon,1}_\cdot; \hat{u}^2_\cdot) - J^1(\hat{u}^1_\cdot; \hat{u}^2_\cdot) = \mathbb{E}\left[\int_0^T \bar{f}^{\varepsilon,1}_t - \hat{f}^1_t dt + \bar{h}^{\varepsilon,1}_0 - \hat{h}^1_0\right]. \tag{3.6}$$

Any derivative of $f : a \to f(a)$ will be denoted $\partial_a f$, indifferent of the space the function is mapping from/to.

Assumption 4. *The functions*

$$\begin{aligned}(y^1, \mu^1, u^1, y^2, \mu^2, u^2, z) &\mapsto b^i(t, y^i, \mu^i, u^i, y^{-i}, \mu^{-i}, u^{-i}, z)\\ (y^1, \mu^1, u^1, y^2, \mu^2, u^2) &\mapsto f^i(t, y^i, \mu^i, u^i, y^{-i}, \mu^{-i}, u^{-i})\\ (y^1, \mu^1, y^2, \mu^2) &\mapsto h^i(y^i, \mu^i, y^{-i}, \mu^{-i})\end{aligned} \tag{3.7}$$

are for all t a.s. differentiable at $(\hat{\Theta}^1_t, \hat{\Theta}^2_t, \hat{Z}_t)$, $(\hat{\Theta}^1_t, \hat{\Theta}^2_t)$ and $(\hat{Y}^1_0, \mathcal{L}(\hat{Y}^1_0), \hat{Y}^2_0, \mathcal{L}(\hat{Y}^2_0))$ respectively. Furthermore,

$$\partial_{y^j} \hat{b}^i_t,\ \partial_{\mu^j} \hat{b}^i_t,\ \partial_{y^j} \hat{f}^i_t,\ \partial_{\mu^j} \hat{f}^i_t, \quad i, j = 1, 2, \tag{3.8}$$

are for all t a.s. uniformly bounded, and

$$\partial_{y^j} \hat{h}^i_0 + \mathbb{E}\left[{}^*(\partial^j_\mu \hat{h}^i_0)\right] \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^d). \tag{3.9}$$

For $i = 1, 2$,

$$\begin{aligned}\bar{h}^{\varepsilon,i}_0 - \hat{h}^i_0 = &\sum_{j=1}^2 \left\{\partial_{y^j} \hat{h}^i_0(\bar{Y}^{\varepsilon,j}_0 - \hat{Y}^j_0) + \mathbb{E}\left[(\partial_{\mu^j} \hat{h}^i_0)^*(\bar{Y}^{\varepsilon,j}_0 - \hat{Y}^j_0)\right]\right\}\\ &+ \sum_{j=1}^2 \left\{o\left(|\bar{Y}^{\varepsilon,j}_0 - \hat{Y}^j_0|\right) + o\left(\mathbb{E}[|\bar{Y}^{\varepsilon,j}_0 - \hat{Y}^j_0|^2]^{1/2}\right)\right\}.\end{aligned} \tag{3.10}$$

A brief overview on differentiation of $\mathcal{P}_2(\mathbb{R}^d)$-valued functions is found in Appendix A, and the notation $(\partial_{\mu^j} \hat{h}^i_0)^*$ is defined in (A.8). Both $\bar{Y}^{\varepsilon,1}_t - \hat{Y}^1_t$ and $\bar{Y}^{\varepsilon,2}_t - \hat{Y}^2_t$ appear in (3.10), this suggests that we need to introduce two first order variation processes. That is, we want $(\widetilde{Y}^i_\cdot, [\widetilde{Z}^{i,1}_\cdot, \widetilde{Z}^{i,2}_\cdot])$, $i = 1, 2$, that for some $C > 0$ satisfies

$$\begin{aligned}&\sup_{0\le t\le T} \mathbb{E}\left[|\widetilde{Y}^i_t|^2 + \sum_{j=1}^2 \int_0^t \|\widetilde{Z}^{i,j}_s\|^2_F\, ds\right] \le C\varepsilon^2,\\ &\sup_{0\le t\le T} \mathbb{E}\left[|\bar{Y}^{\varepsilon,i}_t - \hat{Y}^i_t - \widetilde{Y}^i_t|^2 + \sum_{j=1}^2 \int_0^t \|\bar{Z}^{\varepsilon,i,j}_s - \hat{Z}^{i,j}_s - \widetilde{Z}^{i,j}_s\|^2_F\, ds\right] \le C\varepsilon^2.\end{aligned} \tag{3.11}$$

Let δ_i denote variation in $u^i_\cdot$ so that for $\vartheta \in \{f^i, b^i, i = 1, 2\}$,

$$\delta_i \vartheta(t) := \vartheta(t, \hat{Y}^i_t, \mathcal{L}(\hat{Y}^i)_t, \bar{u}^{\varepsilon,i}_t, \hat{\Theta}^{-i}_t, \hat{Z}_t) - \hat{\vartheta}_t. \tag{3.12}$$

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Assumption 5. *For $y^i, \mu^i \in S$, $i = 1, 2$, $z \in \mathbb{R}^{d\times(2d_1+2d_2)}$ and $(u^1, u^2), (v^1, v^2) \in U^1 \times U^2$, there exists a constant $L > 0$ such that*

$$|b^i(t, y^i, \mu^i, u^i, y^{-i}, \mu^{-i}, u^{-i}, z) - b^i(t, y^i, \mu^i, v^i, y^{-i}, \mu^{-i}, v^{-i}, z)| \le L\sum_{j=1}^{2} d_{U^j}(u^j, v^j), \tag{3.13}$$

a.s. for all $t \in [0, T]$.

Lemma 3.1. *Let assumption 1, 2, 4 and 5 be in force. Then the first order variation processes that satisfy* (3.11) *are given by the following system of BSDEs,*

$$\begin{cases} d\widetilde{Y}^i_t = \left(\sum_{j=1}^{2}\left\{\partial_{y^j}\hat{b}^i_t\widetilde{Y}^j_t + \mathbb{E}\left[(\partial_{\mu^j}\hat{b}^i_t)^*\widetilde{Y}^j_t\right]\right\} + \sum_{j,k=1}^{2}\partial_{z^{j,k}}\hat{b}^i_t\widetilde{Z}^{j,k}_t + \delta_1 b^i(t)\mathbb{I}_{E_\varepsilon}(t)\right)dt \\ \qquad + \sum_{j=1}^{2}\widetilde{Z}^{i,j}_t dW^j_t, \\ \widetilde{Y}^i_T = 0, \\ \quad i = 1, 2. \end{cases} \tag{3.14}$$

A proof is found in the appendix. By Lemma 3.1,

$$\begin{aligned} \mathbb{E}\left[\bar{h}^{\varepsilon,1}_0 - \hat{h}^1_0\right] &= \mathbb{E}\left[\sum_{j=1}^{2}\partial_{y^j}\hat{h}^1_0\widetilde{Y}^j_0 + \mathbb{E}\left[(\partial_{\mu^j}\hat{h}^1_0)^*\widetilde{Y}^j_0\right]\right] + o(\varepsilon) \\ &= \mathbb{E}\left[\sum_{j=1}^{2}p^{1,j}_0\widetilde{Y}^j_0\right] + o(\varepsilon), \end{aligned} \tag{3.15}$$

where the introduced costates $p^{1,j}_\cdot$, $j = 1, 2$, satisfy $p^{1,j}_0 := \partial_{y^j}\hat{h}^1_0 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{h}^1_0)\right]$. The notation ${}^*(\partial_{\mu^j}\hat{h}^1_0)$ is defined in (A.10). Assumption 4 grants us existence and uniqueness to equation (3.16) below.

Lemma 3.2 (Duality relation)**.** *Let assumption 1, 2 and 4 hold and let $p^{1,j}_\cdot$ be given by*

$$\begin{cases} dp^{1,j}_t = -\left\{\partial_{y^j}\hat{H}^1_t + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{H}^1_t)\right]\right\}dt - \sum_{k=1}^{2}\left\{p^{1,1}_t\partial_{z^{j,k}}\hat{b}^1_t + p^{1,2}_t\partial_{z^{j,k}}\hat{b}^2_t\right\}dW^k_t, \\ p^{1,j}_0 = \partial_{y^j}\hat{h}^1_0 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{h}^1_0)\right], \end{cases} \tag{3.16}$$

where for $(y^i, \mu^i) \in S$, $i = 1, 2$, and $(u^1, u^2, z) \in U^1 \times U^2 \times \mathbb{R}^{d\times(2d_1+2d_2)}$,

$$\begin{aligned} &H^1(\omega, t, y^1, \mu^1, u^1, y^2, \mu^2, u^2, z, p^{1,1}_t, p^{1,2}_t) \\ &:= \sum_{j=1}^{2} b^j(\omega, t, y^j, \mu^j, u^j, y^{-j}, \mu^{-j}, u^{-j}, z)p^{1,j}_t - f^1(t, y^1, \mu^1, u^1, y^2, \mu^2, u^2). \end{aligned} \tag{3.17}$$

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Then the following duality relation holds,

$$\mathbb{E}\left[\sum_{j=1}^{2} p_0^{1,j}\widetilde{Y}_0^j\right] = -\mathbb{E}\left[\int_0^T \sum_{j=1}^{2} p_t^{1,j}\delta_1 b^j(t)1_{E_\varepsilon}(t) + \widetilde{Y}_t^j\left(\partial_{y^j}\hat{f}_t^1 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{f}_t^1)\right]\right)dt\right]. \tag{3.18}$$

A proof of the lemma above is found in the appendix. We have that

$$\begin{aligned}\bar{f}_t^{\varepsilon,i} - \hat{f}_t^i = &\sum_{j=1}^{2}\left\{\partial_{y^j}\hat{f}_t^i(\bar{Y}_t^{\varepsilon,j} - \hat{Y}_t^j) + \mathbb{E}\left[(\partial_{\mu^j}\hat{f}_t^i)^*(\bar{Y}_t^{\varepsilon,j} - \hat{Y}_t^j)\right]\right\} + \delta_1 f^i(t)1_{E_\varepsilon}(t)\\ &+\sum_{j=1}^{2}\left\{o\left(|\bar{Y}_t^{\varepsilon,j} - \hat{Y}_t^j|\right) + o\left(\mathbb{E}[|\bar{Y}_t^{\varepsilon,j} - \hat{Y}_t^j|^2]^{1/2}\right)\right\}.\end{aligned} \tag{3.19}$$

By the expansion (3.19) and Lemma 3.1,

$$\begin{aligned}\mathbb{E}\left[\int_0^T \bar{f}_t^{\varepsilon,1} - \hat{f}_t^1 dt\right] = &\mathbb{E}\left[\int_0^T \sum_{j=1}^{2}\widetilde{Y}_t^j\left(\partial_{y^j}\hat{f}_t^1 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{f}_t^1)\right]\right) + \delta_1 f^1(t)1_{E_\varepsilon}(t)dt\right]\\ &+ o(\varepsilon),\end{aligned} \tag{3.20}$$

which yields

$$\begin{aligned}J^1(\bar{u}_\cdot^{\varepsilon,1};\hat{u}_\cdot^2) - J^1(\hat{u}_\cdot^1;\hat{u}_\cdot^2) = &\mathbb{E}\left[\int_0^T\left\{-p_t^{1,1}\delta_1 b^1(t) - p_t^{1,2}\delta_1 b^2(t) + \delta_1 f^1(t)\right\}1_{E_\varepsilon}(t)dt\right]\\ &+ o(\varepsilon).\end{aligned} \tag{3.21}$$

Therefore

$$J^1(\bar{u}_\cdot^{\varepsilon,1};\hat{u}_\cdot^2) - J^1(\hat{u}_\cdot^1;\hat{u}_\cdot^2) = -\mathbb{E}\left[\int_0^T \delta_1 H^1(t)1_{E_\varepsilon}(t)dt\right] + o(\varepsilon). \tag{3.22}$$

From the last identity, we can derive necessary and sufficient conditions for player 1's best response to $\hat{u}_\cdot^2$.

The same argument can be carried out for players 2's best response to $\hat{u}_\cdot^1$. Naturally, we need to impose the corresponding assumptions on player 2's control. For completeness and later reference, we state now the second player's version of Lemma 3.2.

Lemma 3.3. *(Duality relation, player 2) Let Assumptions 1, 2 and 4 hold, and let $p_\cdot^{2,j}$ be given by*

$$\begin{cases} dp_t^{2,j} = -\left\{\partial_{y^j}\hat{H}_t^2 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{H}_t^2)\right]\right\}dt - \sum_{k=1}^{2}\left\{p_t^{2,1}\partial_{z^{j,k}}\hat{b}_t^1 + p_t^{2,2}\partial_{z^{j,k}}\hat{b}_t^2\right\}dW_t^k,\\ p_0^{2,j} = \partial_{y^j}\hat{h}_0^2 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{h}_0^2)\right],\end{cases} \tag{3.23}$$

where for $(y^i,\mu^i)\in S$, $i=1,2$, and $(u^1,u^2,z)\in U^1\times U^2\times\mathbb{R}^{d\times(2d_1+2d_2)}$,

$$\begin{aligned}&H^2(\omega,t,y^2,\mu^2,u^2,y^1,\mu^1,u^1,z,p_t^{2,1},p_t^{2,2})\\ &:= \sum_{j=1}^{2} b^j(\omega,t,y^j,\mu^j,u^j,y^{-j},\mu^{-j},u^{-j},z)p_t^{2,j} - f^2(t,y^2,\mu^2,u^2,y^1,\mu^1,u^1).\end{aligned} \tag{3.24}$$

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Then the following duality relation holds,

$$\mathbb{E}\left[\sum_{j=1}^{2} p_0^{2,j}\widetilde{Y}_0^j\right] = -\mathbb{E}\left[\int_0^T \sum_{j=1}^{2} p_t^{2,j}\delta_2 b^j(t)1_{E_\varepsilon}(t) + \widetilde{Y}_t^j\left(\partial_{y^j}\hat{f}_t^2 + \mathbb{E}\left[{}^*(\partial_{y^j}\hat{f}_t^2)\right]\right)dt\right]. \tag{3.25}$$

Necessary equilibrium conditions can be stated as a system of 6 equations, 2 state BSDEs and 4 costate (adjoint) SDEs. Equilibria can be verified by convexity/concavity assumptions on the 4 functions $H^i, h^i, i = 1, 2$. We let assumptions 1-5 be in place.

Theorem 3.1 (Necessary equilibrium conditions). *Suppose that* $(\hat{Y}_\cdot^i, [\hat{Z}_\cdot^{i,1}, \hat{Z}_\cdot^{i,2}], \hat{u}_\cdot^i)$*,* $i = 1, 2$*, is an equilibrium for the MFTG and that* $p_\cdot^{i,j}$*,* $i, j = 1, 2$*, solve* (3.16) *and* (3.23)*. Then, for* $i = 1, 2$*,*

$$\hat{u}_t^i = \underset{\alpha\in U^i}{argmax}\ H^i(t, \hat{Y}_t^i, \mathcal{L}(\hat{Y}_t^i), \alpha, \hat{\Theta}_t^{-i}, \hat{Z}_t, p_t^{i,1}, p_t^{i,2}), \quad a.e.\ t,\ a.s. \tag{3.26}$$

Proof. Let $E_\varepsilon := [s, s+\varepsilon]$, $u_\cdot \in \mathcal{U}^1$ and $A \in \mathcal{F}_t$ for $t \in [0, T]$. Consider the spike-perturbation

$$u_t^\varepsilon := \begin{cases} u_t 1_A + \hat{u}_t^1 1_{A^c}, & t \in E_\varepsilon, \\ \hat{u}_t^1, & t \in [0, T]\backslash E_\varepsilon. \end{cases} \tag{3.27}$$

Then

$$\begin{gathered}\hat{H}_t^1 - H^1(t, \hat{Y}_t^1, \mathcal{L}(\hat{Y}_t^1), u_t^\varepsilon, \hat{\Theta}_t^2, \hat{Z}_t, p_t^{1,1}, p_t^{1,2}) = \\ \left(\hat{H}_t^1 - H^1(t, \hat{Y}_t^1, \mathcal{L}(\hat{Y}_t^1), u_t, \hat{\Theta}_t^2, \hat{Z}_t, p_t^{1,1}, p_t^{1,2})\right)1_A 1_{E_\varepsilon}(t)\end{gathered} \tag{3.28}$$

Applying (3.22), we obtain

$$\frac{1}{\varepsilon}\mathbb{E}\left[\int_s^{s+\varepsilon}\left(\hat{H}_t^1 - H^1(t, \hat{Y}_t^1, \mathcal{L}(\hat{Y}_t^1), u_t, \hat{\Theta}_t^2, \hat{Z}_t, p_t^{1,1}, p_t^{1,2})\right)1_A dt\right] \geq \frac{1}{\varepsilon}o(\varepsilon) \tag{3.29}$$

Sending ε to zero yields

$$\mathbb{E}\left[\left(\hat{H}_s^1 - H^1(s, \hat{Y}_s^1, \mathcal{L}(\hat{Y}_s^1), u_s, \hat{\Theta}_s^2, \hat{Z}_s, p_s^{1,1}, p_s^{1,2})\right)1_A\right] \geq 0, \quad a.e.\ s \in [0, T]. \tag{3.30}$$

The last inequality holds for all $A \in \mathcal{F}_s$, thus

$$\mathbb{E}\left[\left(\hat{H}_s^1 - H^1(s, \hat{Y}_s^1, \mathcal{L}(\hat{Y}_s^1), u_s, \hat{\Theta}_s^1, \hat{Z}_s, p_s^{1,1}, p_s^{1,2})\right) \mid \mathcal{F}_s\right] \geq 0, \quad a.e.\ s \in [0, T],\ a.s. \tag{3.31}$$

By measurability of the integrand in (3.31),

$$\hat{u}_t^1 = \underset{\alpha\in U^1}{\operatorname{argmax}}\ H^1(t, \hat{Y}_t^1, \mathcal{L}(\hat{Y}_t^1), \alpha, \hat{\Theta}_t^2, \hat{Z}_t, p_t^{1,1}, p_t^{1,2}), \quad a.e.\ t \in [0, T],\ a.s. \tag{3.32}$$

The same argument yields

$$\hat{u}_t^2 = \underset{\alpha\in U^2}{\operatorname{argmax}}\ H^2(t, \hat{Y}_t^2, \mathcal{L}(\hat{Y}_t^2), \alpha, \hat{\Theta}_t^1, \hat{Z}_t, p_t^{2,1}, p_t^{2,2}), \quad a.e.\ t \in [0, T],\ a.s. \tag{3.33}$$

□

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Theorem 3.2 (Sufficient equilibrium conditions). *Suppose that $\hat{u}^1_\cdot$ and $\hat{u}^2_\cdot$ satisfy* (3.26). *Suppose furthermore that for $(t, p^{i,1}, p^{i,2}, z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d\times(2d_1+2d_2)}$, $i = 1,2$,*

$$(y^1, \mu^1, u^1, y^2, \mu^2, u^2) \mapsto H^i(t, y^i, \mu^i, u^i, y^{-i}, \mu^{-i}, u^{-i}, z, p^{i,1}, p^{i,2}) \tag{3.34}$$

is concave a.s. and

$$(y^1, \mu^1, y^2, \mu^2) \mapsto h^i(y^i, \mu^i, y^{-i}, \mu^{-i}) \tag{3.35}$$

is convex a.s. Then $\hat{u}^1_\cdot, \hat{u}^2_\cdot$ constitute an equilibrium control and $(\hat{Y}^i_\cdot, [\hat{Z}^{i,1}_\cdot, \hat{Z}^{i,2}_\cdot], \hat{u}^i_\cdot)$, $i = 1,2$, is an equilibrium for the MFTG.

Proof. By assumption, $\delta_i H^i(t) \le 0$ for any spike variation, almost surely for a.e. t. Applying the convexity and concavity assumptions in the expansion steps results in the inequality

$$0 \le -\mathbb{E}\left[\int_0^T \delta_i H^i(t) 1_{E_\varepsilon}(t)dt\right] \le J^i(u^i_\cdot; \hat{u}^{-i}_\cdot) - J^i(\hat{u}^i_\cdot; \hat{u}^{-i}_\cdot). \tag{3.36}$$

□

4. Problem 2: MFTC

Carrying out a similar argument to that of the previous section, we find necessary optimality conditions for problem (2.8). Also, we readily get a verification theorem. The pair $(\hat{u}^1_\cdot, \hat{u}^2_\cdot) \in \mathcal{U}^1 \times \mathcal{U}^2$ is optimal if

$$J(\hat{u}^1_\cdot, \hat{u}^2_\cdot) \le J(u^1_\cdot, u^2_\cdot), \quad (u^1_\cdot, u^2_\cdot) \in \mathcal{U}^1 \times \mathcal{U}^2. \tag{4.1}$$

Assume from now on that $(\hat{u}^1_\cdot, \hat{u}^2_\cdot)$ is an optimal control. We study the inequality (4.1) when $(\check{u}^{\varepsilon,1}_\cdot, \check{u}^{\varepsilon,2}_\cdot)$ is a spike-perturbation of $(\hat{u}^1_\cdot, \hat{u}^2_\cdot)$,

$$(\check{u}^{\varepsilon,1}_t, \check{u}^{\varepsilon,2}_t) := \begin{cases} (\hat{u}^1_t, \hat{u}^2_t), & t \in [0,T]\backslash E_\varepsilon, \\ (u^1_t, u^2_t), & t \in E_\varepsilon, \end{cases} \tag{4.2}$$

where E_ε is any subset of $[0,T]$ of Lebesgue measure ε and $(u^1_\cdot, u^2_\cdot) \in \mathcal{U}^1 \times \mathcal{U}^2$. When the players use the perturbed control, we denote the state dynamics by

$$\begin{cases} d\check{Y}^{\varepsilon,1}_t = b^1(t, \check{\Theta}^{\varepsilon,1}_t, \check{\Theta}^{\varepsilon,2}_t, \check{Z}^\varepsilon_t)dt + \check{Z}^{\varepsilon,1,1}_t dW^1_t + \check{Z}^{\varepsilon,1,2}_t dW^2_t, \ \check{Y}^1_T = y^1, \\ d\check{Y}^{\varepsilon,2}_t = b^2(t, \check{\Theta}^{\varepsilon,2}_t, \check{\Theta}^{\varepsilon,1}_t, \check{Z}^\varepsilon_t)dt + \check{Z}^{\varepsilon,2,1}_t dW^1_t + \check{Z}^{\varepsilon,2,2}_t dW^2_t, \ \check{Y}^2_T = y^2, \end{cases} \tag{4.3}$$

and we will compare their performance to that of the optimally controlled state dynamics

$$\begin{cases} d\hat{Y}^1_t = b^1(t, \hat{\Theta}^1_t, \hat{\Theta}^2_t, \hat{Z}_t)dt + \hat{Z}^{1,1}_t dW^1_t + \hat{Z}^{1,2}_t dW^2_t, \ \hat{Y}^1_T = y^1, \\ d\hat{Y}^2_t = b^2(t, \hat{\Theta}^2_t, \hat{\Theta}^1_t, \hat{Z}_t)dt + \hat{Z}^{2,1}_t dW^1_t + \hat{Z}^{2,2}_t dW^2_t, \ \hat{Y}^2_T = y^2. \end{cases} \tag{4.4}$$

For simplicity, we write for $\vartheta \in \{b^i, f^i, h^i, i = 1,2\}$,

$$\begin{aligned} \check{\vartheta}^\varepsilon_t &:= \vartheta(t, \check{\Theta}^{\varepsilon,i}_t, \check{\Theta}^{\varepsilon,-i}_t, \check{Z}^\varepsilon_t), \\ \hat{\vartheta}_t &:= \vartheta(t, \hat{\Theta}^i_t, \hat{\Theta}^{-i}_t, \hat{Z}_t), \end{aligned} \tag{4.5}$$

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and in this notation,

$$\begin{aligned} J(\check{u}^{\varepsilon,1}_\cdot, \check{u}^{\varepsilon,2}_\cdot) - J(\hat{u}^1_\cdot, \hat{u}^2_\cdot) &= \mathbb{E}\left[\int_0^T \check{f}^{\varepsilon,1}_t + \check{f}^{\varepsilon,2}_t - \hat{f}^1_t - \hat{f}^2_t dt + \check{h}^{\varepsilon,1}_0 + \check{h}^{\varepsilon,2}_0 - \hat{h}^1_0 - \hat{h}^2_0\right] \\ &= \mathbb{E}\left[\int_0^T \check{f}^{\varepsilon}_t - \hat{f}_t dt + \check{h}^{\varepsilon}_0 - \hat{h}_0\right] \end{aligned} \tag{4.6}$$

where $f_t := f^1_t + f^2_t$ and $h_t := h^1_t + h^2_t$. Again, we want to find first order variation processes $(\widetilde{Y}^i_\cdot, [\widetilde{Z}^{i,1}_\cdot, \widetilde{Z}^{i,2}_\cdot])$, $i = 1, 2$, that satisfy (3.11) with $(\bar{Y}^{\varepsilon,i}_\cdot, [\bar{Z}^{\varepsilon,i,1}_\cdot, \bar{Z}^{\varepsilon,i,2}_\cdot])$ replaced by its 'checked' counterpart $(\check{Y}^{\varepsilon,i}_\cdot, [\check{Z}^{\varepsilon,i,1}_\cdot, \check{Z}^{\varepsilon,i,2}_\cdot])$.

Assumption 6. *The functions*

$$\begin{aligned} (y^1, \mu^1, u^1, y^2, \mu^2, u^2, z) &\mapsto b^i(t, y^i, \mu^i, u^i, y^{-i}, \mu^{-i}, u^{-i}, z) \\ (y^1, \mu^1, u^1, y^2, \mu^2, u^2) &\mapsto f^i(t, y^i, \mu^i, u^i, y^{-i}, \mu^{-i}, u^{-i}) \\ (y^1, \mu^1, y^2, \mu^2) &\mapsto h^i(y^i, \mu^i, y^{-i}, \mu^{-i}) \end{aligned} \tag{4.7}$$

are for all t a.s. differentiable at $(\hat{\Theta}^1_t, \hat{\Theta}^2_t, \hat{Z}_t)$, $(\hat{\Theta}^1_t, \hat{\Theta}^2_t)$ and $(\hat{Y}^1_0, \mathcal{L}(\hat{Y}^1_0), \hat{Y}^2_0, \mathcal{L}(\hat{Y}^2_0))$ respectively. Furthermore,

$$\partial_{y^j}\hat{b}^i_t,\ \partial_{\mu^j}\hat{b}^i_t,\ \partial_{y^j}\hat{f}^i_t,\ \partial_{\mu^j}\hat{f}^i_t, \quad i, j = 1, 2, \tag{4.8}$$

are for all t a.s. uniformly bounded and $\partial_{y^j}\hat{h}^i_0 + \mathbb{E}\left[{}^*(\partial^j_\mu \hat{h}^i_0)\right] \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^d)$.

Notice that the point of differentiability is generally not the same in Assumption 4 and 6. Above, $(\hat{u}^1_\cdot, \hat{u}^2_\cdot)$ is an optimal control while in Assumption 4, it is an equilibrium control. Let δ denote simultaneous variation in controls, for $\vartheta \in \{f^i, b^i, i = 1, 2\}$,

$$\delta\vartheta(t) := \delta_1\vartheta(t) + \delta_2\vartheta(t). \tag{4.9}$$

Lemma 4.1. *Let assumption 1, 2, 5 and 6 be in force. Then the first order variation processes that satisfy the 'checked' version of (3.11) are given by the following system of BSDEs,*

$$\begin{cases} d\widetilde{Y}^i_t = \left(\sum_{j=1}^2 \left\{\partial_{y^j}\hat{b}^i \widetilde{Y}^j_t + \mathbb{E}\left[(\partial_{\mu^j}\hat{b}^i)^* \widetilde{Y}^j_t\right]\right\} + \delta b^i(t) 1_{E_\varepsilon}(t) + \sum_{j,k=1}^2 \partial_{z^{j,k}}\hat{b}^i \widetilde{Z}^{j,k}_t\right) dt \\ \qquad + \sum_{j=1}^2 \widetilde{Z}^{i,j}_t dW^j_t, \\ \widetilde{Y}^i_T = 0, \\ \quad i = 1, 2, \end{cases} \tag{4.10}$$

The proof follows the same steps as the proof of Lemma 3.1. By Lemma 4.1,

$$\mathbb{E}\left[\check{h}^\varepsilon_0 - \hat{h}_0\right] = \mathbb{E}\left[\sum_{j=1}^2 p^j_0 \widetilde{Y}^j_0\right] + o(\varepsilon), \tag{4.11}$$

where $p^j_0 := \partial_{y^j}\hat{h}_0 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{h}_0)\right]$.

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Lemma 4.2 (Duality relation). *Let assumption 1, 2 and 6 hold and let $p^j_\cdot$ solve*

$$\begin{cases} dp^j_t = -\left\{\partial_{y^j}\hat{H}_t + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{H}_t)\right]\right\}dt - \sum_{k=1}^{2}\left\{p^1_t\partial_{z^{j,k}}\hat{b}^1_t + p^2_t\partial_{z^{j,k}}\hat{b}^2_t\right\}dW^k_t, \\ p^j_0 = \partial_{y^j}\hat{h}_0 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{h}_0)\right], \end{cases} \tag{4.12}$$

where for $(y^i,\mu^i)\in S$, $i=1,2$, and $(u^1,u^2,z)\in U^1\times U^2\times\mathbb{R}^{d\times(2d_1+2d_2)}$,

$$\begin{aligned} &H(\omega,t,y^1,\mu^1,u^1,y^2,\mu^2,u^2,z,p^1_t,p^2_t) \\ &:= \sum_{j=1}^{2} b^j(\omega,t,y^j,\mu^j,u^j,y^{-j},\mu^{-j},u^{-j},z)p^j_t - f^j(t,y^j,\mu^j,u^j,y^{-j},\mu^{-j},u^{-j}). \end{aligned} \tag{4.13}$$

Then the following duality relation holds,

$$\mathbb{E}\left[\sum_{j=1}^{2} p^j_0\widetilde{Y}^j_0\right] = -\mathbb{E}\left[\int_0^T \sum_{j=1}^{2} p^j_t\delta b^j(t)1_{E_\varepsilon}(t) + \widetilde{Y}^j_t\left(\partial_{y^j}\hat{f}_t + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{f}_t)\right]\right)dt\right]. \tag{4.14}$$

The proof of Lemma 4.2 is almost identical that of Lemma 3.2. By Lemma 4.1,

$$J(\check{u}^{\varepsilon,1}_\cdot,\check{u}^{\varepsilon,2}_\cdot) - J(\hat{u}^1_\cdot,\hat{u}^2_\cdot) = \mathbb{E}\left[\int_0^T\left\{-p^1_t\delta b^1(t) - p^2_t\delta b^2(t) + \delta f(t)\right\}1_{E_\varepsilon}(t)dt\right] + o(\varepsilon). \tag{4.15}$$

Thus

$$J(\check{u}^{\varepsilon,1}_\cdot,\check{u}^{\varepsilon,2}_\cdot) - J(\hat{u}^1_\cdot,\hat{u}^2_\cdot) = -\mathbb{E}\left[\int_0^T \delta H(t)1_{E_\varepsilon}(t)dt\right] + o(\varepsilon). \tag{4.16}$$

In the following two theorems, assumptions 1-3 and 5-6 are in force.

Theorem 4.1 (Necessary optimality conditions). *Suppose that $(\hat{Y}^i_\cdot,[\hat{Z}^{i,1}_\cdot,\hat{Z}^{i,2}_\cdot])$, $i=1,2$ is an optimal solution to the MFTC and that $p^i_\cdot$, $i=1,2$, solves (4.12). Then, for $i=1,2$,*

$$(\hat{u}^1_t,\hat{u}^2_t) = \underset{(v,w)\in U^1\times U^2}{argmax}\ H(t,\hat{Y}^1_t,\mathcal{L}(\hat{Y}^1_t),v,\hat{Y}^2_t,\mathcal{L}(\hat{Y}^2_t),w,\hat{Z}_t,p^1_t,p^2_t), \quad a.e.t,\ a.s. \tag{4.17}$$

Theorem 4.2 (Sufficient optimality conditions). *Suppose $(\hat{u}^1_\cdot,\hat{u}^2_\cdot)$ satisfy (4.17). Suppose furthermore that for $(t,p^1,p^2,z)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{d\times(2d_1+2d_2)}$, $i=1,2$,*

$$(y^1,\mu^1,u^1,y^2,\mu^2,u^2)\mapsto H(t,y^1,\mu^1,u^1,y^2,\mu^2s,u^2,z,p^1,p^2) \tag{4.18}$$

is concave a.s. and

$$(y^1,\mu^1,y^2,\mu^2)\mapsto h(y^1,\mu^1,y^2,\mu^2) \tag{4.19}$$

is convex a.s. Then $(\hat{u}^1_\cdot,\hat{u}^2_\cdot)$ is an optimal control and $(\hat{Y}^i_\cdot,[\hat{Z}^{i,1}_\cdot,\hat{Z}^{i,2}_\cdot],\hat{u}^i_\cdot)$, $i=1,2$ solves the MFTC.

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5. Example: the linear-quadratic case

In this section we consider a linear-quadratic version of (2.7) and (2.8), in the one dimensional case. Let $a_i, c_{i,j}, q_{i,j}, \bar{q}_{i,j}, \widetilde{q}_{i,j}, \bar{s}_{i,j}, s_i, \bar{s}_i^E, r_i : [0,T] \mapsto \mathbb{R}$, $i,j = 1,2$ be deterministic coefficient functions, uniformly bounded over $[0,T]$. Additionally, $r_i(t) \geq \epsilon > 0$ for $i = 1,2$. Define

$$\begin{aligned} b^i(t, \Theta_t^i, \Theta_t^{-i}, Z_t) &= a_i(t)u_t^i + \sum_{j=1}^2 c_{i,j}(t) W_t^j, \\ f^i(t, \Theta_t^i, \Theta_t^{-i}) &= \sum_{j=1}^2 \Bigg\{ \frac{1}{2} q_{i,j}(t)(Y_t^j)^2 + \frac{1}{2}\bar{q}_{i,j}(t)\mathbb{E}[Y_t^j]^2 + \widetilde{q}_{i,j}(t) Y_t^j \mathbb{E}[Y_t^j] \\ &\quad + \bar{s}_{i,j}(t)\mathbb{E}[Y_t^j] Y_t^{-j} \Bigg\} \\ &\quad + s_i(t) Y_t^i Y_t^{-i} + \bar{s}_i^E(t)\mathbb{E}[Y_t^i]\mathbb{E}[Y_t^{-i}] + \frac{1}{2} r_i(t)(u_t^i)^2. \end{aligned} \tag{5.1}$$

The uniform boundedness of the coefficients implies assumptions 1-6, given nice enough initial costs h^1, h^2, satisfying assumption 4 and 6. Integrability of f^i in assumption 3 follows by classical BSDE estimates [31]. In this setup,

$$\begin{aligned} &H^i(t, \Theta_t^i, \Theta_t^{-i}, Z_t) = \\ &\left(a_1(t)u_t^1 + \sum_{j=1}^2 c_{1,j} W_t^j \right) p^{i,1} + \left(a_2(t)u_t^2 + \sum_{j=1}^2 c_{2,j} W_t^j \right) p^{i,2} \\ &- \sum_{j=1}^2 \left\{ \frac{1}{2} q_{i,j}(t)(Y_t^j)^2 + \frac{1}{2}\bar{q}_{i,j}(t)\mathbb{E}[Y_t^j]^2 + \widetilde{q}_{i,j}(t) Y_t^j \mathbb{E}[Y_t^j] + \bar{s}_{i,j}(t)\mathbb{E}[Y_t^j] Y_t^{-j} \right\} \\ &- s_i(t) Y_t^i Y_t^{-i} - \bar{s}_i^E(t)\mathbb{E}[Y_t^i]\mathbb{E}[Y_t^{-i}] - \frac{1}{2} r_i(t)(u_t^i)^2. \end{aligned} \tag{5.2}$$

The Hessian of $(y^1, \dots, u^2) \mapsto H^1(t, y^1, \dots, u^2, z, p^{1,1}, p^{1,2})$ is

$$\mathcal{H}^1(t) := - \begin{bmatrix} q_{1,1}(t) & \widetilde{q}_{1,1}(t) & 0 & s_1(t) & \bar{s}_{1,2}(t) & 0 \\ \widetilde{q}_{1,1}(t) & \bar{q}_{1,1}(t) & 0 & \bar{s}_{1,1}(t) & \bar{s}_1^E(t) & 0 \\ 0 & 0 & r_1(t) & 0 & 0 & 0 \\ s_1(t) & \bar{s}_{1,1}(t) & 0 & q_{1,2}(t) & \widetilde{q}_{1,2}(t) & 0 \\ \bar{s}_{1,2}(t) & \bar{s}_1^E(t) & 0 & \widetilde{q}_{1,2}(t) & \bar{q}_{1,2}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.3}$$

and the Hessian of $(y^1, \dots, u^2) \mapsto H^2(t, y^1, \dots, u^2, z, p^{2,1}, p^{2,2})$ is

$$\mathcal{H}^2(t) := - \begin{bmatrix} q_{2,1}(t) & \widetilde{q}_{2,1}(t) & 0 & s_2(t) & \bar{s}_{2,2}(t) & 0 \\ \widetilde{q}_{2,1}(t) & \bar{q}_{2,1}(t) & 0 & \bar{s}_{2,1}(t) & \bar{s}_2^E(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ s_2(t) & \bar{s}_{2,1}(t) & 0 & q_{2,2}(t) & \widetilde{q}_{2,2}(t) & 0 \\ \bar{s}_{2,2}(t) & \bar{s}_2^E(t) & 0 & \widetilde{q}_{2,2}(t) & \bar{q}_{2,2}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & r_2(t) \end{bmatrix}. \tag{5.4}$$

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The coefficients are further assumed to be such that $\mathcal{H}^1(t)$ and $\mathcal{H}^2(t)$ are negative semi-definite for all $t \in [0,T]$. Also, we assume that $(y^1,\dots,\mu^2) \mapsto h^i(y^1,\dots,\mu^2)$, yet unspecified, is convex. Theorem 3.2 yields

$$\hat{u}^i_t = a_i(t) r_i^{-1}(t) p^{i,i}_t, \tag{5.5}$$

where $p^{i,i}_\cdot$ solves (3.16) or (3.23), depending on i. In fact the equilibrium is unique in this case, since $\hat{u}^i_\cdot$ is the unique pointwise solution to (3.26) and $p^{i,i}_\cdot$ is unique, see (B.15)-(B.16). By Theorem 4.2,

$$\hat{u}^i_t = a_i(t) r_i^{-1}(t) p^i_t, \tag{5.6}$$

where $p^i_\cdot$ solves (4.12), is an optimal control for the linear-quadratic MFTC and it is unique.

5.1. MFTG

The equilibrium dynamics are

$$\begin{cases} d\hat{Y}^i_t = \left(a_i^2(t) r_i^{-1}(t) p^{i,i}_t + \sum_{j=1}^{2} c_{i,j} W^j_t \right) dt + \hat{Z}^{i,1}_t dW^1_t + \hat{Z}^{i,2}_t dW^2_t, \\ \hat{Y}^i_T = y^i_T. \end{cases} \tag{5.7}$$

We see that only two costate processes, $p^{1,1}_\cdot$ and $p^{2,2}_\cdot$, are relevant here. This is a consequence of the lack of explicit dependence on u^{-i} in the b^i and f^i specified in (5.1). Nevertheless, the running cost f^i depends implicitly on u^{-i} through player $-i$'s state and mean.

We make the following ansatz: there exists deterministic functions $\alpha_i, \bar{\alpha}_i, \beta_i, \bar{\beta}_i, \theta_i : [0,T] \to \mathbb{R}$, $i = 1,2$ and $\gamma_{i,j} : [0,T] \to \mathbb{R}$, $i,j = 1,2$, such that

$$\hat{Y}^i_t = \alpha_i(t) p^{i,i}_t + \bar{\alpha}_i(t) \mathbb{E}[p^{i,i}_t] + \beta_i(t) p^{-i,-i}_t + \bar{\beta}_i(t) \mathbb{E}[p^{-i,-i}_t] + \gamma_{i,1}(t) W^1_t + \gamma_{i,2}(t) W^2_t + \theta_i(t). \tag{5.8}$$

Clearly, we need to impose the terminal conditions

$$\alpha_i(T) = 0, \quad \bar{\alpha}_i(T) = 0, \quad \beta_i(T) = 0, \quad \bar{\beta}_i(T) = 0, \quad \gamma_{i,j}(T) = 0, \quad \theta_i(T) = y^i_T. \tag{5.9}$$

Calculations presented in the appendix identifies coefficients and yields the following system of ODEs determining $\alpha_i(\cdot), \dots, \theta_i(\cdot)$,

$$\begin{cases} \dot{\alpha}_i(t) + \alpha_i(t) P^i(t) + \beta_i(t) R^{-i}(t) = a_i^2(t) r_i^{-1}(t), \\ \dot{\bar{\alpha}}_i(t) + \alpha_i(t) \bar{P}_i(t) + \bar{\alpha}_i(t)(P^i(t) + \bar{P}^i(t)) + \beta_i(t) \bar{R}^{-i}(t) + \bar{\beta}_i(t)(R^{-i}(t) + \bar{R}^{-i}(t)) = 0, \\ \dot{\beta}_i(t) + \alpha_i(t) R^i(t) + \beta_i(t) P^{-i}(t) = 0, \\ \dot{\bar{\beta}}_i(t) + \alpha_i(t) \bar{R}^i(t) + \bar{\alpha}_i(t)(R^i(t) + \bar{R}^i(t)) + \beta_i(t) \bar{P}^{-i}(t) + \bar{\beta}_i(t)(P^{-i}(t) + \bar{P}^{-i}(t)) = 0, \\ \dot{\gamma}_{i,1}(t) + \alpha_i(t) \Phi^i(t) + \beta_i(t) \Phi^{-i}(t) = c_{i,1}(t), \\ \dot{\gamma}_{i,2}(t) + \alpha_i(t) \Psi^i(t) + \beta_i(t) \Psi^{-i}(t) = c_{i,2}(t), \\ \dot{\theta}_i(t) + \theta_i(t) \Big((\alpha_i(t) + \bar{\alpha}_i(t))(Q_i(t) + \bar{Q}_i(t)) + (\beta_i(t) + \bar{\beta}_i(t))(S_{-i}(t) + \bar{S}_{-i}(t)) \Big) \\ \qquad + \theta_{-i}(t) \Big((\alpha_i(t) + \bar{\alpha}_i(t))(S_i(t) + \bar{S}_i(t)) + (\beta_i(t) + \bar{\beta}_i(t))(Q_{-i}(t) + \bar{Q}_{-i}(t)) \Big) = 0, \end{cases} \tag{5.10}$$

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where

$$\begin{aligned}
&P^i(t) := Q_i(t)\alpha_i(t) + S_i(t)\beta_{-i}(t),\\
&\bar{P}^i(t) := Q_i(t)\bar{\alpha}_i(t) + \bar{Q}_i(t)(\alpha_i(t) + \bar{\alpha}_i(t)) + S_i(t)\bar{\beta}_{-i}(t) + \bar{S}_i(t)(\beta_{-i}(t) + \bar{\beta}_{-i}(t)),\\
&R^i(t) := Q_i(t)\beta_i(t) + S_i(t)\alpha_{-i}(t),\\
&\bar{R}^i(t) := Q_i(t)\bar{\beta}_i(t) + \bar{Q}_i(t)(\beta_i(t) + \bar{\beta}_i(t)) + S_i(t)\bar{\alpha}_{-i}(t) + \bar{S}_i(t)(\alpha_{-i}(t) + \bar{\alpha}_{-i}(t)),\\
&\Phi^i(t) := (Q_i(t)\gamma_{i,1}(t) + S_i(t)\gamma_{-i,1}(t)), \quad \Psi^i(t) := (Q_i(t)\gamma_{i,2}(t) + S_i(t)\gamma_{-i,2}(t)),\\
&Q_i(t) := q_{i,i}(t) + \widetilde{q}_{i,i}(t), \qquad \bar{Q}_i(t) := \widetilde{q}_{i,i}(t) + \bar{q}_{i,i}(t),\\
&S_i(t) := s_i(t) + \bar{s}_{i,i}(t), \qquad \bar{S}_i(t) := \bar{s}_{i,-i}(t) + \bar{s}_i^E(t).
\end{aligned} \tag{5.11}$$

Now (5.8)-(5.11) gives us the equilibrium dynamics. In this fashion, it is possible to solve LQ problems more general than (5.1).

5.2. MFTC

The optimally controlled dynamics are

$$\begin{cases} d\hat{Y}_t^i = \left(a_i^2(t)r_i^{-1}(t)p_t^i + \sum_{j=1}^{2} c_{i,j}W_t^j\right)dt + \hat{Z}_t^{i,1}dW_t^1 + \hat{Z}_t^{i,2}dW_t^2, \\ \hat{Y}_T^i = y_T^i. \end{cases} \tag{5.12}$$

We make almost the same ansatz as before, assume that there exists deterministic functions $\alpha_i, \bar{\alpha}_i, \beta_i, \bar{\beta}_i, \theta_i : [0,T] \to \mathbb{R}$, $i = 1,2$ and $\gamma_{i,j} : [0,T] \to \mathbb{R}$, $i,j = 1,2$, with terminal conditions

$$\alpha_i(T) = 0, \quad \bar{\alpha}_i(T) = 0, \quad \beta_i(T) = 0, \quad \bar{\beta}_i(T) = 0, \quad \gamma_{i,j}(T) = 0, \quad \theta_i(T) = y_T^i \tag{5.13}$$

such that

$$\hat{Y}_t^i = \alpha_i(t)p_t^i + \bar{\alpha}_i(t)\mathbb{E}[p_t^i] + \beta_i(t)p_t^{-i} + \bar{\beta}_i(t)\mathbb{E}[p_t^{-i}] + \gamma_{i,1}(t)W_t^1 + \gamma_{i,2}(t)W_t^2 + \theta_i(t). \tag{5.14}$$

By redefining $Q_i, \bar{Q}_i, S_i, \bar{S}_i$ in (5.11),

$$\begin{aligned}
Q_i(t) &:= q_{1,i}(t) + q_{2,i}(t) + \widetilde{q}_{1,i}(t) + \widetilde{q}_{2,i}(t),\\
\bar{Q}_i(t) &:= \widetilde{q}_{1,i}(t) + \widetilde{q}_{2,i}(t) + \bar{q}_{1,i}(t) + \bar{q}_{2,i}(t),\\
S_i(t) &:= s_1(t) + s_2(t) + \bar{s}_{1,i}(t) + \bar{s}_{2,i}(t),\\
\bar{S}_i(t) &:= \bar{s}_{1,-i}(t) + \bar{s}_{2,-i}(t) + \bar{s}_1^E(t) + \bar{s}_2^E(t),
\end{aligned} \tag{5.15}$$

(5.10)-(5.11) and (5.13)-(5.14) gives us the optimally controlled state dynamics.

5.3. Simulation and the price of anarchy

Let $T := 1$, $\xi := (y_0^1, y_0^2) \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^d \times \mathbb{R}^d)$ be preferred initial positions for player 1 and 2 respectively, and

$$f_t^i := \frac{1}{2}\Big(r_i(u_t^i)^2 + \rho_i(Y_t^i - \mathbb{E}[Y_t^{-i}])^2\Big), \quad h_t^i := \frac{\nu_i}{2}(Y_0^i - y_0^i)^2. \tag{5.16}$$

In this setup, $\mathcal{H}^1$ and $\mathcal{H}^2$ are negative semi-definite if $r_i, \rho_i > 0$, h^i is convex if $\nu_i > 0$. In Figure 1 numerical simulations of MFTG and MFTC are presented. In (a), the two players have identical preferences, but different terminal conditions. The situation is symmetric in the sense that we expect the realized paths of player 1 reflected through the line $y = 0$ to be approximately paths of player 2. In (c), preferences are asymmetric and as a consequence, the realized paths are not each others mirrored images.

The central planner in a MFTC uses more information than a single player does. In fact, in our example, $\gamma^{i,j}(t) = 0$ when $i \neq j$ in the MFTG. The interpretation is that in the game, player i does not care about player $-i$'s noise, only its mean state. For the central planner however, $\gamma^{i,j}$ is not identically zero for $i \neq j$. This can be observed in (b), where the central planner makes the player states evolve under some common noise.

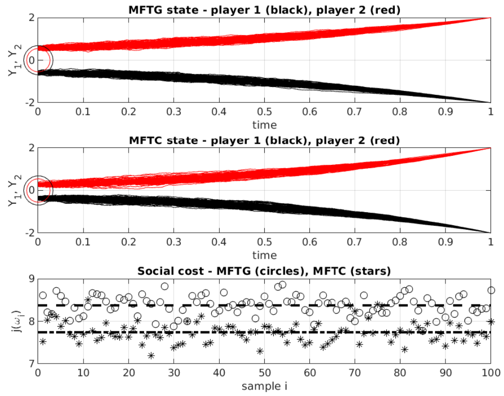
In (c) we see an interesting contrast between the MFTG and the MFTC. Player 1 (black) feels no attraction to player 2 ($\rho_1 = 0$) while player 2 is attracted to the mean position of player 1 ($\rho_2 > 0$). In the game, player 1 travels on the straight line from $(t, y) \approx (0, -1)$ to its terminal position $(t, y) = (1, -2)$. Player 2, on the other hand, deviates far from its preferred initial position at time $t = 0$, only to be in the proximity of player 1. In the MFTC, the central planner makes player 1 linger around $y = 0$ for some time, before turning south towards the terminal position. The result is less movement movement by player 2. Even though player 1 pays a higher individual cost, the social cost is reduced by approximately 33%. The *social cost* J is approximated by

$$J(u^1, u^2) \approx \frac{1}{N}\sum_{i=1}^{N} j(\omega_i), \tag{5.17}$$

where $j(\omega_i) = \sum_{j=1}^{2}\int_0^T f_t^j(\omega_i)dt + h^j(\omega_i)$. In (a) and (c), the outcomes of j (circles for equilibrium control, stars for optimal control) are presented along with the approximation of J (dashed lines) for $N = 100$. The optimal control yields the lower social cost in both cases. This is expected, the general inefficiency of Nash equilibria in nonzero-sum games is well known [16]. The price of anarchy quantifies the inefficiency due to non-cooperation, see for static games [22, 23], for differential games [4] and for linear-quadratic mean-field type games [20]. The price of anarchy in mean-field games has been studied recently in [13, 11]. It is defined as the largest ratio between social cost for an equilibrium (MFTG) to the optimal social cost (MFTC),

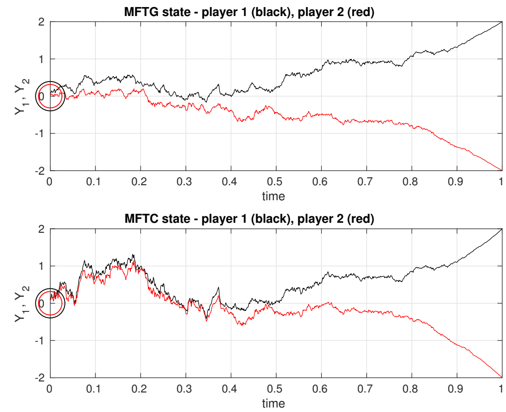
$$PoA := \sup_{\substack{(\hat{u}^1_\cdot, \hat{u}^2_\cdot)\text{ MFTG} \\ \text{equilibrium}}} \frac{J(\hat{u}^1_\cdot, \hat{u}^2_\cdot)}{\min_{u^i_\cdot \in \mathcal{U}^i, i=1,2} J(u^1_\cdot, u^2_\cdot)}. \tag{5.18}$$

Taking the parameter set of (a) as a point of reference, see Table 1, we vary one parameter at the time and study PoA. The result is presented in Figure 2. In the intervals studied, PoA is increasing in ρ_i and T and decreasing in ν_i and r_i. The reason is that the players become less flexible when ν_i and/or r_i are increased, and the improvement a central planner can do decreases. On the other hand, an increased time horizon gives the central planner more time to improve the social cost. Also, an increased preference on attraction rewards the unegoistic behavior in the MFTC model.



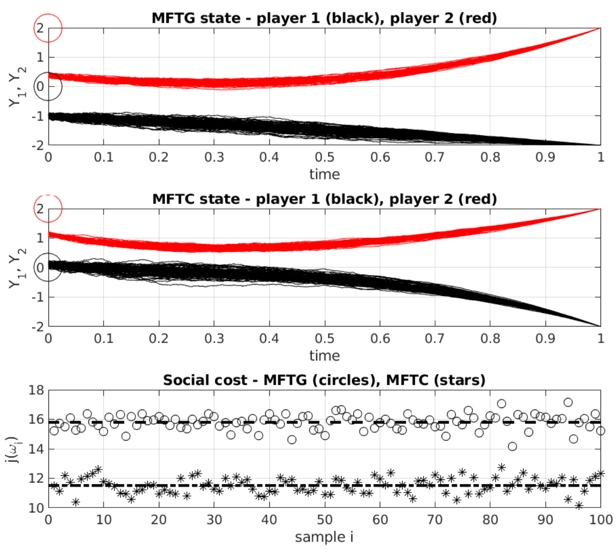
(a)

y_T^1	a_1	c_{11}	c_{12}	r_1	ρ_1	ν_1	y_0^1
-2	1	0.3	0	1	1	1	$\mathcal{N}(0, 0.1)$
y_T^2	a_2	c_{21}	c_{22}	r_2	ρ_2	ν_2	y_0^2
2	1	0	0.3	1	1	1	$\mathcal{N}(0, 0.1)$



(b)

y_T^1	a_1	c_{11}	c_{12}	r_1	ρ_1	ν_1	y_0^1
-2	1	3	0	1	10	1	$\mathcal{N}(0, 0.1)$
y_T^2	a_2	c_{21}	c_{22}	r_2	ρ_2	ν_2	y_0^2
2	1	0	3	1	10	1	$\mathcal{N}(0, 0.1)$



(c)

y_T^1	a_1	c_{11}	c_{12}	r_1	ρ_1	ν_1	y_0^1
-2	1	0.3	0	1	4	1	$\mathcal{N}(0, 0.1)$
y_T^2	a_2	c_{21}	c_{22}	r_2	ρ_2	ν_2	y_0^2
2	1	0	0.3	1	0	1	$\mathcal{N}(2, 0.1)$

Fig 1: Numerical examples: (a) symmetric preference, (b) single path sample, (c) asymmetric attraction and initial position. Circles indicate the preferred initial positions.

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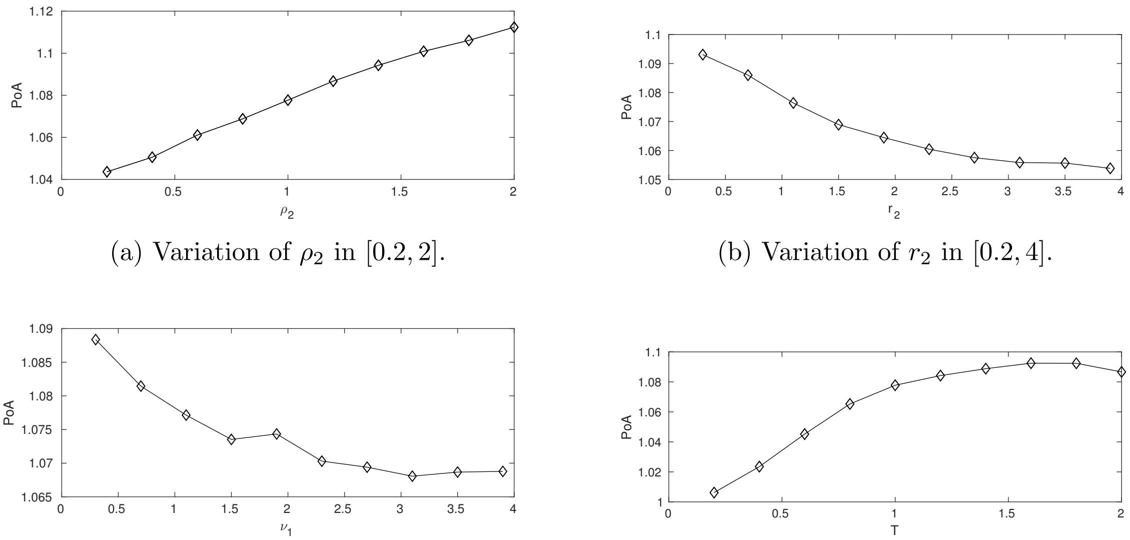
y_T^1	a_1	c_{11}	c_{12}	r_1	ρ_1	ν_1	y_0^1
-2	1	0.3	0	1	1	1	$\mathcal{N}(0, 0.1)$
y_T^2	a_2	c_{21}	c_{22}	r_2	ρ_2	ν_2	y_0^2
2	1	0	0.3	1	1	1	$\mathcal{N}(0, 0.1)$

TABLE 1
Parameter values in the symmetric case (a).



(a) Variation of ρ_2 in $[0.2, 2]$.

(b) Variation of r_2 in $[0.2, 4]$.

(c) Variation of ν_1 in $[0.2, 4]$.

(d) Variation of T in $[0.2, 2]$.

Fig 2: Numerical approximations ($N = 5000$) of the price of anarchy PoA.

6. Concluding remarks

Mean-field type games with backward stochastic dynamics, where the coefficients are allowed to depend on the marginal distributions of the player states, have been defined in this paper. Under regularity assumptions existence of a Nash equilibrium is shown, and a verification theorem is proven. In linear-quadric examples, the player behavior in the MFTG is compared to the centrally planned solution in the MFTC, which minimizes social cost. The efficiency of the MFTG Nash equilibrium, quantified by the price of anarchy, and its dependence on problem parameters is studied in the linear-quadratic case.

The work presented in this paper has many possible extensions. The theory for martingale-driven BSDEs is now standard, and the framework presented above could be extended to include that. Practically, this would mean exchanging $W^1_\cdot, W^2_\cdot$ for two martingales $M^1_\cdot, M^2_\cdot$, which can include jumps, and approach the game with the theory of forward-backward SDEs. Depending on application, the information structure of the problem can change. With our definition of $\mathcal{U}^i$, we have restricted ourselves to open loop adapted controls in this paper. Other types of information structures, such as perfect/partial state- and/or law feedback controls, lagged or noise-perturbed controls are of applied interest. Also, both players have perfect information about each other. This can be relaxed to partial information of the states/laws, as well as a treatment of dependencies on the full state-control distribution.

Appendix A: Differentiation and approximation of measure-valued functions

Derivatives of measure-valued functions will be defined with the lifting technique, outlined for example in [8, 10, 12]. Consider the function $f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$. We assume that our probability space is rich enough, so that for every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, there exists a square-integrable random variable X whose distribution is μ, i.e. $\mu = \mathcal{L}(X)$. For example, $([0,1], \mathcal{B}([0,1]), dx)$ has this property. Then we may write $f(\mu) =: F(X)$ and we can differentiate F in Fréchet-sense whenever there exists a continuous linear functional $DF[X] : L^2(\mathcal{F}; \mathbb{R}^d) \to \mathbb{R}$ such that

$$F(X+Y) - F(X) = \mathbb{E}[DF[X]Y] + o(\|Y\|_2) =: D_Y f(\mu) + o(\|Y\|_2), \tag{A.1}$$

where $\|Y\|_2^2 := \mathbb{E}[Y^2]$. $D_Y f(\mu)$ is the Fréchet derivative of f at μ, in the direction Y and we have that

$$D_Y f(\mu) = \mathbb{E}[DF[X]Y] =: \lim_{t\to 0} \frac{\mathbb{E}[F(X+tY) - F(X)]}{t}, \quad Y \in L^2(\mathcal{F}; \mathbb{R}^d), \mu = \mathcal{L}(X). \tag{A.2}$$

By Riesz' Representation Theorem, $DF[X]$ is unique and it is known [8] that there exists a Borel function $\varphi[\mu] : \mathbb{R}^d \to \mathbb{R}^d$, independent of the version of X, such that $DF[X] = \varphi[\mu](X)$. Therefore, with $\mu' = \mathcal{L}(X')$ for some random variable X', (A.1) can be written as

$$f(\mu') - f(\mu) = E[h[X](X'), X' - X] + o(\|X' - X\|_2), \quad \forall X' \in L^2(\mathcal{F}; \mathbb{R}^d). \tag{A.3}$$

We denote $\partial_\mu f(\mu; x) := h[\mu](x), x \in \mathbb{R}^d$, $\partial_\mu f(\mathcal{L}(X); X) =: \partial_\mu f(\mathcal{L}(X))$, and we have the indentity

$$DF[X] = h[\mathcal{L}(X)](X) = \partial_\mu f(\mathcal{L}(X)). \tag{A.4}$$

Example 1. If $f(\mu) = (\int_{\mathbb{R}^d} x d\mu(x))^2$ then

$$\lim_{t\to 0} \frac{\mathbb{E}[X+tY]^2 - \mathbb{E}[X]^2}{t} = \mathbb{E}[2\mathbb{E}[X]Y], \tag{A.5}$$

and $\partial_\mu f(\mu) = 2 \int_{\mathbb{R}^d} x d\mu(x)$.

Example 2. If $f(\mu) = \int_{\mathbb{R}^d} x d\mu(x)$ then $\partial_\mu f(\mu) = 1$.

The Taylor approximation of a measure-valued function is given by (A.3), and we will write

$$f(\mathcal{L}(X')) - f(\mathcal{L}(X)) = \mathbb{E}\left[\partial_\mu f(\mathcal{L}(X))(X' - X)\right] + o(\|X' - X\|_2). \tag{A.6}$$

Assume now that f takes another argument, ξ. Then

$$f(\xi, \mathcal{L}(X')) - f(\xi, \mathcal{L}(X)) = \mathbb{E}\left[\partial_\mu f(\widetilde{\xi}, \mathcal{L}(X); X)(X' - X)\right] + o(\|X' - X\|_2), \tag{A.7}$$

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where the expectation is **not taken over the tilded variable**. Note that $\mathbb{P}_X$ is deterministic. In situations where the expected value is taken only over the directional argument of $\partial_\mu f$, we will write

$$\mathbb{E}\left[\partial_\mu f(\widetilde{\xi}, \mathcal{L}(X); X)(X' - X)\right] =: \mathbb{E}\left[(\partial_\mu f(\xi, \mathcal{L}(X)))^*(X' - X)\right]. \tag{A.8}$$

The expected value in (A.7) is a random quantity because of $\widetilde{\xi}$. Taking another expected value, and changing the order of integration, leads to

$$\mathbb{E}\left[\widetilde{\mathbb{E}}[\partial_\mu f(\widetilde{\xi}, \mathcal{L}(X); X)](X' - X)\right], \tag{A.9}$$

where the tilded expectation is taken only over the tilded variable. The notation for this will be

$$\widetilde{\mathbb{E}}[\partial_\mu f(\widetilde{\xi}, \mathcal{L}(X); X)] =: \mathbb{E}\left[{}^*(\partial_\mu f(\xi, \mathcal{L}(X)))\right]. \tag{A.10}$$

Appendix B: Proofs

Lemma 3.1

Let

$$\widetilde{b}^i_t := \sum_{j=1}^2 \left\{\partial_{y^j}\hat{b}^i_t \widetilde{Y}^j_t + \mathbb{E}\left[(\partial_{\mu^j}\hat{b}^i_t)^* \widetilde{Y}^j_t\right]\right\} + \sum_{j,k=1}^2 \partial_{z^{j,k}}\hat{b}^i_t \widetilde{Z}^{i,j}_t, \tag{B.1}$$

then $\widetilde{Y}^i_t = -\int_t^T \widetilde{b}^i_s + \delta_1 b^i(s) 1_{E_\varepsilon}(s)ds - \sum_{j=1}^2 \int_t^T \widetilde{Z}^{i,j}_s dW_s$. An application of Ito's formula to $|\widetilde{Y}^1_t|^2 + |\widetilde{Y}^2_t|^2$ yields

$$\begin{aligned}\sum_{i=1}^2 |\widetilde{Y}^i_t|^2 + \int_t^T \sum_{i,j=1}^2 \|\widetilde{Z}^{i,j}_s\|_F^2 ds &= \int_t^T 2\sum_{i=1}^2 \langle \widetilde{Y}^i_s, \widetilde{b}^i_s + \delta_1 b^i(s) 1_{E_\varepsilon}(s)\rangle ds \\ &\quad + \sum_{i,j=1}^2 \int_t^T \langle \widetilde{Y}^i_s, \widetilde{Z}^{i,j}_s dW^j_s\rangle.\end{aligned} \tag{B.2}$$

Let D denote the largest bound for all the derivatives of b^1 and b^2 present. By Jensen's and Young's inequalities,

$$2\sum_{i=1}^2 \langle \widetilde{Y}^i_s, \widetilde{b}^i_s\rangle \le \sum_{i=1}^2 \left\{(6D + 16D^2)|\widetilde{Y}^i_s|^2 + 2D\mathbb{E}[|\widetilde{Y}^i_s|^2]\right\} + \frac{1}{2}\sum_{i,j=1}^2 \|\widetilde{Z}^{i,j}_s\|_F^2. \tag{B.3}$$

The stochastic integrals in (B.2) are local martingales and vanish under an expectation [27]. Therefore, with $K_0 := 8D + 16D^2$,

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^2 |\widetilde{Y}^i_s|^2 + \frac{1}{2}\sum_{i,j=1}^2 \int_t^T \|\widetilde{Z}^{i,j}_s\|_F^2\, ds\right] &\le K_0 \int_t^T \mathbb{E}\left[\sum_{i=1}^2 |\widetilde{Y}^i_s|^2\right] ds \\ &\quad + 2\int_t^T \mathbb{E}\left[\sum_{i=1}^2 \langle \widetilde{Y}^i_s, \delta_1 b^i(s) 1_{E_\varepsilon}(s)\rangle\right] ds.\end{aligned} \tag{B.4}$$

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Let $\tau \in [0,T]$, then

$$\sup_{(T-\tau)\le t\le T} K_0 \int_t^T \mathbb{E}\left[\sum_{i=1}^2 |\widetilde{Y}^i_s|^2\right] ds \le K_0\delta \sup_{(T-\tau)\le t\le T} \mathbb{E}\left[\sum_{i=1}^2 |\widetilde{Y}^i_s|^2\right]. \tag{B.5}$$

and by Hölder's and Young's inequalities,

$$\begin{aligned}
&\sup_{(T-\tau)\le t\le T} \int_t^T \mathbb{E}\left[\sum_{i=1}^2 \langle \widetilde{Y}^i_s, \delta_1 b^i(s) 1_{E_\varepsilon}(s)\rangle\right] ds \\
&\le \sup_{(T-\tau)\le t\le T} \int_t^T \sum_{i=1}^2 \mathbb{E}\left[|\widetilde{Y}^i_s|^2\right]^{1/2} \mathbb{E}\left[|\delta_1 b^i(s) 1_{E_\varepsilon}(s)|^2\right]^{1/2} ds \\
&\le \sum_{i=1}^2 \left\{\sup_{(T-\tau)\le t\le T} \mathbb{E}\left[|\widetilde{Y}^i_s|^2\right]^{1/2}\right\} \int_{T-\tau}^T \mathbb{E}\left[|\delta_1 b^i(s)|^2\right]^{1/2} 1_{E_\varepsilon}(s) ds \\
&\le \sum_{i=1}^2 \frac{\delta}{2} \left\{\sup_{(T-\tau)\le t\le T} \mathbb{E}\left[|\widetilde{Y}^i_s|^2\right]\right\} + \frac{1}{2\delta}\left(\int_{T-\delta}^T \mathbb{E}\left[|\delta_1 b^i(s)|^2\right]^{1/2} 1_{E_\varepsilon}(s) ds\right)^2.
\end{aligned} \tag{B.6}$$

By assumption 5 and the definition of $\mathcal{U}^1$, we have for some $K_1 > 0$,

$$\frac{1}{2\delta}\left(\int_{T-\delta}^T \mathbb{E}\left[|\delta_1 b^i(s)|^2\right]^{1/2} 1_{E_\varepsilon}(s) ds\right)^2 \le K_1 \varepsilon^2 \tag{B.7}$$

Plugging (B.5) and (B.6) into (B.4) yields

$$\sup_{(T-\delta)\le t\le T} \mathbb{E}\left[(1-(K_0+1)\delta)\sum_{i=1}^2 |\widetilde{Y}^i_t|^2 + \frac{1}{2}\sum_{i,j=1}^2 \int_t^T \|\widetilde{Z}^{i,j}_s\|_F^2\, ds\right] \le K_1\varepsilon^2. \tag{B.8}$$

For $\delta < (K_0+1)^{-1}$, we conclude that

$$\sup_{(T-\delta)\le t\le T} \mathbb{E}\left[\sum_{i=1}^2 |\widetilde{Y}^i_t|^2 + \sum_{i,j=1}^2 \int_t^T \|\widetilde{Z}^{i,j}_s\|_F^2\, ds\right] \le K_2\varepsilon^2, \tag{B.9}$$

where $K_2 > 0$ depends on δ, the bound D, the Lipschitz coefficient of b^i and the integration bound in the definition of $\mathcal{U}^1$. The steps above can be repeated for the intervals $[T-2\delta, T-\delta]$, $[T-3\delta, T-2\delta]$, etc. until 0 is reached. After a finite number of iterations, we have

$$\sup_{0\le t\le T} \mathbb{E}\left[\sum_{i=1}^2 |\widetilde{Y}^i_t|^2 + \sum_{i,j=1}^2 \int_t^T \|\widetilde{Z}^{i,j}_s\|_F^2\, ds\right] \le K_3\varepsilon^2, \tag{B.10}$$

where K_3 depends on K_2 and T. This is the first estimate in (3.11). The second estimate follows from similar calculations.

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Lemma 3.2

Integration by parts yields

$$\mathbb{E}\left[\sum_{j=1}^{2}\widetilde{Y}_0^j p_0^{1,j}\right] = -\mathbb{E}\left[\int_0^T \sum_{j=1}^{2}\widetilde{Y}_t^j dp_t^{1,j} + p_t^{1,j} d\widetilde{Y}_t^j + d\langle \widetilde{Y}_t^j, p^{1,j}\rangle_t dt\right]. \tag{B.11}$$

Assume that $dp_t^{1,j} = \beta_t^j dt + \sigma_t^{j,1} dW_t^1 + \sigma_t^{j,2} dW_t^2$, then

$$\begin{aligned}
&\sum_{i=1}^{2}\widetilde{Y}_t^i dp_t^{1,i} + p_t^{1,i} d\widetilde{Y}_t^i + d\langle p^{1,i}, \widetilde{Y}^i\rangle_t = \sum_{i=1}^{2}\Bigg[\widetilde{Y}_t^i\left(\beta_t^i dt + \sigma_t^{i,1} dW_t^1 + \sigma_t^{i,2} dW_t^2\right) \\
&+ p_t^{1,i}\left(\sum_{j=1}^{2}\left\{\partial_{y^j}\hat{b}_t^i \widetilde{Y}_t^j + \mathbb{E}\left[(\partial_{\mu^j}\hat{b}_t^i)^* \widetilde{Y}_t^j\right] + \sum_{k=1}^{2}\partial_{z^{j,k}}\hat{b}_t^i \widetilde{Z}_t^{j,k}\right\} + \delta_1 b^i(t)\mathbb{I}_{E_\varepsilon}(t)\right) \\
&+ \sigma_t^{i,1}\widetilde{Z}_t^{i,1} + \sigma_t^{i,2}\widetilde{Z}_t^{i,2}\Bigg] dt + (\dots)dW_t^1 + (\dots)dW_t^2.
\end{aligned} \tag{B.12}$$

Thus the lemma is equivalent to that, under expectations, we have

$$\begin{aligned}
&-\mathbb{E}\Bigg[\int_0^T \Bigg\{\widetilde{Y}_t^1\beta_t^1 + \widetilde{Y}_t^2\beta_t^2 \\
&\quad + \widetilde{Y}_t^1\left\{p_t^{1,1}\partial_{y^1}\hat{b}_t^1 + p_t^{1,2}\partial_{y^1}\hat{b}_t^2 + \mathbb{E}\left[{}^*(\partial_{\mu^1}\hat{b}_t^1)p_t^{1,1}\right] + \mathbb{E}\left[{}^*(\partial_{\mu^1}\hat{b}_t^2)p_t^{1,2}\right]\right\} \\
&\quad + \widetilde{Y}_t^2\left\{p_t^{1,1}\partial_{y^2}\hat{b}_t^1 + p_t^{1,2}\partial_{y^2}\hat{b}_t^2 + \mathbb{E}\left[{}^*(\partial_{\mu^2}\hat{b}_t^1)p_t^{1,1}\right] + \mathbb{E}\left[{}^*(\partial_{\mu^2}\hat{b}_t^2)p_t^{1,2}\right]\right\} \\
&\quad + (p_t^{1,1}\partial_{z^{1,1}}\hat{b}_t^1 + p_t^{1,2}\partial_{z^{1,1}}\hat{b}_t^2 + \sigma_t^{1,1})\widetilde{Z}_t^{1,1} + (p_t^{1,1}\partial_{z^{1,2}}\hat{b}_t^1 + p_t^{1,2}\partial_{z^{1,2}}\hat{b}_t^2 + \sigma_t^{1,2})\widetilde{Z}_t^{1,2} \\
&\quad + (p_t^{1,1}\partial_{z^{2,1}}\hat{b}_t^1 + p_t^{1,2}\partial_{z^{2,1}}\hat{b}_t^2 + \sigma_t^{2,1})\widetilde{Z}_t^{2,1} + (p_t^{1,1}\partial_{z^{2,2}}\hat{b}_t^1 + p_t^{1,2}\partial_{z^{2,2}}\hat{b}_t^2 + \sigma_t^{2,2})\widetilde{Z}_t^{2,2} \\
&\quad + (p_t^{1,1}\delta_1 b^1(t) + p_t^{1,2}\delta_1 b^2(t))\mathbb{I}_{E_\varepsilon}(t)\Bigg\} dt\Bigg] \\
&= \mathbb{E}\left[\int_0^T \sum_{i=1}^{2}\left(\widetilde{Y}_t^i\left\{\partial_{y^i}\hat{f}_t^1 + \mathbb{E}\left[{}^*(\partial_{\mu^i}\hat{f}_t^1)\right]\right\} - p_t^{1,i}\delta_1 b^i(t)\mathbb{I}_{E_\varepsilon}(t)\right) dt\right].
\end{aligned} \tag{B.13}$$

We match coefficients and get

$$\begin{aligned}
\beta_t^j &= -\left(p_t^{1,1}\partial_{y^j}\hat{b}_t^1 + p_t^{1,2}\partial_{y^j}\hat{b}_t^2 + \mathbb{E}\left[{}^*(\partial_{\mu^i}\hat{b}_t^1)p_t^{1,1}\right] + \mathbb{E}\left[{}^*(\partial_{\mu^i}\hat{b}_t^2)p_t^{1,2}\right]\right) \\
&\quad + \partial_{y^j}\hat{b}_t^1 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{b}_t^1)\right] \\
&= -\left\{\partial_{y^j}\hat{H}_t^1 + \mathbb{E}\left[{}^*(\partial_{\mu^j}\hat{H}_t^1\right]\right\}, \\
\sigma_t^{j,k} &= -\left(p_t^{1,1}\partial_{z^{j,k}}\hat{b}_t^1 + p_t^{1,2}\partial_{z^{j,k}}\hat{b}_t^2\right).
\end{aligned} \tag{B.14}$$

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Linear-quadratic MFTG - derivation of ODE system

Under the ansatz, the adjoint equation is

$$
\begin{aligned}
dp_t^{i,i} &= \Big\{(q_{i,i}(t)+\widetilde{q}_{i,i}(t))\hat{Y}_t^i + (\widetilde{q}_{i,i}(t)+\bar{q}_{i,i}(t))\mathbb{E}[\hat{Y}_t^i] \\
&\qquad (s_{i,1}(t)+s_{i,2}(t)\bar{s}_{i,i}(t))\hat{Y}_t^{-i} + (\bar{s}_{i,-i}(t)+\bar{s}_{i,1}^E+\bar{s}_{i,2}^E(t))\mathbb{E}[\hat{Y}_t^{-i}]\Big\}dt \\
&=: \Big\{Q_i(t)\hat{Y}_t^i + \bar{Q}_i(t)\mathbb{E}[\hat{Y}_t^i] + S_i(t)\hat{Y}_t^{-i} + \bar{S}_i(t)\mathbb{E}[\hat{Y}_t^{-i}]\Big\}dt \\
&= \Big\{Q_i(t)\Big(\alpha_i(t)p_t^{i,i} + \bar{\alpha}_i(t)\mathbb{E}[p_t^{i,i}] + \beta_i(t)p_t^{-i,-i} + \bar{\beta}_i(t)\mathbb{E}[p_t^{-i,-i}] \\
&\qquad\qquad + \gamma_{i,1}(t)W_t^1 + \gamma_{i,2}W_t^2 + \theta_i(t)\Big) \\
&\quad + \bar{Q}_i(t)\Big((\alpha_i(t)+\bar{\alpha}_i(t))\mathbb{E}[p_t^{i,i}] + (\beta_i(t)+\bar{\beta}_i(t))\mathbb{E}[p_t^{-i,-i}] + \theta_i(t)\Big) \\
&\quad + S_i(t)\Big(\alpha_{-i}(t)p_t^{-i,-i} + \bar{\alpha}_{-i}(t)\mathbb{E}[p_t^{-i,-i}] + \beta_{-i}(t)p_t^{i,i} + \bar{\beta}_{-i}(t)\mathbb{E}[p_t^{i,i}] \\
&\qquad\qquad + \gamma_{-i,1}W_t^1 + \gamma_{-i,2}W_t^2 + \theta_{-i}(t)\Big) \\
&\quad + \bar{S}_i(t)\Big((\alpha_{-i}(t)+\bar{\alpha}_{-i}(t))\mathbb{E}[p_t^{-i,-i}] + (\beta_{-i}(t)+\bar{\beta}_{-i}(t))\mathbb{E}[p_t^{i,i}] + \theta_{-i}(t)\Big)\Big\}dt \\
&= \Big\{p_t^{i,i}\Big(Q_i(t)\alpha_i(t) + S_i(t)\beta_{-i}(t)\Big) \\
&\quad + \mathbb{E}[p_t^{i,i}]\Big(Q_i(t)\bar{\alpha}_i(t) + \bar{Q}_i(t)(\alpha_i(t)+\bar{\alpha}_i(t)) + S_i(t)\bar{\beta}_{-i}(t) + \bar{S}_i(t)(\beta_{-i}(t)+\bar{\beta}_{-i}(t))\Big) \\
&\quad + p_t^{-i,-i}\Big(Q_i(t)\beta_i(t) + S_i(t)\alpha_{-i}(t)\Big) \\
&\quad + \mathbb{E}[p_t^{-i,-i}]\Big(Q_i(t)\bar{\beta}_i(t) + \bar{Q}_i(t)(\beta_i(t)+\bar{\beta}_i(t)) + S_i(t)\bar{\alpha}_{-i}(t) + \bar{S}_i(t)(\alpha_{-i}(t)+\bar{\alpha}_{-i}(t))\Big) \\
&\quad + W_t^1(Q_i(t)\gamma_{i,1}(t) + S_i(t)\gamma_{i,2}(t)) + W_t^2(Q_i(t)\gamma_{i,2} + S_i(t)\gamma_{-i,2}) \\
&\quad + \theta_i(t)(Q_i(t)+\bar{Q}_i(t)) + \theta_{-i}(t)(S_i(t)+\bar{S}_i(t))\Big\}dt \\
&=: \Big\{p_t^{i,i}P^i(t) + \mathbb{E}[p_t^{i,i}]\bar{P}^i(t) + p_t^{-i,-i}R^i(t) + \mathbb{E}[p_t^{-i,-i}]\bar{R}^i(t) \\
&\quad + W_t^1\Phi^i(t) + W_t^2\Psi^i(t) + \theta_i(t)(Q_i(t)+\bar{Q}_i(t)) + \theta_{-i}(t)(S_i(t)+\bar{S}_i(t))\Big\}dt,
\end{aligned} \tag{B.15}
$$

and the expected value of $p_\cdot^{i,i}$ solves

$$
\begin{aligned}
d(\mathbb{E}[p_t^{i,i}]) = \Big\{&\mathbb{E}[p_t^{i,i}](P^i(t)+\bar{P}^i(t)) + \mathbb{E}[p_t^{-i,-i}](R^i(t)+\bar{R}^i(t)) \\
&+ \theta_i(t)(Q_i(t)+\bar{Q}_i(t)) + \theta_{-i}(t)(S_i(t)+\bar{S}_i(t))\Big\}dt.
\end{aligned} \tag{B.16}
$$

The initial conditions $p_0^{i,i}$, $\mathbb{E}[p_0^{i,i}]$, $p_0^{-i,-i}$, $\mathbb{E}[p_0^{-i,-i}]$ are given by a system of linear equations, which is derived is the same way as (B.15) and (B.16). Applying Ito's formula to the ansatz, and using (B.15)-(B.16), we get

$$
\begin{aligned}
d\hat{Y}_t^i &= \Big(\dot{\alpha}_i(t)p_t^{i,i} + \dot{\bar{\alpha}}_i(t)\mathbb{E}[p_t^{i,i}] + \dot{\beta}_i(t)p_t^{-i,-i} + \dot{\bar{\beta}}_i(t)\mathbb{E}[p_t^{-i,-i}] \\
&\quad + \dot{\gamma}_{i,1}(t)W_t^1 + \dot{\gamma}_{i,2}(t)W_t^2 + \dot{\theta}_i(t)\Big)dt \\
&\quad + \alpha_i(t)dp_t^{i,i} + \bar{\alpha}_i(t)d(\mathbb{E}[p_t^{i,i}]) + \beta_i(t)dp_t^{-i,-i} + \bar{\beta}_i(t)d(\mathbb{E}[p_t^{-i,-i}]) \\
&\quad + \gamma_{i,1}(t)dW_t^1 + \gamma_{i,2}(t)dW_t^2 \\
&= \Big\{p_t^{i,i}\Big(\dot{\alpha}_i(t) + \alpha_i(t)P^i(t) + \beta_i(t)R^i(t)\Big) \\
&\quad + \mathbb{E}[p_t^{i,i}]\Big(\dot{\bar{\alpha}}_i(t) + \alpha_i(t)\bar{P}^i(t) + \bar{\alpha}_i(t)(P^i(t) + \bar{P}^i(t)) \\
&\qquad + \beta_i(t)\bar{R}^{-i}(t) + \bar{\beta}_i(t)(R^{-i}(t) + \bar{R}^{-i}(t))\Big) \\
&\quad + p_t^{-i,-i}\Big(\dot{\beta}_i(t) + \alpha_i(t)R^i(t) + \beta_i(t)P^{-i}(t)\Big) \\
&\quad + \mathbb{E}[p_t^{-i,-i}]\Big(\dot{\bar{\beta}}_i(t) + \alpha_i(t)\bar{R}^i(t) + \bar{\alpha}_i(t)(R^i(t) + \bar{R}^i(t)) \\
&\qquad + \beta_i(t)\bar{P}^{-i}(t) + \bar{\beta}_i(t)(P^{-i}(t) + \bar{P}^{-i}(t))\Big) \\
&\quad + W_t^1\Big(\dot{\gamma}_{i,1} + \alpha_i(t)\Phi^i(t) + \beta_i(t)\Phi^{-i}(t)\Big) + W_t^2\Big(\dot{\gamma}_{i,2} + \alpha_i(t)\Psi^i(t) + \beta_i(t)\Psi^{-i}(t)\Big) \\
&\quad + \Big(\dot{\theta}_i(t) + \theta_i(t)\big((\alpha_i(t) + \bar{\alpha}_i(t))(Q_i(t) + \bar{Q}_i(t)) + (\beta_i(t) + \bar{\beta}_i(t))(S_{-i}(t) + \bar{S}_{-i}(t))\big) \\
&\qquad + \theta_{-i}\big((\alpha_i(t) + \bar{\alpha}_i(t))(S_i(t) + \bar{S}_i(t)) + (\beta_i(t) + \bar{\beta}_i(t))(Q_i(t) + \bar{Q}_i(t))\big)\Big)\Big\}dt \\
&\quad + \gamma_{i,1}(t)dW_t^1 + \gamma_{i,2}dW_t^2.
\end{aligned}
\tag{B.17}
$$

We can now match these dynamics with the true state dynamics and we get the system of ODEs (5.10) and $\gamma_{i,j}(t) = \hat{Z}_t^{i,j}$.

References

[1] Andersson, D. and Djehiche, B. [2011], 'A maximum principle for SDEs of mean-field type', *Applied Mathematics & Optimization* **63**(3), 341–356.

[2] Aurell, A. and Djehiche, B. [2018*a*], 'Mean-field type modeling of nonlocal crowd aversion in pedestrian crowd dynamics', *SIAM Journal on Control and Optimization* **56**(1), 434–455.

[3] Aurell, A. and Djehiche, B. [2018*b*], 'Modeling tagged pedestrian motion: a mean-field type control approach', *arXiv preprint arXiv:1801.08777* .

[4] Başar, T. and Zhu, Q. [2011], 'Prices of anarchy, information, and cooperation in differential games', *Dynamic Games and Applications* **1**(1), 50–73.

[5] Bensoussan, A., Frehse, J., Yam, P. et al. [2013], *Mean field games and mean field type control theory*, Vol. 101, Springer.

[6] Bismut, J.-M. [1973], 'Conjugate convex functions in optimal stochastic control', *Journal of Mathematical Analysis and Applications* **44**(2), 384–404.

[7] Buckdahn, R., Djehiche, B., Li, J., Peng, S. et al. [2009], 'Mean-field backward stochastic differential equations: a limit approach', *The Annals of Probability* **37**(4), 1524–1565.

[8] Buckdahn, R., Li, J. and Ma, J. [2016], 'A stochastic maximum principle for general mean-field systems', *Applied Mathematics & Optimization* **74**(3), 507–534.

[9] Buckdahn, R., Li, J. and Peng, S. [2009], 'Mean-field backward stochastic differential equations and related partial differential equations', *Stochastic Processes and their Applications* **119**(10), 3133–3154.

[10] Cardaliaguet, P. [2010], Notes on mean field games, Technical report.

[11] Cardaliaguet, P. and Rainer, C. [2018], 'On the (in) efficiency of MFG equilibria', *arXiv preprint arXiv:1802.06637* .

[12] Carmona, R. and Delarue, F. [2018], *Probabilistic Theory of Mean Field Games with Applications I-II*, Springer.

[13] Carmona, R., Graves, C. V. and Tan, Z. [2018], 'Price of anarchy for mean field games', *arXiv preprint arXiv:1802.04644* .

[14] Chen, Z. and Epstein, L. [2002], 'Ambiguity, risk, and asset returns in continuous time', *Econometrica* **70**(4), 1403–1443.

[15] Djehiche, B., Tcheukam, A. and Tembine, H. [2017], 'Mean-field-type games in engineering', *AIMS Electronics and Electrical Engineering* **1**(1), 18–73.

[16] Dubey, P. [1986], 'Inefficiency of nash equilibria', *Mathematics of Operations Research* **11**(1), 1–8.

[17] Duffie, D. and Epstein, L. G. [1992*a*], 'Asset pricing with stochastic differential utility', *The Review of Financial Studies* **5**(3), 411–436.

[18] Duffie, D. and Epstein, L. G. [1992*b*], 'Stochastic differential utility', *Econometrica: Journal of the Econometric Society* pp. 353–394.

[19] Duffie, D., Geoffard, P.-Y. and Skiadas, C. [1994], 'Efficient and equilibrium allocations with stochastic differential utility', *Journal of Mathematical Economics* **23**(2), 133–146.

[20] Duncan, T. E. and Tembine, H. [2018], 'Linear–quadratic mean-field-type games: A direct method', *Games* **9**(1), 7.

[21] El Karoui, N., Peng, S. and Quenez, M. C. [1997], 'Backward stochastic differential equations in finance', *Mathematical finance* **7**(1), 1–71.

[22] Koutsoupias, E. and Papadimitriou, C. [1999], Worst-case equilibria, *in* 'Annual Symposium on Theoretical Aspects of Computer Science', Springer, pp. 404–413.

[23] Koutsoupias, E. and Papadimitriou, C. [2009], 'Worst-case equilibria', *Computer science review* **3**(2), 65–69.

[24] Li, J. and Min, H. [2016], 'Controlled mean-field backward stochastic differential equations with jumps involving the value function', *Journal of Systems Science and Complexity* **29**(5), 1238–1268.

[25] Li, X., Sun, J. and Xiong, J. [2016], 'Linear quadratic optimal control problems for mean-field backward stochastic differential equations', *Applied Mathematics & Optimization* pp. 1–28.

[26] Papadimitriou, C. [2001], Algorithms, games, and the internet, *in* 'Proceedings of the thirty-third annual ACM symposium on Theory of computing', ACM, pp. 749–753.

[27] Pardoux, É. [1999], Bsdes, weak convergence and homogenization of semilinear PDEs, *in* 'Nonlinear analysis, differential equations and control', Springer, pp. 503–549.

[28] Pardoux, É. and Peng, S. [1990], 'Adapted solution of a backward stochastic differential

equation', *Systems & Control Letters* **14**(1), 55–61.

[29] Tang, M. and Meng, Q. [2016], 'Linear-quadratic optimal control problems for mean-field backward stochastic differential equations with jumps', *preprint arXiv:1611.06434* .

[30] Yong, J. [2010], 'Forward-backward stochastic differential equations with mixed initial-terminal conditions', *Transactions of the American Mathematical Society* **362**(2), 1047–1096.

[31] Zhang, J. [2017], *Backward Stochastic Differential Equations: From Linear to Fully Nonlinear Theory*, Vol. 86, Springer.