GENERAL THREE POINTS INEQUALITIES FOR WEIGHTED RIEMANN-STIELTJES INTEGRAL

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ABSTRACT. In this paper we provide amongst others some simple error bounds in approximating the weighted Riemann-Stieltjes integral $\int_a^b f(t) g(t) dv(t)$ by the use of three points formula

$$f(b) \int_c^b g(s) dv(s) + f(a) \int_a^d g(s) dv(s) - f(x) \int_c^d g(t) dv(t)$$

where $x, c, d \in [a, b]$, $g, v : [a, b] \rightarrow \mathbb{C}$ under bounded variation and Lipschitzian assumptions for the function f and such that the involved Riemann-Stieltjes integrals exist.

1. INTRODUCTION

Assume that $u, f : [a, b] \rightarrow \mathbb{C}$ are bounded. If the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ exists, we write for simplicity, like in [1, p. 142] that $f \in \mathcal{R}_{\mathbb{C}}(u, [a, b])$, or $\mathcal{R}_{\mathbb{C}}(u)$ when the interval is implicitly known. If the functions u, f are real valued, then we write $f \in \mathcal{R}(u, [a, b])$, or $\mathcal{R}(u)$.

In order to approximate the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$ by the use of a three points formula, namely to establish bounds for the *error functional*

$$T\Theta(f, u; a, b, x, \alpha) := (1 - \alpha) \{ [u(b) - u(x)] f(b) + [u(x) - u(a)] f(a) \} \\ + \alpha [u(b) - u(a)] f(x) - \int_a^b f(t) du(t),$$

where $\alpha \in [0, 1]$ and $x \in [a, b]$, under bounded variation assumptions for the functions u and f and such that the involved Riemann-Stieltjes integral exists, in the recent paper [26] we have obtained the following result:

Theorem 1. *Let $f, u : [a, b] \rightarrow \mathbb{C}$ and $x \in [a, b]$ are such that $f \in \mathcal{R}_{\mathbb{C}}(u, [a, b])$. If f and u are of bounded variation, then*

$$(1.1) \quad |T\Theta(f, u; a, b, x, \alpha)| \\ \leq \alpha \left[\int_a^x \left(\bigvee_a^t(u) \right) d \left(\bigvee_a^t(f) \right) + \int_x^b \left(\bigvee_x^b(u) \right) d \left(\bigvee_a^t(f) \right) \right] \\ + (1 - \alpha) \left[\int_a^x \left(\bigvee_t^x(u) \right) d \left(\bigvee_a^t(f) \right) + \int_x^b \left(\bigvee_x^t(u) \right) d \left(\bigvee_a^t(f) \right) \right]$$

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$$\begin{aligned}
&\leq \max \{ \alpha, 1 - \alpha \} \left(\bigvee_a^x (u) \bigvee_a^x (f) + \bigvee_x^b (u) \bigvee_x^b (f) \right) \\
&\leq \frac{1}{2} \max \{ \alpha, 1 - \alpha \} \left\{ \begin{aligned} &\left(\bigvee_a^b (u) + \left| \bigvee_a^x (u) - \bigvee_x^b (u) \right| \right) \bigvee_a^b (f) \\ &\left(\bigvee_a^b (f) + \left| \bigvee_a^x (f) - \bigvee_x^b (f) \right| \right) \bigvee_a^b (u) \end{aligned} \right\} \\
&\leq \max \{ \alpha, 1 - \alpha \} \bigvee_a^b (u) \bigvee_a^b (f).
\end{aligned}$$

In [27] we also obtained the following result in the case of Lipschitzian integrands:

Theorem 2. Let $f, u : [a, b] \rightarrow \mathbb{C}$ and $x \in [a, b]$. If f is Lipschitzian with the constant $L > 0$, namely

$$|f(t) - f(s)| \leq L |t - s| \text{ for all } t, s \in [a, b]$$

and u is of bounded variation, then $f \in \mathcal{R}_{\mathbb{C}}(u, [a, b])$ and

$$\begin{aligned}
(1.2) \quad &|T\Theta(f, u; a, b, x, \alpha)| \\
&\leq \alpha L \left[\int_a^x \left(\bigvee_a^t (u) \right) dt + \int_x^b \left(\bigvee_t^b (u) \right) dt \right] \\
&+ (1 - \alpha) L \left[\int_a^x \left(\bigvee_t^x (u) \right) dt + \int_x^b \left(\bigvee_x^t (u) \right) dt \right] \\
&\leq \max \{ \alpha, 1 - \alpha \} L \left((x - a) \bigvee_a^x (u) + (b - x) \bigvee_x^b (u) \right) \\
&\leq \frac{1}{2} \max \{ \alpha, 1 - \alpha \} L \left\{ \begin{aligned} &\left(\bigvee_a^b (u) + \left| \bigvee_a^x (u) - \bigvee_x^b (u) \right| \right) (b - a) \\ &(b - a + |2x - a - b|) \bigvee_a^b (u) \end{aligned} \right\} \\
&\leq \max \{ \alpha, 1 - \alpha \} L (b - a) \bigvee_a^b (u).
\end{aligned}$$

For various bounds on the error functional

$$D(f, u; a, b) := \int_a^b f(t) du(t) - \frac{1}{b - a} [u(b) - u(a)] \cdot \int_a^b f(t) dt$$

where f and u belong to different classes of function for which the Riemann-Stieltjes integral exists, see [22], [21], [20], and [8] and the references therein.

Bounds for the functional

$$\Theta(f, u; a, b, x) := \int_a^b f(t) du(t) - f(x) [u(b) - u(a)]$$

can be found in [15], [16] and [8], while for the functional

$$T(f, u; a, b, x) := \int_a^b f(t) du(t) - [u(b) - u(x)] f(b) - [u(x) - u(a)] f(a)$$

they may be found in [28], [8], [3] and [2]. The details are omitted.

In this paper we provide some simple error bounds in approximating the weighted Riemann-Stieltjes integral $\int_a^b f(t) g(t) dv(t)$ by the use of various general three points formulae out of which we mention the following one

$$f(b) \int_c^b g(s) dv(s) + f(a) \int_a^d g(s) dv(s) - f(x) \int_c^d g(t) dv(t),$$

where $x, c, d \in [a, b]$, $g, v : [a, b] \rightarrow \mathbb{C}$ under bounded variation and Lipschitzian assumptions for the function f and such that the involved Riemann-Stieltjes integral exist.

2. SOME PRELIMINARY FACTS

The following properties of Riemann-Stieltjes integral are well known, [1, p. 158-159]:

Lemma 1. Assume that $f \in \mathcal{R}_{\mathbb{C}}(\alpha, [a, b])$ and $g \in \mathcal{R}_{\mathbb{C}}(\alpha, [a, b])$ where $\alpha \in \mathcal{BV}_{\mathbb{C}}[a, b]$, namely of bounded variation on $[a, b]$. Define

$$F(x) := \int_a^x f(t) d\alpha(t)$$

and

$$G(x) := \int_a^x g(t) d\alpha(t)$$

where $x \in [a, b]$.

Then $f \in \mathcal{R}_{\mathbb{C}}(G, [a, b])$, $g \in \mathcal{R}_{\mathbb{C}}(F, [a, b])$, $fg \in \mathcal{R}_{\mathbb{C}}(\alpha, [a, b])$ and we have

$$\int_a^b f(t) g(t) d\alpha(t) = \int_a^b f(t) dG(t) = \int_a^b g(t) dF(t).$$

If $c \in [a, b]$ and consider the integral $\int_c^t g(s) dv(s)$ that is assumed to exist for any $t \in [a, b]$, then

$$(2.1) \quad \int_c^t g(s) dv(s) = \int_a^t g(s) dv(s) - \int_a^c g(s) dv(s)$$

for any $t \in [a, b]$.

Indeed if $t \in [c, b]$, then (2.1) is obvious. If $t \in [a, c]$, then

$$\int_a^c g(s) dv(s) = \int_a^t g(s) dv(s) + \int_t^c g(s) dv(s),$$

which also gives (2.1).

We start with the following simple fact:

Lemma 2. Let $f, g, v : [a, b] \rightarrow \mathbb{C}$, $\lambda, \mu \in \mathbb{C}$ and $x, c, d \in [a, b]$. If $g, fg \in \mathcal{R}_{\mathbb{C}}(v, [a, b])$, then

$$\begin{aligned}
 (2.2) \quad & \left(\int_d^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \\
 & + \left(\int_c^d g(t) dv(t) + \mu - \lambda \right) f(x) - \int_a^b f(t) g(t) dv(t) \\
 & = \int_a^x \left(\int_c^t g(s) dv(s) - \lambda \right) df(t) + \int_x^b \left(\int_d^t g(s) dv(s) - \mu \right) df(t).
 \end{aligned}$$

In particular, for $\mu = \lambda$ we have

$$\begin{aligned}
 (2.3) \quad & \left(\int_d^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \\
 & + f(x) \int_c^d g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \\
 & = \int_a^x \left(\int_c^t g(s) dv(s) - \lambda \right) df(t) + \int_x^b \left(\int_d^t g(s) dv(s) - \lambda \right) df(t).
 \end{aligned}$$

Proof. Assume that $x, c, d \in [a, b]$. Using the integration by parts formula for the Riemann-Stieltjes integral and Lemma 1, we have

$$\begin{aligned}
 (2.4) \quad & \int_a^x \left(\int_c^t g(s) dv(s) - \lambda \right) df(t) \\
 & = \left(\int_c^t g(t) dv(t) - \lambda \right) f(t) \Big|_a^x - \int_a^x f(t) d \left(\int_c^t g(s) dv(s) - \lambda \right) \\
 & = \left(\int_c^x g(t) dv(t) - \lambda \right) f(x) - \left(\int_c^a g(t) dv(t) - \lambda \right) f(a) - \int_a^x f(t) g(t) dv(t) \\
 & = \left(\int_c^x g(t) dv(t) - \lambda \right) f(x) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) - \int_a^x f(t) g(t) dv(t).
 \end{aligned}$$

In a similar way,

$$\begin{aligned}
 (2.5) \quad & \int_x^b \left(\int_d^t g(s) dv(s) - \mu \right) df(t) \\
 & = \left(\int_d^t g(s) dv(s) - \mu \right) f(t) \Big|_x^b - \int_x^b f(t) d \left(\int_d^t g(s) dv(s) - \mu \right) \\
 & = \left(\int_d^b g(s) dv(s) - \mu \right) f(b) - \left(\int_d^x g(s) dv(s) - \mu \right) f(x) - \int_x^b f(t) g(t) dv(t) \\
 & = \left(\int_d^b g(s) dv(s) - \mu \right) f(b) + \left(\mu - \int_d^x g(s) dv(s) \right) f(x) - \int_x^b f(t) g(t) dv(t).
 \end{aligned}$$

If we add (2.4) and (2.5), we get

$$\begin{aligned} & \int_a^x \left(\int_c^t g(s) dv(s) - \lambda \right) df(t) + \int_x^b \left(\int_d^t g(s) dv(s) - \mu \right) df(t) \\ &= \left(\int_c^x g(t) dv(t) - \lambda \right) f(x) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \\ &+ \left(\int_d^b g(s) dv(s) - \mu \right) f(b) + \left(\mu - \int_d^x g(s) dv(s) \right) f(x) - \int_a^b f(t) g(t) dv(t), \end{aligned}$$

which is equivalent to the desired result (2.2) $\square$

If we take $d = c$ above, we get:

Corollary 1. Let $f, g, v, f : [a, b] \rightarrow \mathbb{C}$, $\lambda, \mu \in \mathbb{C}$ and $x, c \in [a, b]$. If $g, fg \in \mathcal{R}_{\mathbb{C}}(v, [a, b])$, then

$$\begin{aligned} (2.6) \quad & \left(\int_c^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \\ &+ (\mu - \lambda) f(x) - \int_a^b f(t) g(t) dv(t) \\ &= \int_a^x \left(\int_c^t g(s) dv(s) - \lambda \right) df(t) + \int_x^b \left(\int_c^t g(s) dv(s) - \mu \right) df(t). \end{aligned}$$

In particular, for $\mu = \lambda$ we have

$$\begin{aligned} (2.7) \quad & \left(\int_c^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \\ &- \int_a^b f(t) g(t) dv(t) = \int_a^b \left(\int_c^t g(s) dv(s) - \lambda \right) df(t). \end{aligned}$$

Remark 1. If we take $g(t) = 1$, $t \in [a, b]$ in (2.2) and (2.3), then we get

$$\begin{aligned} (2.8) \quad & (v(b) - v(d) - \mu) f(b) + (\lambda + v(c) - v(a)) f(a) \\ &+ (v(d) + \mu - \lambda - v(c)) f(x) - \int_a^b f(t) dv(t) \\ &= \int_a^x (v(t) - v(c) - \lambda) df(t) + \int_x^b (v(t) - v(d) - \mu) df(t). \end{aligned}$$

If we take $\beta = v(d) + \mu$ and $\gamma = \lambda + v(c)$, then by (2.8) we get

$$\begin{aligned} (2.9) \quad & (v(b) - \beta) f(b) + (\gamma - v(a)) f(a) + (\beta - \gamma) f(x) - \int_a^b f(t) dv(t) \\ &= \int_a^x (v(t) - \gamma) df(t) + \int_x^b (v(t) - \beta) df(t). \end{aligned}$$

In particular, for $\beta = \gamma$ we get by (2.9) that

$$(2.10) \quad (v(b) - \beta) f(b) + (\beta - v(a)) f(a) - \int_a^b f(t) dv(t) = \int_a^b (v(t) - \beta) df(t).$$

Remark 2. If we take $c = b$ and $d = a$ in Lemma 2, then we get

$$\begin{aligned}
 (2.11) \quad & \left(\int_a^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^b g(t) dv(t) \right) f(a) \\
 & + \left(\mu - \lambda - \int_a^b g(t) dv(t) \right) f(x) - \int_a^b f(t) g(t) dv(t) \\
 & = \int_x^b \left(\int_a^t g(s) dv(s) - \mu \right) df(t) - \int_a^x \left(\int_t^b g(s) dv(s) + \lambda \right) df(t).
 \end{aligned}$$

In particular, for $\mu = \lambda$ we obtain

$$\begin{aligned}
 (2.12) \quad & \left(\int_a^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^b g(t) dv(t) \right) f(a) \\
 & - f(x) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \\
 & = \int_x^b \left(\int_a^t g(s) dv(s) - \lambda \right) df(t) - \int_a^x \left(\int_t^b g(s) dv(s) + \lambda \right) df(t).
 \end{aligned}$$

If we take $c = a$ and $d = b$ in Lemma 2, then we get

$$\begin{aligned}
 (2.13) \quad & \lambda f(a) - \mu f(b) + (\mu - \lambda) f(x) - \int_a^b f(t) g(t) dv(t) \\
 & = \int_a^x \left(\int_a^t g(s) dv(s) - \lambda \right) df(t) - \int_x^b \left(\int_t^b g(s) dv(s) + \mu \right) df(t).
 \end{aligned}$$

In particular, for $\mu = \lambda$ we obtain

$$\begin{aligned}
 (2.14) \quad & \lambda f(a) - \lambda f(b) - \int_a^b f(t) g(t) dv(t) \\
 & = \int_a^x \left(\int_a^t g(s) dv(s) - \lambda \right) df(t) - \int_x^b \left(\int_t^b g(s) dv(s) + \lambda \right) df(t).
 \end{aligned}$$

Remark 3. If we take $g(t) = 1$, $t \in [a, b]$ in (2.11), then we get

$$\begin{aligned}
 (2.15) \quad & (v(b) - v(a) - \mu) f(b) + (\lambda + v(b) - v(a)) f(a) \\
 & + (\mu - \lambda - v(b) + v(a)) f(x) - \int_a^b f(t) dv(t) \\
 & = \int_x^b (v(t) - v(a) - \mu) df(t) - \int_a^x (v(b) - v(t) + \lambda) df(t).
 \end{aligned}$$

If this equality we take $\mu = \lambda$, then we get

$$\begin{aligned}
 (2.16) \quad & (v(b) - v(a) - \lambda) f(b) + (\lambda + v(b) - v(a)) f(a) \\
 & - (v(b) - v(a)) f(x) - \int_a^b f(t) dv(t) \\
 & = \int_x^b (v(t) - v(a) - \lambda) df(t) - \int_a^x (v(b) - v(t) + \lambda) df(t).
 \end{aligned}$$

Remark 4. If we take $\mu = -\int_a^d g(s) dv(s)$ and $\lambda = \int_c^b g(t) dv(t)$ in (2.2), then we get

$$\begin{aligned} & \int_a^b g(s) dv(s) (f(b) + f(a)) \\ & + \left(\int_c^x g(t) dv(t) - \int_d^x g(s) dv(s) - \int_a^d g(s) dv(s) - \int_c^b g(t) dv(t) \right) f(x) \\ & \quad - \int_a^b f(t) g(t) dv(t) \\ & = \int_a^x \left(\int_c^t g(s) dv(s) - \int_c^b g(t) dv(t) \right) df(t) \\ & \quad + \int_x^b \left(\int_d^t g(s) dv(s) + \int_a^d g(s) dv(s) \right) df(t), \end{aligned}$$

which is equivalent to

$$\begin{aligned} (2.17) \quad & [f(b) + f(a) - f(x)] \int_a^b g(s) dv(s) - \int_a^b f(t) g(t) dv(t) \\ & = \int_x^b \left(\int_a^t g(s) dv(s) \right) df(t) - \int_a^x \left(\int_t^b g(s) dv(s) \right) df(t). \end{aligned}$$

If we take $\lambda = \int_c^d g(t) dv(t)$ and $\mu = -\int_c^d g(t) dv(t)$, then by (2.2) we get

$$\begin{aligned} & \left(\int_d^b g(s) dv(s) + \int_c^d g(t) dv(t) \right) f(b) \\ & \quad + \left(\int_c^d g(t) dv(t) + \int_a^c g(t) dv(t) \right) f(a) \\ & + \left(\int_c^x g(t) dv(t) - \int_d^x g(s) dv(s) - \int_c^d g(t) dv(t) - \int_c^d g(t) dv(t) \right) f(x) \\ & \quad - \int_a^b f(t) g(t) dv(t) \\ & = \int_a^x \left(\int_c^t g(s) dv(s) - \int_c^d g(t) dv(t) \right) df(t) \\ & \quad + \int_x^b \left(\int_d^t g(s) dv(s) + \int_c^d g(t) dv(t) \right) df(t), \end{aligned}$$

which is equivalent to

$$\begin{aligned}
 (2.18) \quad & f(b) \int_c^b g(s) dv(s) + f(a) \int_a^d g(t) dv(t) - f(x) \int_c^d g(t) dv(t) \\
 & - \int_a^b f(t) g(t) dv(t) \\
 & = \int_a^x \left(\int_d^t g(s) dv(s) \right) df(t) + \int_x^b \left(\int_c^t g(s) dv(s) \right) df(t).
 \end{aligned}$$

Remark 5. If we take $\mu = \int_d^b g(s) dv(s)$ and $\lambda = -\int_a^c g(t) dv(t)$ in (2.2), then we get

$$\begin{aligned}
 & \left(\int_c^x g(t) dv(t) + \int_x^d g(s) dv(s) + \int_d^b g(s) dv(s) + \int_a^c g(t) dv(t) \right) f(x) \\
 & - \int_a^b f(t) g(t) dv(t) \\
 & = \int_a^x \left(\int_c^t g(s) dv(s) + \int_a^c g(t) dv(t) \right) df(t) \\
 & + \int_x^b \left(\int_d^t g(s) dv(s) - \int_d^b g(s) dv(s) \right) df(t),
 \end{aligned}$$

which is equivalent to

$$\begin{aligned}
 (2.19) \quad & f(x) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \\
 & = \int_a^x \left(\int_a^t g(s) dv(s) \right) df(t) - \int_x^b \left(\int_t^b g(s) dv(s) \right) df(t).
 \end{aligned}$$

3. INEQUALITIES FOR INTEGRANDS OF BOUNDED VARIATION

We have:

Theorem 3. Assume that $f : [a, b] \rightarrow \mathbb{C}$ is of bounded variation, $g, v : [a, b] \rightarrow \mathbb{C}$, $\lambda, \mu \in \mathbb{C}$ and $x, c, d \in [a, b]$ are such that the Riemann-Stieltjes integrals below exist. Then

$$\begin{aligned}
 (3.1) \quad & \left| \left(\int_d^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\
 & \left. + \left(\int_c^d g(t) dv(t) + \mu - \lambda \right) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \mu \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \mu \right| \right\} \bigvee_a^b(f).
 \end{aligned}$$

In particular, for $\mu = \lambda$ we have

$$\begin{aligned}
 (3.2) \quad & \left| \left(\int_d^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + f(x) \int_c^d g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \lambda \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \lambda \right| \right\} \bigvee_a^b(f).
 \end{aligned}$$

Proof. It is well known that, if $p : [a, b] \rightarrow \mathbb{C}$ is continuous and $v : [a, b] \rightarrow \mathbb{C}$ of bounded variation, then [1]

$$(3.3) \quad \left| \int_a^b p(t) dv(t) \right| \leq \int_a^b |p(t)| d \left(\bigvee_a^t(v) \right) \leq \max_{t \in [a, b]} |p(t)| \bigvee_a^b(v).$$

By using the identity (2.2) and the property (3.3) we get

$$\begin{aligned}
 & \left| \left(\int_d^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + \left(\int_c^d g(t) dv(t) + \mu - \lambda \right) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \left| \int_a^x \left(\int_c^t g(s) dv(s) - \lambda \right) df(t) \right| + \left| \int_x^b \left(\int_d^t g(s) dv(s) - \mu \right) df(t) \right| \\
 & \leq \int_a^x \left| \int_c^t g(s) dv(s) - \lambda \right| d \left(\bigvee_a^t(f) \right) + \int_x^b \left| \int_d^t g(s) dv(s) - \mu \right| d \left(\bigvee_x^t(f) \right) \\
 & \leq \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \mu \right| \bigvee_x^b(f),
 \end{aligned}$$

which proves the first inequality in (3.1).

Observe that

$$\begin{aligned}
 & \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \mu \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \mu \right| \right\} \left(\bigvee_a^x(f) + \bigvee_x^b(f) \right) \\
 & = \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \mu \right| \right\} \bigvee_a^b(f),
 \end{aligned}$$

which proves the last part of (3.1). $\square$

Remark 6. If $m \in (a, b)$ is such that $\bigvee_a^m(f) = \bigvee_m^b(f)$, then under the assumptions of Theorem 3 we have the inequalities

$$(3.4) \quad \left| \left(\int_d^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\ \left. + \left(\int_c^d g(t) dv(t) + \mu - \lambda \right) f(m) - \int_a^b f(t) g(t) dv(t) \right| \\ \leq \frac{1}{2} \left[\max_{t \in [a, m]} \left| \int_c^t g(s) dv(s) - \lambda \right| + \max_{t \in [m, b]} \left| \int_d^t g(s) dv(s) - \mu \right| \right] \bigvee_a^b(f)$$

and

$$(3.5) \quad \left| \left(\int_d^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\ \left. + f(m) \int_c^d g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\ \leq \frac{1}{2} \left[\max_{t \in [a, m]} \left| \int_c^t g(s) dv(s) - \lambda \right| + \max_{t \in [m, b]} \left| \int_d^t g(s) dv(s) - \lambda \right| \right] \bigvee_a^b(f).$$

Corollary 2. With the assumptions of Theorem 3, and if $d = c$, then

$$(3.6) \quad \left| \left(\int_c^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\ \left. + (\mu - \lambda) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\ \leq \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) - \mu \right| \bigvee_x^b(f) \\ \leq \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) - \mu \right| \right\} \bigvee_a^b(f).$$

In particular, for $\mu = \lambda$, we get

$$(3.7) \quad \left| \left(\int_c^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\ \left. - \int_a^b f(t) g(t) dv(t) \right| \\ \leq \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) - \lambda \right| \bigvee_x^b(f) \\ \leq \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) - \lambda \right| \right\} \bigvee_a^b(f).$$

Remark 7. If we take $c = x$ in Corollary 2, then we get

$$\begin{aligned}
 (3.8) \quad & \left| \left(\int_x^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^x g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + (\mu - \lambda) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_x^t g(s) dv(s) - \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) - \mu \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_x^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) - \mu \right| \right\} \bigvee_a^b(f).
 \end{aligned}$$

In particular, for $\mu = \lambda$, we get

$$\begin{aligned}
 (3.9) \quad & \left| \left(\int_x^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^x g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_x^t g(s) dv(s) - \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) - \lambda \right| \bigvee_x^b(f) \\
 & \leq \max_{t \in [a, b]} \left| \int_x^t g(s) dv(s) - \lambda \right| \bigvee_a^b(f).
 \end{aligned}$$

Corollary 3. With the assumptions of Theorem 3, and if $c = b$ and $d = a$, then

$$\begin{aligned}
 (3.10) \quad & \left| \left(\int_a^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^b g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + \left(\mu - \lambda - \int_a^b g(t) dv(t) \right) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) + \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) - \mu \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) + \lambda \right|, \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) - \mu \right| \right\} \bigvee_a^b(f).
 \end{aligned}$$

In particular, for $\mu = \lambda$ we have

$$\begin{aligned}
 (3.11) \quad & \left| \left(\int_a^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^b g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. - \int_a^b g(t) dv(t) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) + \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) - \lambda \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) + \lambda \right|, \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) - \lambda \right| \right\} \bigvee_a^b(f).
 \end{aligned}$$

Corollary 4. With the assumptions of Theorem 3 and if the Riemann-Stieltjes integrals below exist, then

$$\begin{aligned}
 (3.12) \quad & \left| f(b) \int_c^b g(s) dv(s) + f(a) \int_a^d g(t) dv(t) - f(x) \int_c^d g(t) dv(t) \right. \\
 & \quad \left. - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_d^t g(s) dv(s) \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_d^t g(s) dv(s) \right|, \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) \right| \right\} \bigvee_a^b(f).
 \end{aligned}$$

In particular, for $c = a$ and $d = b$, we have

$$\begin{aligned}
 (3.13) \quad & \left| [f(b) + f(a) - f(x)] \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) \right|, \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) \right| \right\} \bigvee_a^b(f)
 \end{aligned}$$

for $c = b$ and $d = a$, we have

$$\begin{aligned}
 (3.14) \quad & \left| f(x) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_a^t g(s) dv(s) \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_t^b g(s) dv(s) \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_a^t g(s) dv(s) \right|, \max_{t \in [x, b]} \left| \int_t^b g(s) dv(s) \right| \right\} \bigvee_a^b(f)
 \end{aligned}$$

and for $c = d = x$

$$\begin{aligned}
 (3.15) \quad & \left| f(b) \int_x^b g(s) dv(s) + f(a) \int_a^x g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_t^x g(s) dv(s) \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) \right| \bigvee_x^b(f) \\
 & \leq \max_{t \in [a, b]} \left| \int_t^x g(s) dv(s) \right| \bigvee_a^b(f).
 \end{aligned}$$

Remark 8. If $m \in (a, b)$ is such that $\bigvee_a^m(f) = \bigvee_m^b(f)$, then under the assumptions of Corollary 4, we have

$$\begin{aligned}
 (3.16) \quad & \left| f(b) \int_c^b g(s) dv(s) + f(a) \int_a^d g(t) dv(t) - f(m) \int_c^d g(t) dv(t) \right. \\
 & \quad \left. - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} \left[\max_{t \in [a, m]} \left| \int_d^t g(s) dv(s) \right| + \max_{t \in [m, b]} \left| \int_c^t g(s) dv(s) \right| \right] \bigvee_a^b(f),
 \end{aligned}$$

$$\begin{aligned}
 (3.17) \quad & \left| [f(b) + f(a) - f(m)] \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} \left[\max_{t \in [a, m]} \left| \int_t^b g(s) dv(s) \right| + \max_{t \in [m, b]} \left| \int_a^t g(s) dv(s) \right| \right] \bigvee_a^b(f),
 \end{aligned}$$

$$\begin{aligned}
 (3.18) \quad & \left| f(m) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} \max \left\{ \max_{t \in [a, m]} \left| \int_a^t g(s) dv(s) \right|, \max_{t \in [m, b]} \left| \int_t^b g(s) dv(s) \right| \right\} \bigvee_a^b(f)
 \end{aligned}$$

and

$$\begin{aligned}
 (3.19) \quad & \left| f(b) \int_m^b g(s) dv(s) + f(a) \int_a^m g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} \left[\max_{t \in [a, m]} \left| \int_t^m g(s) dv(s) \right| + \max_{t \in [m, b]} \left| \int_m^t g(s) dv(s) \right| \right] \bigvee_a^b(f).
 \end{aligned}$$

Using the equalities (2.9) and (2.10) one can obtain various inequalities as in the recent paper [26]. The details are omitted.

4. INEQUALITIES FOR LIPSCHITZIAN INTEGRANDS

We say that the function $f : [a, b] \rightarrow \mathbb{C}$ is *Lipschitzian* with constant $L > 0$ if

$$(4.1) \quad |f(t) - f(s)| \leq L|t - s| \text{ for all } t, s \in [a, b].$$

Theorem 4. Assume that $f : [a, b] \rightarrow \mathbb{C}$ is Lipschitzian with constant $L > 0$, $g, v : [a, b] \rightarrow \mathbb{C}$, $\lambda, \mu \in \mathbb{C}$ and $x, c, d \in [a, b]$ are such that the Riemann-Stieltjes integrals below exist. Then

$$(4.2) \quad \left| \left(\int_d^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\ \left. + \left(\int_c^d g(t) dv(t) + \mu - \lambda \right) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\ \leq L \left[\max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| (x - a) + \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \mu \right| (b - x) \right] \\ \leq L \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \mu \right| \right\} (b - a).$$

In particular, for $\mu = \lambda$ we have

$$(4.3) \quad \left| \left(\int_d^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\ \left. + f(x) \int_c^d g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\ \leq L \left[\max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| (x - a) + \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \lambda \right| (b - x) \right] \\ \leq L \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \lambda \right| \right\} (b - a).$$

Proof. It is well known that, if $p : [a, b] \rightarrow \mathbb{C}$ is Riemann integrable and $v : [a, b] \rightarrow \mathbb{C}$ is Lipschitzian with the constant $L > 0$, then

$$(4.4) \quad \left| \int_a^b p(t) dv(t) \right| \leq L \int_a^b |p(t)| dt.$$

By using the identity (2.2) and the property (4.4) we get

$$\begin{aligned}
 & \left| \left(\int_d^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + \left(\int_c^d g(t) dv(t) + \mu - \lambda \right) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \left| \int_a^x \left(\int_c^t g(s) dv(s) - \lambda \right) df(t) \right| + \left| \int_x^b \left(\int_d^t g(s) dv(s) - \mu \right) df(t) \right| \\
 & \leq L \int_a^x \left| \int_c^t g(s) dv(s) - \lambda \right| dt + L \int_x^b \left| \int_d^t g(s) dv(s) - \mu \right| dt \\
 & \leq L \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| (x - a) + \max_{t \in [x, b]} \left| \int_d^t g(s) dv(s) - \mu \right| (b - x),
 \end{aligned}$$

which proves the first inequality in (4.2).

The rest is obvious. $\square$

Remark 9. For $x = \frac{a+b}{2}$ in (4.2) and (4.3) we get, under the assumptions of Theorem 4, that

$$\begin{aligned}
 (4.5) \quad & \left| \left(\int_d^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + \left(\int_c^d g(t) dv(t) + \mu - \lambda \right) f\left(\frac{a+b}{2}\right) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} L \left[\max_{t \in [a, \frac{a+b}{2}]} \left| \int_c^t g(s) dv(s) - \lambda \right| + \max_{t \in [\frac{a+b}{2}, b]} \left| \int_d^t g(s) dv(s) - \mu \right| \right] (b - a).
 \end{aligned}$$

In particular, for $\mu = \lambda$ we have

$$\begin{aligned}
 (4.6) \quad & \left| \left(\int_d^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + f\left(\frac{a+b}{2}\right) \int_c^d g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} L \left[\max_{t \in [a, \frac{a+b}{2}]} \left| \int_c^t g(s) dv(s) - \lambda \right| + \max_{t \in [\frac{a+b}{2}, b]} \left| \int_d^t g(s) dv(s) - \lambda \right| \right] (b - a).
 \end{aligned}$$

Corollary 5. *With the assumptions of Theorem 4, and if $d = c$, then*

$$\begin{aligned}
 (4.7) \quad & \left| \left(\int_c^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + (\mu - \lambda) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) - \mu \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) - \mu \right| \right\} \bigvee_a^b(f).
 \end{aligned}$$

In particular, for $\mu = \lambda$, we get

$$\begin{aligned}
 (4.8) \quad & \left| \left(\int_c^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^c g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right| (x - a) + \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) - \lambda \right| (b - x) \right] \\
 & \leq L \max \left\{ \max_{t \in [a, x]} \left| \int_c^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) - \lambda \right| \right\} (b - a).
 \end{aligned}$$

Remark 10. *If we take $c = x$ in Corollary 5, then we get*

$$\begin{aligned}
 (4.9) \quad & \left| \left(\int_x^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^x g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + (\mu - \lambda) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_x^t g(s) dv(s) - \lambda \right| (x - a) + \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) - \mu \right| (b - x) \right] \\
 & \leq L \max \left\{ \max_{t \in [a, x]} \left| \int_x^t g(s) dv(s) - \lambda \right|, \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) - \mu \right| \right\} (b - a).
 \end{aligned}$$

In particular, for $\mu = \lambda$, we get

$$\begin{aligned}
 (4.10) \quad & \left| \left(\int_x^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^x g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_x^t g(s) dv(s) - \lambda \right| (x - a) + \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) - \lambda \right| (b - x) \right] \\
 & \leq L \max_{t \in [a, b]} \left| \int_x^t g(s) dv(s) - \lambda \right| (b - a).
 \end{aligned}$$

Corollary 6. With the assumptions of Theorem 4, and if $c = b$ and $d = a$, then

$$\begin{aligned}
 (4.11) \quad & \left| \left(\int_a^b g(s) dv(s) - \mu \right) f(b) + \left(\lambda + \int_a^b g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. + \left(\mu - \lambda - \int_a^b g(t) dv(t) \right) f(x) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) + \lambda \right| (x - a) + \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) - \mu \right| (b - x) \right] \\
 & \leq L \max \left\{ \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) + \lambda \right|, \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) - \mu \right| \right\} (b - a).
 \end{aligned}$$

In particular, for $\mu = \lambda$ we have

$$\begin{aligned}
 (4.12) \quad & \left| \left(\int_a^b g(s) dv(s) - \lambda \right) f(b) + \left(\lambda + \int_a^b g(t) dv(t) \right) f(a) \right. \\
 & \quad \left. - f(x) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) + \lambda \right| (x - a) + \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) - \lambda \right| (b - x) \right] \\
 & \leq L \max \left\{ \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) + \lambda \right|, \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) - \lambda \right| \right\} (b - a).
 \end{aligned}$$

Corollary 7. *With the assumptions of Theorem 4 and if the Riemann-Stieltjes integrals below exist, then*

$$\begin{aligned}
 (4.13) \quad & \left| f(b) \int_c^b g(s) dv(s) + f(a) \int_a^d g(t) dv(t) - f(x) \int_c^d g(t) dv(t) \right. \\
 & \quad \left. - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_d^t g(s) dv(s) \right| (x - a) + \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) \right| (b - x) \right] \\
 & \leq L \max \left\{ \max_{t \in [a, x]} \left| \int_d^t g(s) dv(s) \right|, \max_{t \in [x, b]} \left| \int_c^t g(s) dv(s) \right| \right\} (b - a).
 \end{aligned}$$

In particular, for $c = a$ and $d = b$, we have

$$\begin{aligned}
 (4.14) \quad & \left| [f(b) + f(a) - f(x)] \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) \right| \bigvee_x^b(f) \\
 & \leq \max \left\{ \max_{t \in [a, x]} \left| \int_t^b g(s) dv(s) \right|, \max_{t \in [x, b]} \left| \int_a^t g(s) dv(s) \right| \right\} \bigvee_a^b(f)
 \end{aligned}$$

for $c = b$ and $d = a$, we have

$$\begin{aligned}
 (4.15) \quad & \left| f(x) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_a^t g(s) dv(s) \right| (x - a) + \max_{t \in [x, b]} \left| \int_t^b g(s) dv(s) \right| (b - x) \right] \\
 & \leq L \max \left\{ \max_{t \in [a, x]} \left| \int_a^t g(s) dv(s) \right|, \max_{t \in [x, b]} \left| \int_t^b g(s) dv(s) \right| \right\} (b - a)
 \end{aligned}$$

and for $c = d = x$ we get

$$\begin{aligned}
 (4.16) \quad & \left| f(b) \int_x^b g(s) dv(s) + f(a) \int_a^x g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_t^x g(s) dv(s) \right| (x - a) + \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) \right| (b - x) \right] \\
 & \leq L \max_{t \in [a, b]} \left| \int_t^x g(s) dv(s) \right| (b - a).
 \end{aligned}$$

Remark 11. If we take $x = \frac{a+b}{2}$, then under the assumptions of Corollary 7, we have

$$(4.17) \quad \left| f(b) \int_c^b g(s) dv(s) + f(a) \int_a^d g(t) dv(t) - f\left(\frac{a+b}{2}\right) \int_c^d g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\ \leq \frac{1}{2} L \left[\max_{t \in [a, \frac{a+b}{2}]} \left| \int_d^t g(s) dv(s) \right| + \max_{t \in [\frac{a+b}{2}, b]} \left| \int_c^t g(s) dv(s) \right| \right] (b-a),$$

$$(4.18) \quad \left| \left[f(b) + f(a) - f\left(\frac{a+b}{2}\right) \right] \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\ \leq \frac{1}{2} L \left[\max_{t \in [a, \frac{a+b}{2}]} \left| \int_t^b g(s) dv(s) \right| + \max_{t \in [\frac{a+b}{2}, b]} \left| \int_a^t g(s) dv(s) \right| \right] (b-a),$$

$$(4.19) \quad \left| f\left(\frac{a+b}{2}\right) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\ \leq \frac{1}{2} L \max \left\{ \max_{t \in [a, \frac{a+b}{2}]} \left| \int_a^t g(s) dv(s) \right|, \max_{t \in [\frac{a+b}{2}, b]} \left| \int_t^b g(s) dv(s) \right| \right\} (b-a)$$

and

$$(4.20) \quad \left| f(b) \int_{\frac{a+b}{2}}^b g(s) dv(s) + f(a) \int_a^{\frac{a+b}{2}} g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\ \leq \frac{1}{2} L \left[\max_{t \in [a, \frac{a+b}{2}]} \left| \int_t^{\frac{a+b}{2}} g(s) dv(s) \right| + \max_{t \in [\frac{a+b}{2}, b]} \left| \int_{\frac{a+b}{2}}^t g(s) dv(s) \right| \right] (b-a).$$

5. SOME SIMPLER ERROR BOUNDS

If $g : [a, b] \rightarrow \mathbb{C}$ is continuous and $v : [a, b] \rightarrow \mathbb{C}$ is of bounded variation, then the Riemann-Stieltjes integrals $\int_a^t g(s) dv(s)$ and $\int_t^b g(s) dv(s)$ exist for $t \in [a, b]$ and

$$\left| \int_a^t g(s) dv(s) \right| \leq \int_a^t |g(s)| d\left(\bigvee_a^s(v)\right) \leq \max_{s \in [a, t]} |g(s)| \bigvee_a^t(v)$$

and

$$\left| \int_t^b g(s) dv(s) \right| \leq \int_t^b |g(s)| d\left(\bigvee_t^s(v)\right) \leq \max_{s \in [t, b]} |g(s)| \bigvee_t^b(v),$$

which implies that

$$\max_{t \in [a, x]} \left| \int_a^t g(s) dv(s) \right| \leq \max_{s \in [a, x]} |g(s)| \bigvee_a^x(v)$$

and

$$\max_{t \in [x, b]} \left| \int_t^b g(s) dv(s) \right| \leq \max_{s \in [x, b]} |g(s)| \bigvee_x^b(v),$$

for $x \in (a, b)$.

Therefore, by (3.14) we get for $x \in (a, b)$ that

$$\begin{aligned}
 (5.1) \quad & \left| f(x) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_a^t g(s) dv(s) \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_t^b g(s) dv(s) \right| \bigvee_x^b(f) \\
 & \leq \max_{s \in [a, x]} |g(s)| \bigvee_a^x(v) \bigvee_a^x(f) + \max_{s \in [x, b]} |g(s)| \bigvee_x^b(v) \bigvee_x^b(f) \\
 & \leq \begin{cases} \max \left\{ \max_{s \in [a, x]} |g(s)| \bigvee_a^x(v), \max_{s \in [x, b]} |g(s)| \bigvee_x^b(v) \right\} \bigvee_a^b(f), \\ \max \left\{ \max_{s \in [a, x]} |g(s)| \bigvee_a^x(f), \max_{s \in [x, b]} |g(s)| \bigvee_x^b(f) \right\} \bigvee_a^b(v) \end{cases}
 \end{aligned}$$

provided f, v are of bounded variation and g is continuous and such that the integral $\int_a^b f(t) g(t) dv(t)$ exists.

If $m \in (a, b)$ is such that $\bigvee_a^m(f) = \bigvee_m^b(f)$, then from the first inequality in (5.1) we get

$$\begin{aligned}
 (5.2) \quad & \left| f(m) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} \left[\max_{t \in [a, m]} \left| \int_a^t g(s) dv(s) \right| + \max_{t \in [m, b]} \left| \int_t^b g(s) dv(s) \right| \right] \bigvee_a^b(f).
 \end{aligned}$$

If $p \in (a, b)$ is such that $\bigvee_a^p(v) = \bigvee_p^b(v)$, then from the inequality (5.1) we get

$$\begin{aligned}
 (5.3) \quad & \left| f(p) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, p]} \left| \int_a^t g(s) dv(s) \right| \bigvee_a^p(f) + \max_{t \in [p, b]} \left| \int_t^b g(s) dv(s) \right| \bigvee_p^b(f) \\
 & \leq \frac{1}{2} \left[\max_{s \in [a, p]} |g(s)| \bigvee_a^p(f) + \max_{s \in [p, b]} |g(s)| \bigvee_p^b(f) \right] \bigvee_a^b(v).
 \end{aligned}$$

By (4.15) we also get for $x \in (a, b)$ that

$$\begin{aligned}
 (5.4) \quad & \left| f(x) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_a^t g(s) dv(s) \right| (x-a) + \max_{t \in [x, b]} \left| \int_t^b g(s) dv(s) \right| (b-x) \right] \\
 & \leq L \left[\max_{s \in [a, x]} |g(s)| \bigvee_a^x(v) (x-a) + \max_{s \in [x, b]} |g(s)| \bigvee_x^b(v) (b-x) \right] \\
 & \leq L \begin{cases} \max \left\{ \max_{s \in [a, x]} |g(s)| \bigvee_a^x(v), \max_{s \in [x, b]} |g(s)| \bigvee_x^b(v) \right\} (b-a), \\ \max \left\{ \max_{s \in [a, x]} |g(s)| (x-a), \max_{s \in [x, b]} |g(s)| (b-x) \right\} \bigvee_a^b(v) \end{cases}
 \end{aligned}$$

provided v is of bounded variation, f is Lipschitzian with the constant $L > 0$ and g is continuous on $[a, b]$.

In particular, for $x = \frac{a+b}{2}$ we get from the first inequality in (5.4) that

$$\begin{aligned}
 (5.5) \quad & \left| f\left(\frac{a+b}{2}\right) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} L \left[\max_{t \in [a, \frac{a+b}{2}]} \left| \int_a^t g(s) dv(s) \right| + \max_{t \in [\frac{a+b}{2}, b]} \left| \int_t^b g(s) dv(s) \right| \right] (b-a).
 \end{aligned}$$

If $p \in (a, b)$ is such that $\bigvee_a^p(v) = \bigvee_p^b(v)$, then from the inequality (5.4) we get

$$\begin{aligned}
 (5.6) \quad & \left| f(p) \int_a^b g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, p]} \left| \int_a^t g(s) dv(s) \right| (p-a) + \max_{t \in [p, b]} \left| \int_t^b g(s) dv(s) \right| (b-p) \right] \\
 & \leq \frac{1}{2} L \left[\max_{s \in [a, p]} |g(s)| (p-a) + \max_{s \in [p, b]} |g(s)| (b-p) \right] \bigvee_a^b(v).
 \end{aligned}$$

Similarly, by (3.15) we have for $x \in (a, b)$ that

$$\begin{aligned}
 (5.7) \quad & \left| f(b) \int_x^b g(s) dv(s) + f(a) \int_a^x g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, x]} \left| \int_t^x g(s) dv(s) \right| \bigvee_a^x(f) + \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) \right| \bigvee_x^b(f) \\
 & \leq \max_{s \in [a, x]} |g(s)| \bigvee_a^x(v) \bigvee_a^x(f) + \max_{s \in [x, b]} |g(s)| \bigvee_x^b(v) \bigvee_x^b(f) \\
 & \leq \begin{cases} \max \left\{ \max_{s \in [a, x]} |g(s)| \bigvee_a^x(v), \max_{s \in [x, b]} |g(s)| \bigvee_x^b(v) \right\} \bigvee_a^b(f) \\ \max \left\{ \max_{s \in [a, x]} |g(s)| \bigvee_a^x(f), \max_{s \in [x, b]} |g(s)| \bigvee_x^b(f) \right\} \bigvee_a^b(v) \end{cases}
 \end{aligned}$$

provided f, v are of bounded variation and g is continuous and such that the integral $\int_a^b f(t) g(t) dv(t)$ exists.

If $m \in (a, b)$ is such that $\bigvee_a^m(f) = \bigvee_m^b(f)$, then from the first inequality in (5.7) we get

$$\begin{aligned}
 (5.8) \quad & \left| f(b) \int_m^b g(s) dv(s) + f(a) \int_a^m g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} \left[\max_{t \in [a, m]} \left| \int_t^m g(s) dv(s) \right| + \max_{t \in [m, b]} \left| \int_m^t g(s) dv(s) \right| \right] \bigvee_a^b(f).
 \end{aligned}$$

If $p \in (a, b)$ is such that $\bigvee_a^p(v) = \bigvee_p^b(v)$, then from the inequality (5.1) we get

$$\begin{aligned}
 (5.9) \quad & \left| f(b) \int_p^b g(s) dv(s) + f(a) \int_a^p g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \max_{t \in [a, p]} \left| \int_t^p g(s) dv(s) \right| \bigvee_a^p(f) + \max_{t \in [p, b]} \left| \int_p^t g(s) dv(s) \right| \bigvee_p^b(f) \\
 & \leq \frac{1}{2} \left[\max_{s \in [a, p]} |g(s)| \bigvee_a^p(f) + \max_{s \in [p, b]} |g(s)| \bigvee_p^b(f) \right] \bigvee_a^b(v).
 \end{aligned}$$

From (4.16) we have

$$\begin{aligned}
 (5.10) \quad & \left| f(b) \int_x^b g(s) dv(s) + f(a) \int_a^x g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, x]} \left| \int_t^x g(s) dv(s) \right| (x-a) + \max_{t \in [x, b]} \left| \int_x^t g(s) dv(s) \right| (b-x) \right] \\
 & \leq L \left[\max_{s \in [a, x]} |g(s)| \bigvee_a^x(v) (x-a) + \max_{s \in [x, b]} |g(s)| \bigvee_x^b(v) (b-x) \right] \\
 & \leq L \begin{cases} \max \left\{ \max_{s \in [a, x]} |g(s)| \bigvee_a^x(v), \max_{s \in [x, b]} |g(s)| \bigvee_x^b(v) \right\} (b-a), \\ \max \left\{ \max_{s \in [a, x]} |g(s)| (x-a), \max_{s \in [x, b]} |g(s)| (b-x) \right\} \bigvee_a^b(v) \end{cases}
 \end{aligned}$$

provided that v is of bounded variation, f is Lipschitzian with the constant $L > 0$ and g is continuous on $[a, b]$.

In particular, for $x = \frac{a+b}{2}$ we get from the first inequality in (5.10) that

$$\begin{aligned}
 (5.11) \quad & \left| f(b) \int_{\frac{a+b}{2}}^b g(s) dv(s) + f(a) \int_a^{\frac{a+b}{2}} g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq \frac{1}{2} L \left[\max_{t \in [a, \frac{a+b}{2}]} \left| \int_t^{\frac{a+b}{2}} g(s) dv(s) \right| + \max_{t \in [\frac{a+b}{2}, b]} \left| \int_{\frac{a+b}{2}}^t g(s) dv(s) \right| \right] (b-a).
 \end{aligned}$$

If $p \in (a, b)$ is such that $\bigvee_a^p(v) = \bigvee_p^b(v)$, then from the inequality (5.10) we get

$$\begin{aligned}
 (5.12) \quad & \left| f(b) \int_p^b g(s) dv(s) + f(a) \int_a^p g(t) dv(t) - \int_a^b f(t) g(t) dv(t) \right| \\
 & \leq L \left[\max_{t \in [a, p]} \left| \int_t^p g(s) dv(s) \right| (p-a) + \max_{t \in [p, b]} \left| \int_p^t g(s) dv(s) \right| (b-p) \right] \\
 & \leq \frac{1}{2} L \left[\max_{s \in [a, x]} |g(s)| (p-a) + \max_{s \in [x, b]} |g(s)| (b-p) \right] \bigvee_a^b(v).
 \end{aligned}$$

Using the equalities (2.9) and (2.10) one can obtain various inequalities as in the recent paper [27]. The details are omitted.

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