 doi:10.20944/preprints201808.0376.v1

CONTINUITY AND MOTION (NAVIER-STOKES) EQUATIONS

Provorov D.A., *Russia, Saint-Petersburg*[1]

[1] *Russia, 197198, Saint-Petersburg, Dobrolubova, 21, 6*

Abstract

The paper is considering of the basic differential equations of hydrodynamics: the equation of continuity and motion. On the simplest example, it is shown that the equation of continuity in a system with the equation of motion leads to contradictions and erroneous results of modeling. A more correct form of the continuity equation is described. It is shown that the equations of motion can be written in the form of complete differentials. Three possible integral forms of the equations of motion are presented. As a conclusion, the existence and smoothness of the solution of the Navier-Stokes equations are considered.

Keywords: continuity equation, Navier-Stokes equation, Bernoulli equation, existence and smoothens of the solutions of Navier-Stokes equation

1. Introduction

The bases of the mathematical apparatus for a flow modeling are the differential equations that originate in physical conservation laws. Such laws are: the law of conservation of mass in a closed system, the law of conservation of momentum and the law of conservation of energy. Each of the laws corresponds to its differential equation: the equation of continuity, motion and energy. The two most commonly used forms of the equations of motion are the Navier-Stokes and Euler equations (a special case of the Navier-Stokes equation). More than one hundred works are devoted to solving the system of differential equations of continuity and motion. Each of these papers is a particular solution of a particular problem with its own assumptions and boundary conditions. However, no general integral solution of this system of equations has been found to date. Moreover, it is not proved that the system has smooth solutions in three-dimensional space.

The present paper is an attempt to answer the questions: are there smooth solutions to the Navier-Stokes equations, how do they look and under what conditions do they exist? The work is based on the derivation of the equations of continuity and motion.

2. Continuity equation

2.1. Classical derivation

Classical derivation of the continuity equation usually looks like described below. And it can be find in (R.B. Bird, 2002). We consider an abstract fluid flow of density ρ, within which we select a cube of infinitesimal volume ΔV with sides $\Delta x, \Delta y,\ \Delta z$. Let the flow through the left-hand face $\Delta y \Delta z$ move with velocity v_x. Then in a time equal to Δt through the face $\Delta y \Delta z$ inside the cube a mass of liquid equal to $\rho v_x \Delta y \Delta z \Delta t$ will be introduced into the flow. In this case, a mass equal to $(\rho v_x + \Delta x\, \partial \rho v_x/\partial x)\Delta y \Delta z \Delta t$ is carried out through the right side of the cube. Due to the flow through all faces of the cube, the change in mass inside the cube will be:

$$\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)\Delta x \Delta y \Delta z \Delta t$$

Since the volume ΔV is fixed (unchangeable), in a time equal Δt the change in mass inside the cube under consideration will lead to a change in the density of the liquid to the value $\rho + \Delta t\, \partial \rho/\partial t$. And the amount of change (accumulation) of the mass will be equal to $\Delta x \Delta y \Delta z \Delta t\, \partial \rho/\partial t$. Then we can write:

$$\left(\frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z}\right)\Delta x \Delta y \Delta z \Delta t = -\frac{\partial \rho}{\partial t}\Delta x \Delta y \Delta z \Delta t$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_x}{\partial x} + \frac{\partial \rho v_y}{\partial y} + \frac{\partial \rho v_z}{\partial z} = 0 \qquad (1)$$

In the case of incompressible flow, i.e. $\rho = const$, the equation takes the form:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \qquad (2)$$

Or in another form of record:

$$\nabla \vec{v} = 0 \qquad (3)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 August 2018 doi:10.20944/preprints201808.0376.v1

The last expression is called the continuity equation for an incompressible fluid.

2.2. Contradictions

As an example, consider a two-dimensional unidirectional flow of an incompressible fluid under the effect of a pressure drop (see Figure 1). The flow will be considered isothermal, stationary. The flow will be considered from the point of view of two known equations: the Navier-Stokes equation and the Bernoulli equation. The effects of gravity and inertia forces will be neglected.

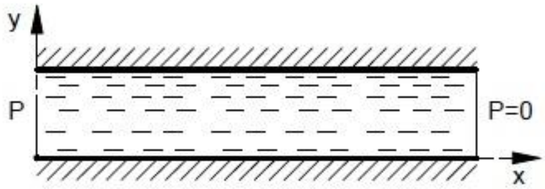

Figure 1. Unidirectional isothermal incompressible flow due to pressure drop

Navier-Stokes equation

Since the flow is unidirect, there are only one nonzero velocity component v_x. Then the continuity equation takes the form:

$$\frac{\partial v_x}{\partial x} = 0$$

The equation of motion (Navier-Stokes) in projections onto the corresponding coordinate axes is written as follows:

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}$$

$$0 = -\frac{\partial P}{\partial y}$$

It is assumed that the equations of motion in this form show that the pressure drop exists if any component of the shear stresses is not constant with respect to the spatial coordinates. It should be said that in the case of a low-viscosity liquid ($\tau_{xy} = 0$), the system of equations does not give a solution, it shows that the pressure and velocity are constants, which in turn contradicts the conditions of the problem posed. In this case, the problem is solved using the Bernoulli law.

Bernoulli law

The Bernoulli equation for the problem under consideration (in the case of $\tau_{xy} = 0$) takes the form:

$$\frac{\rho}{2}\left(v^2(x_1) - v^2(x_2)\right) = P$$

The above equation shows that the pressure gradient exists when there is a gradient of the specific kinetic energy in the flow direction, which in the case of incompressibility can exist only in the case $v_x(x) \neq const$, i.e.:

$$\frac{\partial v_x}{\partial x} \neq 0$$

In practice, it is customary to divide the problems of hydrodynamics into problems of the flow of viscous liquids and inviscid fluids. The former are solved by the system of equations of continuity and motion (Navier-Stokes), the latter by means of the Bernoulli and Euler equations (a particular case of the Navier-Stokes equations). It is worth paying attention to the fact that if for the problem under consideration the equations of motion are written in the form of Euler's equations, without taking into account the continuity equation, then they will take the form identical to the Bernoulli equation:

$$\rho v_x \frac{\partial v_x}{\partial x} = -\frac{\partial P}{\partial x} \quad \rightarrow$$

$$\rho v_x \frac{d v_x}{dx} = -\frac{dP}{dx} \quad \rightarrow$$

$$\frac{\rho}{2}\left(v^2(x_1) - v^2(x_2)\right) = P$$

Thus, we can conclude that the equation of continuity in the form: $\nabla \vec{v} = 0$, is not fair. At least, it completely contradicts the Bernoulli law for inviscid fluids. Perhaps the inaccuracy of the classical derivation is that the volume of the cube is assumed unchanged of the time. However, it can vary like the volume of water in the bath varies, depending on the flow rate of inflow and outflow of liquid.

2.3. New continuity equation

Suppose that the fluid is incompressible, i.e. its density does not change with time and does not depend on the position in space. This means that the derivative of density in the total differentials (the total derivative) is 0:

$$\frac{d\rho}{dt} = 0 \tag{4}$$

The total derivative can be written as the sum of partial derivatives. In the Cartesian coordinate system, we have:

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt} \tag{5}$$

It is obvious that:

$$\frac{dx}{dt} = v_x; \frac{dy}{dt} = v_y; \frac{dz}{dt} = v_z \tag{6}$$

Then:

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}v_x + \frac{\partial\rho}{\partial y}v_y + \frac{\partial\rho}{\partial z}v_z = 0 \tag{7}$$

The last expression is similar to the following record:

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \nabla\rho = 0 \tag{8}$$

In the case of uniaxial flow (the task from the previous section), the equation takes the form:

$$\frac{\partial\rho}{\partial x}v_x = 0 \quad => \quad \frac{\partial\rho}{\partial x} = 0$$

The equation of continuity in this case shows that the density remains unchanged in the direction of the flow. In general, the continuity equation indicates that the density, as a function of $\rho = \rho(t; x; y; z)$, is smooth, i.e. continuously differentiable at any point of the region of the problem under consideration.
Now we assume that the fluid is compressible, expandable, etc., such that its density is a function of other scalar parameters h_i, such as pressure, temperature, and so on, the total number of which is n. And the parameters h_i are functions of coordinates and time:

$$\rho = \rho(t; x; y; z; h_1; h_2; \dots; h_i; \dots; h_n)$$
$$h_1 = h_1(t; x; y; z)$$
$$h_2 = h_2(t; x; y; z)$$
$$\dots$$
$$h_i = h_i(t; x; y; z)$$
$$\dots$$
$$h_n = h_n(t; x; y; z)$$

Then the total density derivative is

$$\frac{d\rho}{dt} = \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\frac{dh_i}{dt} \tag{9}$$

It should be said that ρ and any h_i from the number n must be smooth, continuously differentiable functions. Otherwise, the equation (3) is not fair. Thus, the general continuity equation has the form:

$$\frac{d\rho}{dt} - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\frac{dh_i}{dt} = 0$$
$$\{\rho; h_1; h_2; \dots; h_i; \dots; h_n\} \in C(\Omega) \tag{10}$$

It is also worth noting that the notation d/dt is a full derivative operator.
So we can said, that the density is a function of other parameters (arguments). Continuity of density means that it, as a function, is differentiable, provided that all its arguments (parameters) are differentiable

3. Equation of motion (Navier-Stokes equation)

3.1. Derivation of the equation for an incompressible and compressible fluid

We rewrite the continuity equation for an incompressible fluid again:

$$\rho = \rho(t; x; y; z)$$
$$\frac{d\rho}{dt} = 0 \tag{11}$$

Then we multiply the equation by the differential of the velocity dv and integrate:

$$\int \frac{d\rho}{dt} dv = c; c = const$$

Since the density is formally independent from the velocity, we can write:

$$\frac{d}{dt}\left(\int \rho\, dv\right) = c \quad => \quad \frac{d}{dt}(\rho v) = c \quad => \quad \rho\, \frac{dv}{dt} = c \tag{12}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 August 2018 doi:10.20944/preprints201808.0376.v1

Now, similarly integrating (12) over the entire volume V of the problem under consideration, we can write:

$$\oint \rho \frac{dv}{dt} dV = \oint c\, dV \qquad (13)$$

Obviously, equation (13) expresses Newton's second law. The right-hand side of the equation is the sum of the mass $\sum F_m$ and the surface $\sum F_p$ forces:

$$\oint c\, dV = \sum F_m + \sum F_p \qquad (14)$$

The sum of the mass forces can be represented as a volume integral of the product of the density ρ and the main vector $\vec{g}$ of mass forces:

$$\sum F_m = \oint \rho\vec{g}\, dV \qquad (15)$$

The sum of the surface forces in turn is an integral of the product of the total stress vector $\vec{\pi}$ distributed on the elementary area dS and the unit vector n normal to dS over the entire surface S that bounds the volume of the problem under consideration:

$$\sum F_p = \oint \vec{\pi}\vec{n}\, dS \qquad (16)$$

In accordance with the Ostrogradskii-Gauss theorem on the divergence of a vector field, the sum of the surface forces can be represented in the form:

$$\sum F_p = \oint \vec{\pi}\vec{n}\, dS = \oint \nabla\vec{\pi}\, dV \qquad (17)$$

Then equation (13) will be written as:

$$\oint \rho \frac{d\vec{v}}{dt} dV = \oint (\rho\vec{g} + \nabla\vec{\pi})\, dV \qquad (18)$$

Obviously, the integrands are equal to each other:

$$\rho \frac{dv}{dt} = \rho g + \nabla\pi \qquad (19)$$

Equation (19) is a well-known equation of motion or the Navier-Stokes equation for an incompressible fluid. In the case of compressibility, the equation of motion can also be obtained by integrating the continuity equation with respect to dv. The right-hand side of the equation is similarly formed. The left-hand side of the equation multiplied by the differential of the velocity takes the form:

$$\int \left(\frac{d\rho}{dt} - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\frac{dh_i}{dt}\right) dv = const$$

We call attention to the fact that the functions $\rho = \rho(t; x; y; z; h_1; h_2; \dots; h_i; \dots; h_n)$, $h_i = h_i(t; x; y; z)$ are formally independent of the velocity. Then we can write:

$$\frac{d}{dt}\left(\int \rho\, dv\right) - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\frac{d}{dt}\left(\int h_i\, dv\right) = const$$

$$\frac{d}{dt}(\rho v) - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\frac{d}{dt}(h_i v) = \rho g + \nabla\pi \qquad (20)$$

Expression (20) is the equation of motion of a compressible fluid. It is worth noting that the equation of motion was derived from the equation of continuity by integrating it over velocity. Thus, it can be said that the equation of motion is valid when the continuity equation is valid and there is a velocity differential on the entire region of the problem considered Ω. Or vice versa: if there exists a velocity differential on the whole domain Ω of the problem under consideration and the equation of continuity on this region is valid, then the equation of motion is also valid.

3.2. The Navier-Stokes equation in the form of full differentials

If we scalarly multiply the equation of motion by the differential of the displacement vector $d\vec{l}$, which is obviously equal to $\vec{v}dt$ and is equal to the differential of the radius vector $d\vec{r}$, then we obtain:

$$\left(\frac{d}{dt}(\rho\vec{v}) - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\frac{d}{dt}(h_i\vec{v})\right)\vec{v}dt = (\rho\vec{g} + \nabla\vec{\pi})d\vec{r} \qquad (21)$$

Or

$$\vec{v}d(\rho\vec{v}) - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\vec{v}d(h_i\vec{v}) = \rho\vec{g}d\vec{r} + d\vec{r}\nabla\vec{\pi} \qquad (22)$$

We consider separately each term of the given equation:

$$\vec{v}d(\rho\vec{v}) = \vec{v}\rho d\vec{v} + \vec{v}\cdot\vec{v}d\rho \qquad (23)$$

$$\sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\vec{v}d(h_i\vec{v}) = \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}(\vec{v}h_i d\vec{v} + \vec{v}\cdot\vec{v}dh_i) \quad (24)$$

$$d\vec{r}\nabla\vec{\pi} = \frac{\partial\pi_x}{\partial x}dx + \frac{\partial\pi_y}{\partial y}dy + \frac{\partial\pi_z}{\partial z}dz = d\vec{\pi} \quad (25)$$

We note that:

$$d\rho = \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}dh_i \quad (26)$$

Then

$$\rho\vec{v}d\vec{v} - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}h_i\vec{v}d\vec{v} = \rho\vec{g}d\vec{r} + d\vec{\pi} \quad (27)$$

Thus, it is seen that the equation of motion can be obtained in the form of full differentials and formally be integrated.

3.3. The integral form of the Navier-Stokes equation

To integrate equation (27), it is necessary to determine the boundary conditions. Three types of standard boundary conditions will be selected in a framework of this paper.

The first type of boundary conditions is the equality to zero of velocity. Such boundary conditions are very often used in practice and are called "no slip" conditions. That is, if there exists a set $\vec{r}_{0v}$ such that $\vec{v}(\vec{r}_{0v}) = 0$, then the integral form of the equations of motion has the form:

$$\rho\int_0^v \vec{v}\,d\vec{v} - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}h_i\int_0^v \vec{v}\,d\vec{v} = \int_{r0v}^{r} \rho\vec{g}\,d\vec{r} + \int_{\pi 0v}^{\pi} d\vec{\pi}$$

As a result we have:

$$\exists\{\vec{r}_{0v}\}, \vec{v}(r_{0v}) = 0 \rightarrow$$

$$\frac{\rho}{2}\vec{v}\cdot\vec{v} - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\frac{h_i}{2}\vec{v}\cdot\vec{v} = \rho\vec{g}\cdot\vec{r} - \rho\vec{g}\cdot\vec{r}_{0v} + \vec{\pi} - \vec{\pi}(r_{0v}) \quad (28)$$

The second type of boundary conditions follows from Newton's first law on inertial frames of reference. Those if an inertial frame of reference is given and there exists a set $\vec{r}_{0f}$ such that $\vec{v}(\vec{r}_{0v}) = const$ and $\vec{\pi}(r_{0f}) = 0$, then the integral form of the equations of motion has the form:

$$\rho\int_{v(r0f)}^{v} \vec{v}\,d\vec{v} - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}h_i\int_{v(r0f)}^{v} \vec{v}\,d\vec{v} = \int_{r0f}^{r} \rho\vec{g}\,d\vec{r} + \int_0^{\pi} d\vec{\pi}$$

As a result we have:

$$\exists\{\vec{r}_{0f}\}, \vec{v}(\vec{r}_{0f}) = const, \vec{\pi}(r_{0f}) = 0 \rightarrow$$

$$\rightarrow \frac{\rho}{2}\left(\vec{v}\cdot\vec{v} - \vec{v}(\vec{r}_{0f})\cdot\vec{v}(\vec{r}_{0f})\right) - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\frac{h_i}{2}\left(\vec{v}\cdot\vec{v} - \vec{v}(\vec{r}_{0f})\cdot\vec{v}(\vec{r}_{0f})\right) =$$
$$= \rho\vec{g}\cdot\vec{r} - \rho\vec{g}\cdot\vec{r}_{0f} + \vec{\pi} \quad (29)$$

The third types of boundary conditions are the two positions of the radius vector ($\vec{r}_1$ and $\vec{r}_2$) in the given coordinate system. The integral form of the equations of motion in this case takes the form:

$$\rho\int_{v(r1)}^{v(r2)} \vec{v}\,d\vec{v} - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}h_i\int_{v(r1)}^{v(r2)} \vec{v}\,d\vec{v} = \int_{r1}^{r2} \rho\vec{g}\,d\vec{r} + \int_{\pi(r1)}^{\pi(r2)} d\vec{\pi}$$

As a result we have:

$$\frac{\rho}{2}\left(\vec{v}(\vec{r}_2)\cdot\vec{v}(\vec{r}_2) - \vec{v}(\vec{r}_1)\cdot\vec{v}(\vec{r}_1)\right) - \sum_{i=1}^{n} \frac{\partial\rho}{\partial h_i}\frac{h_i}{2}\left(\left(\vec{v}(\vec{r}_2)\cdot\vec{v}(\vec{r}_2) - \vec{v}(\vec{r}_1)\cdot\vec{v}(\vec{r}_1)\right)\right) =$$
$$= \rho\vec{g}\cdot\vec{r}_2 - \rho\vec{g}\cdot\vec{r}_1 + \vec{\pi}(\vec{r}_2) - \vec{\pi}(\vec{r}_1) \quad (30)$$

In the case of an inviscid fluid, equations (28), (29), (30) acquire the meaning of the Bernoulli law. Thus, the equations (28), (29), (30) obtained are a generalization of the Bernoulli law for cases of viscous compressible or incompressible flows.

4. The existence and smoothness of the Navier-Stokes equations (as a conclusion)

The task of the existence and smoothness of the Navier-Stokes equation was published by Charles Fefferman in 2000 on the web site of the Clay Institute (Fefferman, 2000). Now we consider this task taking into account all that was shown above. We continue the subsequent numbering of conditions and problems similarly to the publication of Fefferman.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 August 2018 doi:10.20944/preprints201808.0376.v1

Conditions (3) and (4). Let at the initial instant of time the velocity field be smooth and it is defined on the whole space $\mathbb{R}^n$:

$$v(x,0) = v^0(x), \qquad x \in \mathbb{R}^n$$

Condition (5). The force field f is also determined at the initial time and at any other time and is smooth:

$$f(x,t) \in C^\infty(\mathbb{R}^n \times [0,\infty))$$

Condition (2). It is assumed that for any $t \geq 0$ the density is continuous and does not depend on time. Flow incompressible:

$$\frac{d\rho}{dt} = 0$$

In view of what was said above, condition (2) is written differently than in Fefferman's publication.

Condition (6). It is required to show that for any $t \geq 0$ there exist smooth solutions $v(x,t), p(x,t) \in C^\infty(\mathbb{R}^n \times [0,\infty))$ corresponding to the *condition (1)*:

$$\frac{dv}{dt} = f(x,t) + \nabla\pi, \;\; p(x,t) \in \pi(x,t)$$

As it is said in conditions (3), (4) at the initial instant $t = 0$, the fields of forces and velocities are smooth. Then, under the conditions of continuity and incompressibility, in accordance with (28) we can write:

$$v^0 dv^0 = f(x,0)dx + d\pi(x,0), \;\; p(x,0) \in \pi(x,0) \tag{31}$$

Or

$$d\big(v^0 \cdot v^0 - \pi(x,0)\big) = f(x,0)dx \tag{32}$$

Proceeding from conditions (3), (5) and equation (32), it is seen that the stress vector function $\pi(x,0)$ is smooth, respectively:

$$p(x,0) \in \pi(x,0) \in C^\infty(\mathbb{R}^n \times [0,\infty)) \tag{33}$$

Now consider the case of $t > 0$. For this case, condition (5) is preserved. Then:

$$d\big(v(x,t) \cdot v(x,t) - \pi(x,t)\big) = f(x,t)dx \tag{34}$$

From (34) clear that differential $d\big(v(x,t) \cdot v(x,t) - \pi(x,t)\big)$ exists. Hence:

$$\forall t > 0 \;\; \exists d\big(v(x,t) \cdot v(x,t) - \pi(x,t)\big) \rightarrow \;\; v(x,t), \pi(x,t) \in C^\infty(\mathbb{R}^n \times [0,\infty)) \tag{35}$$

And as a result we can write down:

$$\forall t \geq 0 \;\; v(x,t), p(x,t) \in C^\infty(\mathbb{R}^n \times [0,\infty)) \tag{36}$$

The condition (6) is satisfied.

Condition (7). The essence of the condition is to show that the velocity does not increase infinitely as $x \rightarrow \infty$.. We assume that $f(x,t) = 0$, as stated in problems (A) and (B). Since the entire space $\mathbb{R}^n$ has no finite limits, at least one inertial frame of reference can be found in it, such that the velocity for $x = 0$

$$v(0,t) = v_I = const, \qquad \pi(0,t) = 0$$

Then, in accordance with (29), we can write:

$$v(x,t) \cdot v(x,t) - v_I \cdot v_I = \pi(x,t) \tag{37}$$

Or

$$v_I \cdot v_I = v(x,t) \cdot v(x,t) - \pi(x,t) \tag{38}$$

From the equation obtained, we see that the difference $v(x,t) \cdot v(x,t) - \pi(x,t)$ is constant, so neither $v(x,t)$ nor $\pi(x,t)$ not increase infinitely for $x \rightarrow \infty$. The condition is satisfied. However, in the task of Charles Fefferman, condition (7) is written as

$$\int_{\mathbb{R}^n} v(x,t) \cdot v(x,t) \, dx < C, \forall t \geq 0 \tag{39}$$

Such a condition, even if the velocity does not increase infinitely, may not be fulfilled if the velocity field does not tend to 0 as $x \rightarrow \infty$.. However, it can be seen that for x increasing from zero to infinity the stresses $\pi(x,t)$ also increase from the zero value. And if $\pi(x,t)$ increase continuously with increasing x, then the velocity tends to 0. In this case, the integral (39) becomes improper:

$$\int_{\mathbb{R}^n} v(x,t) \cdot v(x,t) \, dx = \lim_{b \to \infty} \int_0^b v(x,t) \cdot v(x,t) \, dx = E < C, \forall t \geq 0 \tag{40}$$

The condition (7) can be satisfied strictly in accordance with the formulation of Fefferman under certain conditions.

Thus, it was shown that solutions of the Navier-Stokes equation exist. The smoothness of the solutions was also proved.

 doi:10.20944/preprints201808.0376.v1

References

Fefferman, Ch. 2000. Existance and smoothness of the Navier-Stokes equation. *Clay Mathematics Institute.* [Online] 2000. http://claymath.org/millenium/Navier-Stokes_Equations.

R.B. Bird, W.E. Stewart, E.N. Lightfoot. 2002. *Transport phenomena.* John Willey & Sons, Inc. s.l. : John Willey & Sons, Inc, 2002.