THE PARTIALLY DEGENERATE CHANGHEE-GENOCCHI POLYNOMIALS AND NUMBERS

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ABSTRACT. In this paper, we introduce the partially degenerate Changhee-Genocchi polynomials and numbers and investigated some identities of these polynomials. Furthermore, we investigate some explicit identities and properties of the partially degenerate Changhee-Genocchi arising from the nonlinear differential equations.

1. Introduction

As is well known, the Genocchi polynomials $G_n(x)$ are defined by the generating function as follows:

$$
\frac{2t}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} \frac{G_n(x)}{n!} t^n \quad \text{(see [1, 3, 6, 17]).}
$$

(1.1)

When $x = 0$, $G_n = G_n(0)$ are called the Genocchi numbers.

The Changhee polynomials $Ch_n(x)$ are defined by the generating function to be

$$
\frac{2}{2 + t} (1 + t)^x = \sum_{n=0}^{\infty} \frac{Ch_n(x)}{n!} t^n \quad \text{(see [5, 8, 13, 16, 18]).}
$$

(1.2)

When $x = 0$, $Ch_n = Ch_n(0)$ are called the Changhee numbers.

By replacing $t$ by $e^t - 1$, we get

$$
\sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n = \frac{2}{e^t + 1}e^{xt} = \sum_{m=0}^{\infty} \frac{Ch_m(x)}{m!} (e^t - 1)^m
$$

(1.3)

$$
= \sum_{m=0}^{\infty} \frac{Ch_m(x)}{n!} \sum_{n=m}^{\infty} \frac{1}{m!} S_2(n, m) t^n
$$

$$
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \frac{Ch_m(x)S_2(n, m)}{n!} \right) t^n,
$$

where $E_n(x)$ are ordinary Euler polynomials.

Thus, we have

$$
E_n(x) = \sum_{m=0}^{n} Ch_m(x)S_2(n, m).
$$

(1.4)

Now, we define the degenerate exponential function as follow:
When $x = 0$, $CG_n = CG_n(0)$ are called the Changhee-Genocchi numbers.

The Genocchi-Changhee polynomials $GCh_n(x)$ are defined by the generating function to be

$$\frac{2t}{2 + t}(1 + t)^x = \sum_{n=0}^{\infty} GCh_n(x) \frac{t^n}{n!}.$$ (1.10)

When $x = 0$, $GCh_n = GCh_n(0)$ are called the Genocchi-Changhee numbers.

The degenerate Changhee-Genocchi polynomials $CG_n(x \mid \lambda)$ are defined by the generating function to be

$$\frac{2\log(1 + \log e^{t \lambda})}{2 + \log e^{t \lambda}}(1 + \log e^{t \lambda})^x = \sum_{n=0}^{\infty} CG_n(x \mid \lambda) \frac{t^n}{n!} \ (\text{see [11]}).$$ (1.11)

When $x = 0$, $CG_n = CG_n(0)$ are called the degenerate Changhee-Genocchi numbers.

We recall the Stirling numbers of the first kind $S_1(n, m)$ and $S_2(n, m)$ are defined by

$$\frac{1}{m!}(\log(1 + t))^m = \sum_{n=m}^{\infty} S_1(n, m) \frac{t^n}{n!} \ (\text{see [4, 7, 14]}).$$ (1.12)

and

$$\frac{1}{m!}(e^t - 1)^m = \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} \ (\text{see [10, 12, 15]}).$$ (1.13)

Recently, B-M. Kim et al. studied Changhee-Genocchi polynomials and some identities of these polynomials. They also introduced Changhee-Genocchi polynomials and investigated some identities of these polynomials ([?] ). Also, H.-I. Kwon et al. introduced degenerate Changhee-Genocchi polynomials and some identities of these polynomials and investigated some identities of these polynomials ([?]). In this paper, we introduce the partially degenerate Changhee-Genocchi polynomials and numbers and investigated some identities of these polynomials. Furthermore, we investigate some explicit identities and properties of the partially degenerate Changhee-Genocchi arising from the nonlinear differential equations.
The partially degenerate Changhee-Genocchi polynomials

2. The partially degenerate Changhee-Genocchi polynomials and numbers

In this section, we define the partially degenerate Changhee-Genocchi polynomials and numbers and investigate some identities of the partially degenerate Changhee-Genocchi polynomials.

Now, we consider the degenerate Genocchi polynomials which are given by the generating function to be

\[ \frac{2t}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} G_{n,\lambda}(x) \frac{t^n}{n!}. \]  

(2.1)

When \( x = 0 \), \( G_{n,\lambda} = G_{n,\lambda}(0) \) are called the degenerate Genocchi numbers.

It is not difficult to show that \( G_{0,\lambda}(0) = 0 \).

So,

\[ \frac{2t}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} G_{n+1,\lambda}(x) \frac{t^{n+1}}{n+1} \frac{t^n}{n!}. \]  

(2.2)

Thus,

\[ \sum_{n=0}^{\infty} G_{n+1,\lambda}(x) \frac{t^{n+1}}{n+1} \frac{t^n}{n!} = \frac{2t}{e^t + 1} e^{tx} = \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^{n+1}}{n!}. \]  

(2.3)

Comparing the coefficients on the both sides in (2.3), we have the following result.

**Theorem 2.1.** Let \( \lambda \in \mathbb{C}_p \) with \( 0 < |\lambda|_p < 1 \). Then

\[ \frac{G_{n+1,\lambda}(x)}{n+1} = E_{n,\lambda}(x), \quad (n \geq 0). \]  

(2.4)

In [4], the degenerate Changhee polynomials which are given by

\[ \frac{2}{2 + \log e^t} (1 + \log e^t)^x = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x) \frac{t^n}{n!}. \]  

(2.5)
By replacing $t$ by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (??), we get

$$\frac{2}{2 + t}(1 + t)^x = \sum_{m=0}^{\infty} Ch_{m, \lambda}(x) \frac{1}{m!} \left( \frac{1}{\lambda}(e^{\lambda t} - 1) \right)^m$$

$$= \sum_{m=0}^{\infty} Ch_{m, \lambda}(x) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n}{n!} t^n$$

$$= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} Ch_{m, \lambda}(x) \lambda^{n-m} S_2(n, m) \right) \frac{t^n}{n!}. \tag{2.6}$$

Thus, we obtain the following result.

**Theorem 2.2.** Let $\lambda \in \mathbb{C}_p$ with $0 < |\lambda|_p < 1$. Then

$$Ch_n(x) = \sum_{m=0}^{n} Ch_{m, \lambda}(x) \lambda^{n-m} S_2(n, m). \tag{2.7}$$

Now, we define the partially degenerate Changhee-Genocchi polynomials which are given by

$$2 \log(1 + t) \frac{2}{2 + \log e^{\lambda t}} (1 + \log e^{\lambda t})^x = \sum_{n=0}^{\infty} \hat{CG}_{n, \lambda}(x) \frac{t^n}{n!}. \tag{2.8}$$

When $x = 0$, $\hat{CG}_{n, \lambda} = \hat{CG}_{n, \lambda}(0)$ are called the partially degenerate Changhee-Genocchi numbers.

Also, we define the higher-order partially degenerate Changhee-Genocchi numbers which are given by the generating function to be

$$\left(2 \log(1 + t) \frac{2}{2 + \log e^{\lambda t}} \right)^k = \sum_{n=0}^{\infty} \hat{CG}^{(k)}_{n, \lambda} \frac{t^n}{n!}. \tag{2.9}$$

Now, we observe that

$$\lim_{\lambda \to 0} \frac{2}{2 + \log e^{\lambda t}} (1 + \log e^{\lambda t})^x = \frac{2 \log(1 + t)}{2 + t}(1 + t)^x$$

$$= \frac{2t}{2 + t}(1 + t)^x \frac{\log(1 + t)}{t}$$

$$= \left( \sum_{l=0}^{\infty} GCh_l(x) \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} D_m \frac{t^m}{m!} \right)$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} GCh_l(x)D_{n-l} \right) \frac{t^n}{n!}. \tag{2.10}$$
Comparing the coefficients on the both sides in (2.9), we have the following result.

**Theorem 2.3.** Let \( \lambda \in \mathbb{C}_p \) with \( 0 < |\lambda|_p < 1 \) and \( \lambda \to 0 \). Then

\[
\hat{C}G_{n,0}(x) = \sum_{l=0}^{n} \binom{n}{l} GCh_l(x)D_{n-l}. \tag{2.11}
\]

Now, we observe that

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{C}G_{n,\lambda} = 2 \log(1 + t) \frac{2 \log(1 + e^t)}{2 + \log e^t} \frac{1 + \log e^t}{t} \tag{2.12}
\]

Comparing the coefficients on the both sides in (2.12), we have the following result.

**Theorem 2.4.** Let \( \lambda \in \mathbb{C}_p \) with \( 0 < |\lambda|_p < 1 \). Then

\[
\hat{C}G_{n,\lambda} = \sum_{l=0}^{n} \binom{n}{l} Ch_l(x)D_{n-l}. \tag{2.13}
\]

We observe that

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \hat{C}G_{n,\lambda}(x) = 2 \log(1 + t) \frac{2 \log(1 + e^t)}{2 + \log e^t} \frac{1 + \log e^t}{t} \tag{2.14}
\]
Theorem 2.5. Let \( \lambda \in \mathbb{C}_p \) with \( 0 < |\lambda|_p < 1 \). Then

\[
\widehat{CG}_{n,\lambda}(x) = \sum_{m=0}^{n} \sum_{k=0}^{m} \binom{n}{m} (x)_{k} \lambda^{m-k} S_1(m,k) \widehat{CG}_{n-m,\lambda}.
\] (2.15)

3. The partially degenerate Changhee-Genocchi numbers arising from differential equations

In this section, we investigate some identities of the partially degenerate Changhee-Genocchi numbers arising from differential equations. Let

\[
F = F(t) = \frac{1}{2\lambda + \log(1 + \lambda t)}.
\] (3.1)

Then, by taking the derivative with respect to \( t \) of (3.1), we obtain

\[
F^{(1)} = \frac{d}{dt} F(t) = \frac{1}{(2\lambda + \log(1 + \lambda t))^2} \left( -\frac{\lambda}{1 + \lambda t} \right)
\]

\[= \left( -\frac{\lambda}{1 + \lambda t} \right) F^2.
\] (3.2)

From (3.1), we have

\[\lambda F^2 = -(1 + \lambda t) F^{(1)}.
\] (3.3)

By taking the derivative with respect to \( t \) in (3.1), we note that

\[2\lambda F F^{(1)} = -\lambda F^{(1)} - (1 + \lambda t) F^{(2)}.
\] (3.4)

Thus, by multiple \( (1 + \lambda t) \) on the both sides of (3.1), we obtain

\[2(1 + \lambda t) F F^{(1)} = -\lambda(1 + \lambda t) F^{(1)} - (1 + \lambda t)^2 F^{(2)}.
\] (3.5)

From (3.1) and (3.1), we get

\[2\lambda^2 F^3 = \lambda(1 + \lambda t) F^{(1)} + (1 + \lambda t)^2 F^{(2)}.
\] (3.6)

From the above equation, we have

\[3\lambda^2 F^2 F^{(1)} = \lambda^2 F^{(1)} + \lambda(1 + \lambda t) F^{(2)} + 2\lambda(1 + \lambda t) F^{(2)} + (1 + \lambda t)^2 F^{(3)}
\]

\[= \lambda^2 F^{(1)} + 3\lambda(1 + \lambda t) F^{(2)} + (1 + \lambda t)^2 F^{(3)}.
\] (3.7)

Multiply \( (1 + \lambda t) \) on the both sides of (3.1), we get

\[3\lambda^2(1 + \lambda t) F^2 F^{(1)} = \lambda^2(1 + \lambda t) F^{(1)} + 3\lambda(1 + \lambda t)^2 F^{(2)} + (1 + \lambda t)^3 F^{(3)}.
\] (3.8)

From (3.1) and (3.1), we obtain

\[3\lambda^3 F^4 = -\lambda^2(1 + \lambda t) F^{(1)} - 3\lambda(1 + \lambda t)^2 F^{(2)} - (1 + \lambda t)^3 F^{(3)}.
\] (3.9)
Continuing this process, we get

\[ N! \lambda^N F^{N+1} = (-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k}(1 + \lambda t)^k F^{(k)}. \]  

(3.10)

Let us take the derivative on the both sides of (3.10) with respect to \( t \). Then we obtain

\[ (N + 1)! \lambda^N F^{(1)} = (-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k-1} k(1 + \lambda t)^{k-1} F^{(k)} \]

\[ + (-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k}(1 + \lambda t)^k F^{(k+1)}. \]  

(3.11)

Multiply \((1 + \lambda t)\) on the both sides of (3.10), we have

\[ (N + 1)! \lambda^N (1 + \lambda t) F^{N} F^{(1)} = (-1)^N \sum_{k=1}^{N} k a_k(N) \lambda^{N-k} (1 + \lambda t)^k F^{(k)} \]

\[ + (-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k} (1 + \lambda t)^{k+1} F^{(k+1)}. \]  

(3.12)

Then, by (3.10) and (3.11), we obtain

\[ (N + 1)! \lambda^{N+1} F^{N+2} = (-1)^{N+1} \sum_{k=1}^{N} k a_k(N) \lambda^{N-k+1} (1 + \lambda t)^k F^{(k)} \]

\[ + (-1)^{N+1} \sum_{k=1}^{N} a_k(N) \lambda^{N-k} (1 + \lambda t)^{k+1} F^{(k+1)} \]

\[ = (-1)^{N+1} \sum_{k=1}^{N} k a_k(N) \lambda^{N-k+1} (1 + \lambda t)^k F^{(k)} \]

\[ + (-1)^{N+1} \sum_{k=2}^{N+1} a_{k-1}(N) \lambda^{N-k+1} (1 + \lambda t)^k F^{(k)} \]

\[ = (-1)^{N+1} a_1(N) \lambda^N (1 + \lambda t) F^{(1)} \]

\[ + (-1)^{N+1} a_N(N)(1 + \lambda t)^{N+1} F^{(N+1)} \]

\[ + (-1)^{N+1} \sum_{k=2}^{N} (k a_k(N) + a_k(N)(1 + \lambda t)^{k} F^{(k)}. \]  

(3.13)
By substituting $N$ by $N + 1$ given in (3.3), we have another equation.

\[
(N + 1)! N^{N+1} F^{N+2} = (-1)^{N+1} \sum_{k=1}^{N+1} a_k (N + 1) N^{-k+1} (1 + \lambda t)^k F^{(k)}
\]

\[
= (-1)^{N+1} a_1 (N + 1) N^1 (1 + \lambda t) F^{(1)}
\]

\[
+ (-1)^{N+1} a_{N+1} (N + 1) (1 + \lambda t)^{N+1} F^{(N+1)}
\]

\[
+ (-1)^{N+1} \sum_{k=2}^{N} a_k (N + 1) N^{-k+1} (1 + \lambda t)^k F^{(k)}.
\]

Comparing the coefficients on the both sides of (3.3) and (3.4), we have

\[
a_1(N + 1) = a_1(N), \quad a_{N+1}(N + 1) = a_N(N),
\]

and

\[
a_k(N + 1) = k a_k(N) + a_{k-1}(N), \quad \text{for} \quad 2 \leq k \leq N.
\]

From (3.14) and (3.15), for $N = 1$, we obtain

\[
\lambda^2 F^2 = -\sum_{k=1}^{1} a_k (1) \lambda^{-k}(1 + \lambda t)^k F^{(k)}
\]

\[
= -a_1(1)(1 + \lambda t) F^{(1)}
\]

\[
= -(1 + \lambda t) F^{(1)}.
\]

From (3.16), we get

\[
a_1(1) = 1.
\]

From (3.17), we have the following result using (3.16).

\[
a_1(N + 1) = a_1(N) = a_1(N - 1) = \cdots = a_1(1) = 1.
\]

and

\[
a_{N+1}(N + 1) = a_N(N) = a_{N-1}(N - 1) = \cdots = a_1(1) = 1.
\]

From (3.16), for $2 \leq k \leq N$, we have

\[
a_k(N + 1) = k a_k(N) + a_{k-1}(N)
\]

\[
= k(ka_k(N - 1) + a_{k-1}(N - 1)) + a_{k-1}(N)
\]

\[
= k^2 a_k(N - 1) + ka_{k-1}(N - 1) + a_{k-1}(N)
\]

\[
= \cdots
\]

\[
= k^{N-k+1} a_k(k) + k^{N-k} a_{k-1}(k) + \cdots + a_{k-1}(N)
\]

Therefore by (3.14) and (3.15), we get
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\[ a_k(N + 1) = k^{N-k+1} a_k(k) + k^{N-k} a_{k-1}(k) + \cdots + a_k(N) \]

\[ = \sum_{i_1=0}^{N-k+1} k^{N-k+1-i_1} a_{k-1}(k - 1 + i_1) \]

\[ = \sum_{i_1=0}^{N-k+1} k^{N-k+1-i_1} \sum_{i_2=0}^{i_1} (k-1)^{i_1-i_2} a_{k-2}(k - 2 + i_2) \]

\[ = \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} k^{N-k+1-i_1} (k-1)^{i_1-i_2} a_{k-2}(k - 2 + i_2) \]

\[ = \cdots \]

\[ = \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-1}=0}^{i_{k-2}} k^{N-k+1-i_1} (k-1)^{i_1-i_2} \cdots 2^{i_{k-2}-i_{k-1}} a_1(1 + i_{k-1}). \]  

(3.22)

From (3.22) and (3.22), we obtain

\[ a_k(N + 1) = \sum_{i_1=0}^{N-k+1} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-1}=0}^{i_{k-2}} k^{N-k+1-i_1} (k-1)^{i_1-i_2} \cdots 2^{i_{k-2}-i_{k-1}} a_1(1 + i_{k-1}). \]  

(3.23)

Thus, we have the following theorem.

**Theorem 3.1.** Let \( N \in \mathbb{N} \). Then the following differential equation,

\[ N!\lambda^N F^{N+1} = (-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k} (1 + \lambda t)^k F^{(k)} \]

have a solution \( F = F(t) = \frac{1}{2\lambda + \log(1 + \lambda t)}, \) where \( a_N(N) = 1, \ a_1(N) = 1 \)

and

\[ a_k(N) = \sum_{i_1=0}^{N-k} \sum_{i_2=0}^{i_1} \cdots \sum_{i_{k-1}=0}^{i_{k-2}} k^{N-k-i_1} (k-1)^{i_1-i_2} \cdots 2^{i_{k-2}-i_{k-1}}, \text{ for } 2 \leq k \leq N. \]
From (3.24), we get

\[
F = \frac{1}{2\lambda + \log(1 + \lambda t)}
\]

\[
= \frac{t}{\log(1 + t)} \times \frac{1}{2\lambda} \times 2 \log(1 + t)
\]

\[
= \left( \sum_{l_1=0}^{\infty} b_{l_1} \frac{t^{l_1}}{l_1!} \right) \times \left( \sum_{l_2=1}^{\infty} \frac{CG_{l_2\lambda} t^{l_2-1}}{2\lambda l_2!} \right)
\]

\[
= \sum_{l_3=0}^{\infty} \left( \sum_{l_2=0}^{l_3} \frac{(l_3 - l_2) CG_{l_2+1\lambda} 1}{2\lambda(l_2 + 1) l_2!} \right) t^{l_3} \frac{1}{l_3!}
\]

(3.24)

From the above equation, we get

\[
F^{(k)} = \left( \frac{d}{dt} \right)^k F(t)
\]

\[
= \left( \frac{d}{dt} \right)^k \left( \sum_{l_1=0}^{\infty} \left( \sum_{l_2=0}^{l_3} \frac{(l_3 - l_2) CG_{l_2+1\lambda} 1}{2\lambda(l_2 + 1) l_2!} \right) t^{l_3} \frac{1}{l_3!} \right)
\]

\[
= \sum_{l_3=0}^{\infty} \left( \sum_{l_2=0}^{l_3} \frac{(l_3 + k) CG_{l_2+k+1\lambda} 1}{2\lambda(l_2 + 1) l_2!} \right) t^{l_3} \frac{1}{l_3!}
\]

(3.25)

Multiply $2^{N+1}\lambda(\log(1 + t))^{N+1}$ on the right sides of (3.24), we get

\[
(-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k+1} 2^{N+1} (\log(1 + t))^{N+1} (1 + \lambda t)^k F^{(k)}
\]

\[
= (-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k+2} 2^{N+1} \left( \sum_{M_1=N+1}^{\infty} S_1(M_1, N + 1) \frac{t^{M_1}}{M_1!} \right)
\]

\[
\times \left( \sum_{M_2=0}^{k} \frac{(k)_{M_2} \lambda^{M_2} t^{M_2}}{M_2!} \right) F^{(k)}
\]

\[
= (-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k+2} 2^{N+1} \sum_{M_3=N+1}^{\infty} \left( \sum_{M_1=N+1}^{M_3} \frac{M_3}{M_1!} \right)
\]

\[
\times S_1(M_1, N + 1)(k)_{M_3-M_1} \lambda^{M_3-M_1} \frac{t^{M_3}}{M_3!} F^{(k)}.
\]

(3.26)
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Where \( S_1(n, k) \) is the Stirling number of the first kind.

Thus, by (3.26) and (3.27), we get

\[
(-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k+1} 2^{N+1} (\log(1 + t))^{N+1} (1 + \lambda t)^k F^{(k)}
\]

\[
= (-1)^N \sum_{k=1}^{N} a_k(N) \lambda^{N-k+1} 2^{N+1} \sum_{M_3=N+1}^{\infty} \left( \sum_{M_1=N+1}^{M_3} \binom{M_3}{M_1} \right) t^{M_3} \frac{1}{M_3!} 
\]

\[
\times S_1(M_1, N+1) (k)_{M_3-M_1} \lambda^{M_3-M_1} \left( \sum_{l_2=0}^{n} \sum_{l_3=0}^{l_3+k} \binom{l_3+k}{l_2} b_{l_3-l_2+k} CG_{l_2+1, \lambda} \frac{1}{2 \lambda (l_2+1)} \right) \frac{t^{l_3}}{l_3!}
\]

\[
= (-1)^N \sum_{n=N+1}^{\infty} \left( \sum_{k=1}^{n} \sum_{M_3=N+1}^{M_3=M_1=N+1} \sum_{l_2=0}^{n-M_3+k} \binom{n}{M_3} \binom{M_3}{M_1} \binom{n-M_3+k}{l_2} \right)
\]

\[
\times a_k(N) \lambda^{N-k+M_3-M_1} 2^N S_1(M_1, N+1) (k)_{M_3-M_1} b_{n-M_3-l_2+k} \times CG_{l_2+1, \lambda} \frac{1}{l_2+1} \times \frac{t^n}{n!}.
\]

(3.27)

Also, multiply \( 2^{N+1} \lambda (\log(1 + t))^{N+1} \) on the left sides of (3.26), we get

\[
N! 2^{N+1} \lambda^{N+1} (\log(1 + t))^{N+1} F^{N+1} = N! \left( \frac{2 \log(1 + t)}{2 + \log e_\lambda} \right)^{N+1}
\]

\[
= N! \sum_{n=N+1}^{\infty} CG_{n, \lambda}^{(N+1)} \frac{t^n}{n!}.
\]

(3.28)

By equation (3.17), (3.26) and (3.27), we finally get the explicit expression arising from nonlinear differential equation.
**Theorem 3.2.** For $n \geq N + 1$, we have

$$
\overline{CG}_{n, \lambda}^{(N+1)} = \frac{(-1)^N}{N!} \sum_{k=1}^{N} \sum_{M_3=M_1=N+1}^{n} \sum_{M_1=M_2}^{n-M_3+k} \sum_{l_2=0}^{M_3} \binom{n}{M_3} \binom{M_3}{M_1} \binom{n-M_3+k}{l_2} a_k(N)
\times \lambda^{N-k+M_3-M_1} 2^N S_1(M_1, N+1)(k)_{M_3-M_1} b_{n-M_3-l_2+k} CG_{l_2+1, \lambda} \frac{1}{l_2+1}.
$$

4. Conclusion

T. Kim have studied some identities of Changhee numbers which are derived from generating function using nonlinear differential equation (see [8]). In this paper, we study some identities of the partially degenerate Changhee-Genocchi polynomials and the partially degenerate Changhee-Genocchi number arising from nonlinear differential equation. In **Theorem 2.3** and **Theorem 2.4**, we get the same identities of the partially degenerate Changhee-Genocchi polynomials. In **Theorem 3.1**, we get the solution of nonlinear differential equation arising from generating function of the partially degenerate Changhee-Genocchi numbers. In **Theorem 3.2**, we have explicit expression of the partially degenerate Changhee-Genocchi number from the result of **Theorem 3.1** using generating function and nonlinear differential equations.

**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
All authors contributed equally to this work. All authors read and approved the final manuscript.

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**References**
The partially degenerate Changhee-Genocchi polynomials


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