An intrinsic formulations for incompressible Navier-Stokes turbulent flow

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This paper proposes an explicit and simple representation of velocity fluctuation and the Reynolds stress tensor in terms of the mean velocity field. The proposed formulations reveal that the mean vorticity is the key source of producing turbulence. It is found that there is no turbulence if there were no vorticity. As a natural consequence, the laminar-turbulence transition condition was obtained in a rational way.

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INTRODUCTION

Turbulence is a difficulty subject, which pervades so many aspects of peoples' daily lives [1–16]. It is believe that the turbulence flow are govern by the Navier-Stokes momentum equation is $\rho\boldsymbol{u}_{,t}+\boldsymbol{\nabla}\cdot\boldsymbol{\Pi}=0$, continuity equation of incompressible flow is $\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$, where the energy-moementum tensor given by $\boldsymbol{\Pi}=p\boldsymbol{I}+\rho\boldsymbol{u}\otimes\boldsymbol{u}-\mu(\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{u}\boldsymbol{\nabla})$, dynamic viscosity μ, gradient operator $\boldsymbol{\nabla}=\boldsymbol{e}_i\partial_i$, base vector in the i-coordinate $\boldsymbol{e}_i$, and tensor product $\otimes$.

To solve the problem, in 1895 Reynolds published a seminal work on turbulence [30], in which he proposed that flow velocity $\boldsymbol{u}$ and pressure p are decomposed into its time-averaged quantities, $\bar{\boldsymbol{u}}, t$, $\bar{p}$, and fluctuating quantities, $\boldsymbol{u}'$, p'; thus, the Reynolds decompositions are: $\boldsymbol{u}=\bar{\boldsymbol{u}}(\boldsymbol{x},t)+\boldsymbol{u}'(\boldsymbol{x},t)$ and $p(\boldsymbol{x},t)=\bar{p}(\boldsymbol{x},t)+p'(\boldsymbol{x},t)$, where coordinates and times are $(\boldsymbol{x},t)$.

Applying the Reynolds decomposition and averaging operation, we have the Reynolds averaged Naviers-Stiokes turbulence equations (RANS): $\rho\bar{\boldsymbol{u}}_{,t}+\rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}})+\boldsymbol{\nabla}\bar{p}=\mu\boldsymbol{\nabla}^2\bar{\boldsymbol{u}}-\rho\boldsymbol{\nabla}\cdot(\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'})$ and continuity equation of the mean velocity: $\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}}=0$, in which the Reynolds stress tensor is defined as $\boldsymbol{\tau}=-\rho\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'}=-\rho\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}(\boldsymbol{u}'\otimes\boldsymbol{u}')dt$, and T is the period of time over which the averaging takes place and must be sufficiently large to give meaningful averages. For a time average to make sense, the mean flow has to be steady, namely, $\bar{\boldsymbol{u}}_{,t}=0$. Without this constraint, the Reynolds averaged turbulence equations would be meaningless [2]. With respect to the time, generally speaking, the average of the derivative $\partial u/\partial t$, like the average of the derivative of any bounded quantity, is zero. Let $f(t)$ be such a quantity. Then the average value of the derivative df/dt over a certain time interval T is $\overline{\frac{df}{dt}}=\frac{1}{T}\int_0^T\frac{df}{dt}dt=\frac{f(T)-f(0)}{T}$. Since $f(t)$ varies only within finite limits, then as T increase without limit, the average value of df/dt clearly goes zero [19].

For a general three-dimensional flow, there are four independent equations governing the mean velocity field; namely three components of the Reynolds equations together with one mean continuity equation. However, these four equations contain more than four unknowns. In addition to $\bar{\boldsymbol{u}}$ and $\bar{p}$ (four quantities), there are also the Reynolds stresses. The Reynolds average Navier-Stokes (RANS) equations are unclosed. This is a manifestation of the turbulence closure problem. This closure issue has eluded scientists and mathematicians for ages.

All literature state the Reynolds stress tensor has six unknowns, which make up to ten unknowns in total for the Reynolds turbulence equations [1–8, 10–16]. Later we will show that the Reynolds stress tensor has actually only three unknowns.

In 1940 and 1945, P.-Y. Chou [31, 32] published a remarkable result and pointed out that because the Navier-Stokes equations are the basic dynamical equations of fluid motion, it is insufficient to consider only the mean turbulent motion. The turbulent fluctuations are as important as the mean motion and the equations for turbulent fluctuations also need to be considered. Subtracting the mean motions from the Navier-Stokes equation and continuity equation, Chou [31, 32] obtained the Chou Naviers-Stokes turbulence equations (CNS): $\rho\boldsymbol{u}'_{,t}+\rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\boldsymbol{u}'+\boldsymbol{u}'\otimes\bar{\boldsymbol{u}}+\boldsymbol{u}'\otimes\boldsymbol{u}')+\boldsymbol{\nabla}p'=\mu\boldsymbol{\nabla}^2\boldsymbol{u}'+\rho\boldsymbol{\nabla}\cdot(\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'})$ and $\boldsymbol{\nabla}\cdot\boldsymbol{u}'=0$.

Although Chou [32] mentioned that the rigorous way of treating the turbulence problem is probably to solve the Reynolds' equations of mean motion and the equations of turbulent fluctuation simultaneously. However, from the presentation of [32] and all his subsequent publications [33–37], we noticed that Chou together with all other researchers [1–8, 10–16] did not realise that the fluctuation equations together with the mean equations already can

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form a closed equations system.

But researchers are making progress on understanding the physics of the Reynolds stresses. In a Preprint published on 28 June 2018 in Preprints.org, a new perspectives proposed by Sun [38–41] could help to solve the long-standing puzzle over the turbulence closure issue. He proven that the Reynolds stress tensor is not a general second order tensor with six independent elements, while its each element is the product of two fluctuation velocity components. There are 3 velocity components in 3D flow, therefore the number of independent parameters is 3 rather than 6, namely the three components of the fluctuation velocity. For three dimensional flow, we can only have three components of fluctuation velocity $\boldsymbol{u}'$ as unknowns. It means that the Reynolds stress tensor has only three unknowns, namely u'_1, u'_2, u'_3. For two dimensional flow, of course, the 2D Reynolds stress tensor has only two unknowns, namely u'_1, u'_2.

With this new understanding, Sun [38–41] showed the integral-differential equations of the Reynolds mean and fluctuation equations have same number of unknowns That is why Sun [38–41] claim that the Reynolds Navier-Stokes turbulence equations of incompressible flow are closed rather than unclosed.

Although the closed turbulence equations are obtained, however they are not easy to be solved. For simplification purpose, some kind of approximation should be introduced. To simplify the closed turbulence equations, Sun [40] proposed an explicit velocity fluctuation by using dimensional analysis, namely $\boldsymbol{u}' = \frac{\mu}{\rho U}\boldsymbol{\nabla} \times \bar{\boldsymbol{u}}$. However, since the characteristic flow velocity U is not an intrinsic quantity, the proposed model in [40] should be enhanced further. In this article, we will utilize the fluctuation equation to propose an intrinsic formulation of the velocity fluctuation, and to construct the Reynolds stress tensor and to establish the laminar-turbulence transition condition as well.

Section II gives the closed turbulence equations; Section III proposes the first order approximation of velocity fluctuation; Section IV obtains the Reynolds stress in terms of mean vorticity; Section V lists the formulations in the Cartesian coordinates; Section VI presents some important consequences and laminar-turbulence transition condition is formulated; finally concluded in Section VII.

THE CLOSED TURBULENCE EQUATIONS

A closed turbulence equations of incompressible flow proposed by Sun [38–41] can be given as follows

$$\rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}}) + \boldsymbol{\nabla}\bar{p} = \mu\boldsymbol{\nabla}^2\bar{\boldsymbol{u}} - \rho\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\boldsymbol{\nabla}\cdot(\boldsymbol{u}'\otimes\boldsymbol{u}')dt, \tag{1}$$

$$\rho\boldsymbol{u}'_{,t} + \rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\boldsymbol{u}' + \boldsymbol{u}'\otimes\bar{\boldsymbol{u}} + \boldsymbol{u}'\otimes\boldsymbol{u}') + \boldsymbol{\nabla}p' = \mu\boldsymbol{\nabla}^2\boldsymbol{u}' + \rho\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\boldsymbol{\nabla}\cdot(\boldsymbol{u}'\otimes\boldsymbol{u}')dt, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \tag{3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}' = 0. \tag{4}$$

Denoting kinematic viscosity $\nu = \mu/\rho$, the above equations be equivalently rewritten in a conventional form

$$\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}} + \frac{1}{\rho}\boldsymbol{\nabla}\bar{p} = \nu\boldsymbol{\nabla}^2\bar{\boldsymbol{u}} - \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}(\boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}')dt, \tag{5}$$

$$\boldsymbol{u}'_{,t} + \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{u}' + \boldsymbol{u}'\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}} + \boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}' + \frac{1}{\rho}\boldsymbol{\nabla}p' = \nu\boldsymbol{\nabla}^2\boldsymbol{u}' + \lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}(\boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}')dt, \tag{6}$$

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \tag{7}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}' = 0. \tag{8}$$

Applying the divergence operation $\boldsymbol{\nabla}$ on both sides of the Eqs.(5,6), we can obtain equations for both mean and fluctuation pressure as follows

$$\boldsymbol{\nabla}^2\bar{p} = -\rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}}) - \rho\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\boldsymbol{\nabla}\cdot(\boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}')dt, \tag{9}$$

$$\boldsymbol{\nabla}^2 p' = -\rho\boldsymbol{\nabla}\cdot[\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{u}' + \boldsymbol{u}'\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}} + \boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}'] + \rho\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\boldsymbol{\nabla}\cdot(\boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}')dt. \tag{10}$$

For later use, dot multiplying Eq.(6) with $\bar{\boldsymbol{u}}'$, we can obtain fluctuation kinetic equation as follows

$$\boldsymbol{u}'\cdot[\boldsymbol{u}'_{,t} + \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\boldsymbol{u}' + \boldsymbol{u}'\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}} + \boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}' + \frac{1}{\rho}\boldsymbol{\nabla}p'] = \nu\boldsymbol{u}'\cdot\boldsymbol{\nabla}^2\boldsymbol{u}' + \boldsymbol{u}'\cdot\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}(\boldsymbol{u}'\cdot\boldsymbol{\nabla}\boldsymbol{u}')dt. \tag{11}$$

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Note tensor identity $(u'_{i,k})^2 = \boldsymbol{\nabla u'} : \boldsymbol{\nabla u'}$. The above equation can be rearranged equivalently as follows

$$\begin{aligned}&\frac{1}{2}(\boldsymbol{u}')^2_{,t} + \boldsymbol{u}' \cdot (\boldsymbol{u}' \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}}) + \nu \boldsymbol{\nabla u}' : \boldsymbol{\nabla u}' \\ &- \boldsymbol{u}' \cdot \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} (\boldsymbol{u}' \cdot \boldsymbol{\nabla u}')dt \\ &= \boldsymbol{\nabla} \cdot [-\frac{1}{2}(\boldsymbol{u}')^2(\bar{\boldsymbol{u}} + \boldsymbol{u}') - \frac{1}{\rho} p' \boldsymbol{u}' + \frac{1}{2}\nu(\boldsymbol{u}')^2],\end{aligned} \tag{12}$$

where $(\boldsymbol{u}')^2 = u_1'^2 + u_2^2 + u_3^2$. The terms on the right gives zero on integration over the whole region Ω of the flow, since $\bar{\boldsymbol{u}} = \boldsymbol{u}' = 0$ on the boundary surfaces of the region or at infinity. This gives as the required equation

$$\begin{aligned}&\int_\Omega [\frac{1}{2}(\boldsymbol{u}')^2_{,t} + \boldsymbol{u}' \cdot (\boldsymbol{u}' \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}}) + \nu \boldsymbol{\nabla u}' : \boldsymbol{\nabla u}' \\ &- \boldsymbol{u}' \cdot \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} (\boldsymbol{u}' \cdot \boldsymbol{\nabla u}')dt]dV = 0.\end{aligned} \tag{13}$$

This is an integrated invariant of turbulence incompressible flow. Note that the term $-\frac{1}{2}(\boldsymbol{u}')^2(\bar{\boldsymbol{u}} + \boldsymbol{u}') - \frac{1}{\rho}p'\boldsymbol{u}' + \frac{1}{2}\nu(\boldsymbol{u}')^2$ in Eq.(12) non-linear in velocity fluctuation $\boldsymbol{u}'$ does not contribute to the relation Eq.(13).

THE FIRST ORDER APPROXIMATION OF VELOCITY FLUCTUATION

Although the turbulence problem can be numerically calculated by Eqs.(5,6,7,8), it would be useful to have some theoretical consideration from the equations. Since the key issue of turbulence analysis is to find the mean field of velocity and pressure, therefore one approximation strategy is to find the estimation of the velocity fluctuation field under the incompressibility condition $\boldsymbol{\nabla} \cdot \boldsymbol{u}' = 0$ and time average conditions $\bar{\boldsymbol{u}'} = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \boldsymbol{u}'dt = 0$ as well.

The question is: how does one construct velocity fluctuation $\boldsymbol{u}'$? Since there are no definitive stances on what turbulence is, or no scientific definition of turbulence, it is hard to guess the weight of velocity fluctuation $\boldsymbol{u}'$ within the flow velocity $\boldsymbol{u}$. But what is apparent is that the mean field and fluctuation counterpart of velocity is interconnected, for instance, the faster $\bar{\boldsymbol{u}}$ goes, the higher the frequency $\boldsymbol{u}'$, which implies that the velocity fluctuation $\boldsymbol{u}'$ is dependent on the mean velocity field $\bar{\boldsymbol{u}}$. With this understanding one can propose that the velocity fluctuation $\boldsymbol{u}'$ is a function of mean velocity $\bar{\boldsymbol{u}}$, namely $\boldsymbol{u}' = \boldsymbol{u}'(\bar{\boldsymbol{u}}, t)$. Therefore, the Reynolds velocity decomposition can be rewritten as $\boldsymbol{u} = \bar{\boldsymbol{u}} + \boldsymbol{u}'(\bar{\boldsymbol{u}}, t)$. However, the closure problem would still be there if $\boldsymbol{u}'(\bar{\boldsymbol{u}}, t)$ cannot be proposed.

Numerous observations [1–16] have shown that turbulence is caused by excessive kinetic energy in parts of a fluid flow, which overcomes the damping effect of the fluid's viscosity. Hence, turbulence is easier to create in low viscosity fluids, but more difficult in highly viscous fluids. The dynamic balance between kinetic energy and viscous damping in a fluid flow can be perceived as flow symmetries that are broken by mechanisms, which produce turbulence, and are restored by the chaotic character of the cascade to small scales [11]. This dynamic balance process is the key source to generating velocity fluctuation $\boldsymbol{u}'$. In particular, it is found that the mean velocity vorticity $\bar{\boldsymbol{\omega}} = \boldsymbol{\nabla} \times \bar{\boldsymbol{u}}$ plays an essential role in producing turbulence. This means that velocity fluctuation $\boldsymbol{u}'$ should be a function of both mean velocity $\bar{\boldsymbol{u}}$ and its vorticity $\bar{\boldsymbol{\omega}}$.

To satisfy the incompressibility condition $\boldsymbol{\nabla} \cdot \boldsymbol{u}' = 0$, the velocity fluctuation $\boldsymbol{u}'$ must be divergence-free, hence we can introduce a vector function $\boldsymbol{\psi}$ and let

$$\boldsymbol{u}' = \boldsymbol{\nabla} \times \boldsymbol{\psi}. \tag{14}$$

Based on the above understanding and mathematical requirements, we like propose the following conjecture for the vector $\boldsymbol{\psi}$:

$$\boldsymbol{\psi} = f(\bar{\boldsymbol{u}})A(\boldsymbol{x}, t), \tag{15}$$

where $A(\boldsymbol{x}, t)$ is a scalar function and represents the nature of fluctuation. How should one determine the function $f(\bar{\boldsymbol{u}})$? Considering the vector as a first order tensor, if $f(\bar{\boldsymbol{u}})$ is a homogenous function of $\bar{\boldsymbol{u}}$, we should express that the function $f(\bar{\boldsymbol{u}}) = \boldsymbol{b} + \alpha\bar{\boldsymbol{u}}$, in which the constant vector $\boldsymbol{a}$ can be omitted, since $\boldsymbol{\nabla} \times \boldsymbol{b} = 0$. Physically, there is no velocity fluctuation $\boldsymbol{u}'$ if there is no mean velocity $\bar{\boldsymbol{u}}$, namely $\boldsymbol{u}'_{\bar{\boldsymbol{u}}=\boldsymbol{0}} = 0$.

If one substitutes $\phi = 0$ and Eq. (15) into Eq. (16), it will lead to the following velocity fluctuation:

$$\boldsymbol{u}' = \alpha \boldsymbol{\nabla} \times [\bar{\boldsymbol{u}} A(\boldsymbol{x}, t)] = \alpha \left(A \boldsymbol{\nabla} \times \bar{\boldsymbol{u}} + \boldsymbol{\nabla} A \times \bar{\boldsymbol{u}}\right), \tag{16}$$

where the α is a constant with the length dimension. The time mean condition $\bar{\boldsymbol{u}'} = 0$, which requires $\bar{A} = 0$ and $\overline{\boldsymbol{\nabla} A} = 0$.

Analogues to the study of flow stability by Landau [42], we can assume $A(\boldsymbol{x}, t)$ is space independent function $A(t)$, which represents the velocity fluctuation. Hence, the velocity fluctuation can be expressed as:

$$\boldsymbol{u}' = A(t) \boldsymbol{\nabla} \times \bar{\boldsymbol{u}} = A(t) \bar{\boldsymbol{\omega}}. \tag{17}$$

The beauty of the velocity fluctuation in Eq. (17) is that both the incompressibility $\boldsymbol{\nabla} \cdot \boldsymbol{u}' = 0$ and time average

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conditions $\bar{\boldsymbol{u}'} = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} \boldsymbol{u}' dt = 0$ can be satisfied simultaneously under the time mean conditions $\bar{A}(t) = 0$.

Since the velocity fluctuation is proportional to the vorticity $\bar{\boldsymbol{\omega}} = \boldsymbol{\nabla}\times\bar{\boldsymbol{u}}$, and Curl of the mean velocity, $\boldsymbol{\nabla}\times\bar{\boldsymbol{u}}$ is a three dimensional quantity; therefore, the turbulence is always rotational and three dimensional, whilst characterized by high levels of fluctuation vorticity. Hence, vorticity dynamics play an essential role in the description of turbulent velocity fluctuations [2].

To determine the function $A(t)$, substituting Eq.(17) into Eq.(13),this gives as the required equation about $A(t)$

$$A_{,t} = aA - b\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}[A(\xi)]^2 d\xi, \tag{18}$$

where the coefficients, $a = \frac{1}{a_0}(a_1 - \nu a_2)$ and integration constants $a_0 = \int_\Omega \bar{\boldsymbol{\omega}}\cdot\bar{\boldsymbol{\omega}} dV$ is always positive ; $a_1 = -\int_\Omega \bar{\boldsymbol{u}}\cdot(\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})dV$ represents the energy exchange between the mean flow and fluctuation, and may have either sign; $a_2 = \int_\Omega (\bar{\omega}_{i,k})^2 dV = \int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} : \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} dV$ is the fluctuation dissipative loss, and is also always positive; $b = \int_\Omega \bar{\boldsymbol{\omega}}\cdot(\bar{\boldsymbol{\omega}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{\omega}})dV$.

Eq.(18) is a nonlinear integral-differential equation, which is very hard to solve due to the limits of $T\to\infty$. Here we are going to propose two ways and see which one works.

(1) Option one

Eq.(18) can be expressed as its equivalent form

$$A_{,t} = aA - \frac{b}{T}\int_t^{t+T}[A(\xi)]^2 d\xi, \quad \text{for a large } T. \tag{19}$$

Its iterative scheme is

$$\frac{d}{dt}(e^{-at}A_{n+1}) = -\frac{b}{T}e^{-at}\int_t^{t+T}(A_n)^2 d\xi. \tag{20}$$

Using iterative method, for the given observation interval T, the solution of Eq.(20) can be obtained, for instance, the 3rd order solution is

$$A_1(t,T) = A(0)e^{at}, \tag{21}$$
$$A_2(t,T) = c + c_2 e^{at}, \tag{22}$$

where $c = -\frac{bA^2(0)}{2Ta^2}(e^{2aT}-1)$, $c_2 = A(0) - c$; and the 3rd order solution

$$\begin{aligned} A_3(t,T) = &-\frac{bc^2}{a} - \frac{2cc_2}{aT}(e^{aT}-1)te^{at} \\ &- \frac{bc_2^2}{2a^2T}(e^{2aT}-1)e^{2at} + c_3 e^{at}. \end{aligned} \tag{23}$$

From $t = 0$, $A_3 = A(0)$, we have $c_3 = \frac{bc^2}{a} + \frac{bc_2^2}{2Ta^2}(e^{2aT} - 1)$. These iterative process can be carried on and more accurate solutions can be obtained.

If we take the limits of $T\to\infty$, the above solution tend to infinite. We can not use those solution and alternative way has to be find out.

(2) Option two

From turbulence physics, the mean velocity is a slow function while the velocity fluctuation is a fast function, however, its square can be considered as slow function. So that we can make an approximation $\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}[A(\xi)]^2 d\xi \approx \lim_{T\to\infty}\frac{1}{T}[A(\xi)]^2(t+T-t) = A^2(t)$, hence the Eq.(18) can be approximated as follows

$$A_{,t} = aA - bA^2. \tag{24}$$

This is the Riccati equation, whose solution is

$$A(t) = \frac{A(t_0)e^{at}}{[1-\frac{b}{a}A(t_0)]e^{at} + \frac{b}{a}A(t_0)e^{at}}. \tag{25}$$

Ensure to have finite value of $A(t)$, the exponent a must be negative, i.e., $a < 0$. In this case, $\lim_{T\to\infty} A(t) = \frac{a}{b}$, thus $\bar{A}(t) = 0$ is secured, and $\bar{A}^2 = (\frac{a}{b})^2$.

In summary, we have velocity fluctuation as follows

$$\boldsymbol{u}' = A(t)\boldsymbol{\nabla}\times\bar{\boldsymbol{u}} = A(t)\bar{\boldsymbol{\omega}} \tag{26}$$
$$A(t) = -\frac{\int_\Omega \bar{\boldsymbol{u}}\cdot(\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})dV + \nu\int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} : \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} dV}{(\int_\Omega \bar{\boldsymbol{\omega}}\cdot\bar{\boldsymbol{\omega}} dV)[\int_\Omega \bar{\boldsymbol{\omega}}\cdot(\bar{\boldsymbol{\omega}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{\omega}})dV]} \tag{27}$$

Comparing with the formulation of velocity fluctuation proposed by Sun [40], the velocity fluactuation in Eq.(26) has no external adjustable parameter, and all quantities can be calculated by the mean field of velocity.

THE REYNOLDS STRESS TENSOR

The Reynolds stress tensor $\boldsymbol{\tau} = -\rho\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'}$ can be obtained as follows [49]:

$$\begin{aligned} \boldsymbol{\tau} &= -\overline{[\boldsymbol{\nabla}\times(A\bar{\boldsymbol{u}})]\otimes[\boldsymbol{\nabla}\times(A\bar{\boldsymbol{u}})]} \\ &= -\left(\overline{A^2}\bar{\boldsymbol{\omega}}\otimes\bar{\boldsymbol{\omega}} + \overline{(\boldsymbol{\nabla}A\times\bar{\boldsymbol{u}})\otimes(\boldsymbol{\nabla}A\times\bar{\boldsymbol{u}})}\right) \\ &= -\left[\overline{A^2}\bar{\boldsymbol{\omega}}\otimes\bar{\boldsymbol{\omega}} + \overline{(\boldsymbol{\nabla}A\otimes\boldsymbol{\nabla}A)}\,{}^{\times}_{\times}\,(\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}})\right]. \end{aligned} \tag{28}$$

If the scalar function A is only a function of time, the Reynolds stress tensor is given by

$$\boldsymbol{\tau} = -\bar{A}^2\bar{\boldsymbol{\omega}}\otimes\bar{\boldsymbol{\omega}} \tag{29}$$
$$\bar{A}^2 = \left(\frac{\int_\Omega \bar{\boldsymbol{u}}\cdot(\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}})dV + \nu\int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} : \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} dV}{(\int_\Omega \bar{\boldsymbol{\omega}}\cdot\bar{\boldsymbol{\omega}} dV)[\int_\Omega \bar{\boldsymbol{\omega}}\cdot(\bar{\boldsymbol{\omega}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{\omega}})dV]}\right)^2. \tag{30}$$

In this special case, the Reynolds averaged stress tensor is produced fully by the mean vorticity.

This expression of the averaged Reynolds stress tensor reveals that the mean vorticity $\bar{\boldsymbol{\omega}}$ is a key source in producing turbulence, and it is worth commenting here in this regard. The non-linearity between the averaged Reynolds stress tensor and the mean vorticity and velocity is the key feature of turbulence phenomena, and is totally different from molecular diffusivity [2]. Thus, non-linearity of the averaged Reynolds stress tensor is the turbulence mechanism behind rapid mixing.

CLOSED TURBULENCE EQUATIONS

With the explicit velocity fluctuation in Eq. (17), one can formulate the Reynolds averaged Navier-Stokes equations as follows [50]:

$$\rho\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{u}} = -\boldsymbol{\nabla}\bar{p} + \mu\nabla^2\bar{\boldsymbol{u}} - \bar{A}^2\rho\bar{\boldsymbol{\omega}}\cdot\boldsymbol{\nabla}\bar{\boldsymbol{\omega}}, \tag{31}$$
$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0. \tag{32}$$

It is clear that Eq. (31,34,35) are a closed equation, in which the mean velocity field $\bar{\boldsymbol{u}}$ is the only unknown vector.

The above formulations show that although the specific expression of the function $A(\boldsymbol{x}, t)$ is not known, one can still approximately calculate the Reynolds stress tensor $\boldsymbol{\tau}$, the mean velocity $\bar{\boldsymbol{u}}$, mean pressure $\bar{p}$, as well as the kinetic energy of the velocity fluctuation. However, the pressure fluctuation and the vorticity fluctuation cannot be determined without knowing $A(\boldsymbol{x}, t)$. This imperfection does not affect the turbulence study too much, since one of the central issues is to find mean field quantities such as the mean velocity and the pressure, which can been formulated within the current theoretical framework.

FORMULATIONS IN THE CARTESIAN COORDINATES

For a better understanding, the formulations in the Cartesian coordinates are listed below:

The vorticity components are: $\bar{\omega}_x = \frac{\partial\bar{w}}{\partial y} - \frac{\partial\bar{v}}{\partial z}$, $\bar{\omega}_y = \frac{\partial\bar{u}}{\partial z} - \frac{\partial\bar{w}}{\partial x}$, and $\bar{\omega}_z = \frac{\partial\bar{v}}{\partial x} - \frac{\partial\bar{u}}{\partial y}$.

The velocity fluctuation components are:

$$\begin{aligned}
u' &= A(t)(\frac{\partial\bar{w}}{\partial y} - \frac{\partial\bar{v}}{\partial z}),\\
v' &= A(t)(\frac{\partial\bar{u}}{\partial z} - \frac{\partial\bar{w}}{\partial x}),\\
w' &= A(t)(\frac{\partial\bar{v}}{\partial x} - \frac{\partial\bar{u}}{\partial y}).
\end{aligned} \tag{33}$$

The averaged Reynolds stress components are:

$$\begin{aligned}
\tau_{xx} &= -\bar{A}^2(\frac{\partial\bar{w}}{\partial y} - \frac{\partial\bar{v}}{\partial z})^2,\\
\tau_{xy} &= \bar{A}^2(\frac{\partial\bar{w}}{\partial y} - \frac{\partial\bar{v}}{\partial z})(\frac{\partial\bar{u}}{\partial z} - \frac{\partial\bar{w}}{\partial x}),\\
\tau_{xz} &= \bar{A}^2(\frac{\partial\bar{w}}{\partial y} - \frac{\partial\bar{v}}{\partial z})(\frac{\partial\bar{v}}{\partial x} - \frac{\partial\bar{u}}{\partial y}),\\
\tau_{yx} &= \tau_{xy},\\
\tau_{yy} &= -\bar{A}^2(\frac{\partial\bar{u}}{\partial z} - \frac{\partial\bar{w}}{\partial x})^2,\\
\tau_{yz} &= \bar{A}^2(\frac{\partial\bar{u}}{\partial z} - \frac{\partial\bar{w}}{\partial x})(\frac{\partial\bar{v}}{\partial x} - \frac{\partial\bar{u}}{\partial y}),\\
\tau_{zx} &= \tau_{xz},\\
\tau_{zy} &= \tau_{yz},\\
\tau_{zz} &= -\bar{A}^2(\frac{\partial\bar{v}}{\partial x} - \frac{\partial\bar{u}}{\partial y})^2.
\end{aligned} \tag{34}$$

It is clear that the Reynolds stress is proportional to the square mean velocity gradient, which has been proved by experiments. Tennekes and Lumley [2] pointed out that diagonal components of $\boldsymbol{\tau}$, their values ρu_1^2, ρu_2^2, ρu_3^2, in many flows, contribute little to the transport of mean momentum. The off-diagonal components of $\boldsymbol{\tau}$ are shear stresses; they play a dominate role in the theory of mean momentum transfer by turbulent motion. Therefore the diagonal components of the Reynolds stresses, ρu_1^2, ρu_2^2, ρu_3^2, can be omitted if needed.

The closed Navier-Stokes turbulence equations are:

$$\begin{aligned}
\bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y} + \bar{w}\frac{\partial\bar{u}}{\partial z} + \frac{1}{\rho}\frac{\partial\bar{p}}{\partial x} &= \nu\nabla^2\bar{u} - (\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_x,\\
\bar{u}\frac{\partial\bar{v}}{\partial x} + \bar{v}\frac{\partial\bar{v}}{\partial y} + \bar{w}\frac{\partial\bar{v}}{\partial z} + \frac{1}{\rho}\frac{\partial\bar{p}}{\partial x} &= \nu\nabla^2\bar{u} - (\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_y,\\
\bar{u}\frac{\partial\bar{w}}{\partial x} + \bar{v}\frac{\partial\bar{w}}{\partial y} + \bar{w}\frac{\partial\bar{w}}{\partial z} + \frac{1}{\rho}\frac{\partial\bar{p}}{\partial x} &= \nu\nabla^2\bar{u} - (\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_z,
\end{aligned} \tag{35}$$

where the kinematic viscosity is $\nu = \mu/\rho$, and the Laplace operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, and $(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_x = \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}$, $(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_y = \frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\tau_{yy}}{\partial y} + \frac{\partial\tau_{yz}}{\partial z}$, $(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_z = \frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{zy}}{\partial y} + \frac{\partial\tau_{zz}}{\partial z}$.

SOME IMPORTANT CONSEQUENCES

As a byproduct, we can get following important outcomes from the above formulations.

Vorticity is essential, no vorticity no velocity fluctuation and no turbulence

From both $\boldsymbol{u}' = A\boldsymbol{\nabla}\times\bar{\boldsymbol{u}} = A\bar{\boldsymbol{\omega}}$ and $\boldsymbol{\tau} = -\bar{A}^2\bar{\boldsymbol{\omega}}\otimes\bar{\boldsymbol{\omega}}$, it is obviously to see that the mean vorticity $\bar{\boldsymbol{\omega}} = \boldsymbol{\nabla}\times\bar{\boldsymbol{u}}$

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is an essential quantity. If there were no mean vorticity, namely $\bar{\boldsymbol{\omega}} = \mathbf{0}$, then there would be no both the velocity fluctuation $\boldsymbol{u}'$ and the Reynolds stress tensor $\boldsymbol{\tau}$. Therefore, we can even say that no vorticity no turbulence.

Turbulence is three-dimensional, there is no 1-D and 2-D turbulence

From Eq.(34) we can see that the Reynolds stresses are zero in the case of 1-D and 2-D flow. This stems from the fact of three-dimensionality of vorticity $\bar{\boldsymbol{\omega}} = \boldsymbol{\nabla} \times \bar{\boldsymbol{u}}$. Although this consequence is a consensus on turbulence [2], however its formulation is obtained for the first time in this article.

Laminar-turbulence transition condition

Another important issue is about laminar-turbulence transition condition, luckily which is a consequence of this study. It is natural to define the laminar-turbulence transition condition as follows: no velocity fluctuation no turbulence, that is no laminar-turbulence transition.

In the limits of $t \to \infty$, the finite value of $A(t)$ occurs at $a < 0$, while $A(t)$ tend to be zero at $a > 0$. Therefore, turbulence onset condition and or laminar-turbulence transition condition must be $a = 0$, namely

$$-\int_\Omega \bar{\boldsymbol{u}} \cdot (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\bar{\boldsymbol{u}})dV = \nu \int_\Omega \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} : \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} dV. \tag{36}$$

Due to the arbitrary nature of the region Ω, the above onset condition can be reduced to

$$\bar{\boldsymbol{u}} \cdot (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\bar{\boldsymbol{u}}) + \nu \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} : \boldsymbol{\nabla}\bar{\boldsymbol{\omega}} = 0. \tag{37}$$

equivalently in coordinates format

$$\bar{u}_i \bar{u}_j \bar{u}_{i,j} + \nu[(\bar{u}_{i,j} - \bar{u}_{j,i})_{,k}]^2 = 0, \quad (i,j,k = 1,2,3). \tag{38}$$

Its expanding is given by

$$\begin{aligned}
&\bar{u}_1\bar{u}_1\bar{u}_{1,1} + \bar{u}_2\bar{u}_1\bar{u}_{2,1} + \bar{u}_3\bar{u}_1\bar{u}_{3,1} \\
&+ \bar{u}_1\bar{u}_2\bar{u}_{1,2} + \bar{u}_2\bar{u}_2\bar{u}_{2,2} + \bar{u}_3\bar{u}_2\bar{u}_{3,2} \\
&+ \bar{u}_1\bar{u}_3\bar{u}_{1,3} + \bar{u}_2\bar{u}_3\bar{u}_{2,3} + \bar{u}_3\bar{u}_3\bar{u}_{3,3} \\
&+ \nu[(\bar{u}_{1,1} - \bar{u}_{1,1})_{,1}]^2 + \nu[(\bar{u}_{1,2} - \bar{u}_{2,1})_{,1}]^2 + \nu[(\bar{u}_{1,3} - \bar{u}_{3,1})_{,1}]^2 \\
&+ \nu[(\bar{u}_{2,1} - \bar{u}_{1,2})_{,2}]^2 + \nu[(\bar{u}_{2,2} - \bar{u}_{2,2})_{,2}]^2 + \nu[(\bar{u}_{2,3} - \bar{u}_{3,2})_{,2}]^2 \\
&+ \nu[(\bar{u}_{3,1} - \bar{u}_{1,3})_{,3}]^2 + \nu[(\bar{u}_{3,2} - \bar{u}_{2,3})_{,3}]^2 + \nu[(\bar{u}_{3,3} - \bar{u}_{3,3})_{,3}]^2 \\
&= 0.
\end{aligned} \tag{39}$$

Remove the null teams $\bar{u}_{1,1} - \bar{u}_{1,1}$, $\bar{u}_{2,2} - \bar{u}_{2,2}$ and $\bar{u}_{3,3} - \bar{u}_{3,3}$, we have the final laminar-turbulence onset condition as follows

$$\begin{aligned}
&\bar{u}_1\bar{u}_1\bar{u}_{1,1} + \bar{u}_2\bar{u}_1\bar{u}_{2,1} + \bar{u}_3\bar{u}_1\bar{u}_{3,1} \\
&+ \bar{u}_1\bar{u}_2\bar{u}_{1,2} + \bar{u}_2\bar{u}_2\bar{u}_{2,2} + \bar{u}_3\bar{u}_2\bar{u}_{3,2} \\
&+ \bar{u}_1\bar{u}_3\bar{u}_{1,3} + \bar{u}_2\bar{u}_3\bar{u}_{2,3} + \bar{u}_3\bar{u}_3\bar{u}_{3,3} \\
&+ \nu[(\bar{u}_{1,2} - \bar{u}_{2,1})_{,1}]^2 + \nu[(\bar{u}_{1,3} - \bar{u}_{3,1})_{,1}]^2 \\
&+ \nu[(\bar{u}_{2,1} - \bar{u}_{1,2})_{,2}]^2 + \nu[(\bar{u}_{2,3} - \bar{u}_{3,2})_{,2}]^2 \\
&+ \nu[(\bar{u}_{3,1} - \bar{u}_{1,3})_{,3}]^2 + \nu[(\bar{u}_{3,2} - \bar{u}_{2,3})_{,3}]^2 \\
&= 0.
\end{aligned} \tag{40}$$

Introducing parameter $\beta = \frac{\bar{u}_i \bar{u}_j \bar{u}_{i,j}}{\nu(\bar{u}_{i,j} - \bar{u}_{j,i})_{,k}(\bar{u}_{i,j} - \bar{u}_{j,i})_{,k}}$, then the above condition can be expressed in a simpler form:

$$\beta \begin{cases} < 1 & \text{laminar flow} \\ = 0 & \text{turbulent flow} \\ > 0 & \text{instability} \end{cases} \tag{41}$$

It is worth to mention that the turbulence onset condition (37) has nothing to do with any characteristic velocity. Condition (37) is intrinsic since there is no external characteristic velocity be used as in the Reynolds number definition, which means that the turbulence onset condition is fully controlled by local mean velocity and corresponding voticity.

CONCLUSIONS

In conclusion, the main results are listed in the Table below

TABLE I: Turbulence modelling

Items	Formulations
Closed turbulence equations	Eqs.(5,6)
Velocity fluctuation	Eqs.(26,27)
The Reynolds stress tensor	Eqs.(29,30)
Closed RANS equation	Eqs.(31,32)
Turbulence onset condition	Eqs.(37)

This study has attempted to propose a simplification of the velocity fluctuations that can simultaneously satisfy both incompressibility and time-average conditions. The velocity fluctuation and the Reynolds stress tensor has been constructed, the turbulence onset condition is obtained. The simplified closed turbulence formulations show that the mean vorticity has a strong influence on the velocity fluctuation and the Reynolds stress tensor, as well as on the mean pressure. This fact reveals that three-dimensional vorticity fluctuation is a fundamental mechanism of producing turbulence. We can even say that no vorticity no velocity fluctuation and hence no turbulence.

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[50] According to [2], to be compatible the mean velocity should be independent from time, namely $\bar{u}_{,t} = 0$