 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

TL-moments for Type-I Censored Data with an Application to the Weibull Distribution

Hager A. Ibrahim[2], Mahmoud Riad Mahmoud[1], Fatma A. Khalil[2] and Ghada A. El-Kelany[2]

Abstract

This paper aims to provide an adaptation of the TL-moments method to censored data. The present study concentrates on Type-I censored data. The idea of using TL-moments with censored data may seem conflicting. But our perspective in that, we may use data censored from one side and trimmed from the other side. This study is applied to estimate the two unknown parameters of the Weibull distribution. The suggested point is compared with Direct L-moments and ML methods. A Monte Carlo simulation study is carried out to compare these method in terms of estimate average, root of mean square error (RMSE) and relative absolute biases (RAB).

Keywords: Censored Data; Estimation; Direct L-moments; TL-moments; Maximum likelihood; Weibull Distribution.

1 Introduction

In the second half of the last century, there has been a great attention paid for using unconventional estimation methods in the theory of estimation in addition to the classical methods. Classical estimation methods (e.g, method of moments, method of least squares, and maximum likelihood method) work well in cases where the distribution belongs to the exponential family. However, in some applications, the data contain there are some extreme observations which may influence the values of the estimator greatly. Therefore, if there is a concern about outliers, one should use a robust method of estimation which has been developed to reduce the influence of outliers on the final estimates. Using a robust estimation techniques for estimating unknown parameters has a great importance to the investigator in many fields, such as in industrial, medical applications and occasionally in business

[1]Dept. of Mathematical Statistics, Institute of Statistics Studies and Research, Cairo University, Egypt.
[2]Dept. of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

applications. In recent Decades, a great attention for dealing with outliers has been focused on robust estimation methods; see, for example, Barnett and Lewis (1994).

L-moments method has been noticed as appealing alternative to the conventional moments method, see Hosking (1990). To avert the effect of outliers, Elamir and Seheult (2003) introduced an alternative robust approach of L-moments which they called trimmed L-moments (TL-moments). TL-moments have some advantages over L-moments and the method of moments; Tl-moments exist whether or not the mean exist (for example, the Cauchy distribution) and they are more robust to the presence of outliers.

The idea of TL-moments, the expected value $E(X_{r-k:r})$ is replaced with the expected value $E(X_{r+t_1-k:r+t_1+t_2})$. Thus, for each r, we increase the sample size of a random sample from the original r to $r+t_1+t_2$, working only with the expected values of these r modified order statistics $X_{t_1+1:r+t_1+t_2}, X_{t_1+2:r+t_1+t_2}, ..., X_{t_1+r:r+t_1+t_2}$ by trimming the smallest t_1 and largest t_2 from the conceptual random sample. This modification is called the r^{th} trimmed L-moment (TL-moment) and marked as $\lambda_r^{(t_1,t_2)}$.

TL-moment of the r^{th} order of the random variable X is defined as:

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E(X_{r+t_1-k:r+t_1+t_2}), \quad r = 1, 2, \ldots. \tag{1.1}$$

The expectation of the order statistics are given by:

$$E(X_{i:r}) = \frac{r!}{(i-1)!(r-i)!}\int_0^1 u^{i-1}(1-u)^{r-i}q(u)du.$$

Its basic idea for the method of expectation is to take the expected values of some functions of the random variable of interest and extend them to a sample and equate the corresponding results and solve for the unknown parameters.

This paper is concerned with comparing the performance of three estimating methods; namely, TL-moments, Direct L-moments, and maximum likelihood (ML); with Type-I censored data. This study is applied to estimate the two unknown parameters of the Weibull distribution by quantile function takes the form:

$$q(u) = a[-\log(1-u)]^{\frac{1}{b}}, \quad 0 \le u \le 1. \tag{1.2}$$

This article is organized as follows; TL-moments for censored data, in general case, is introduced in Section 2. TL-moments for the Weibull distribution is presented in Section 3. Simulation study and concluding remarks are presented in section 4 and 5 respectively.

2 TL-moments for censored data

For the analysis of censored samples; Wang (1990a, b, 1996a) introduced the concept of partial probability-weighted moments (PPWMs). Hosking (1995) defined two variants of L-moments, which he used with right-censored data. Zafirakou-Koulouris *et al.* (1998) extended the applicability of L-moments to left-censored data. Mahmoud *et al.* (2017) introduced two variant of what they termed the method of Direct L-moments and used them of right and left censored data from the Kumaraswamy distribution.

The aim of this section is introducing an adaptation of the TL-moments method to censored data. In fact, the idea of using TL-moment with censored data may seem conflicting, but the idea is that we may use data censored from one side and trimmed from the other side.

2.1 Right Censoring for Left Trim

Let $x_1, x_2, ..., x_n$ be a Type-I censored random sample of size n from a population with distribution function $F(x)$ and quantile function $q(u)$. We know, from formula of TL-moments (1.1), that TL-moments are defined as:

$$\lambda_r^{(t_1,t_2)} = \frac{(r+t_1+t_2)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!(t_2+k)!} \binom{r-1}{k} \int_0^1 u^{r+t_1-k-1}(1-u)^{t_2+k} q(u)du. \quad (2.1)$$

When we suppose left trim t_1, i.e. $t_2 = 0$. From formula (2.1) we get

$$\lambda_r^{(t_1,0)} = \frac{(r+t_1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} \int_0^1 u^{r+t_1-k-1}(1-u)^{k} q(u)du. \quad (2.2)$$

In this case, let the censoring time T satisfy $F(T) = c$ and c is the fraction of observed data. The random sample takes the form $x_{t_1+1}, x_{t_1+2}, ..., x_n$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

$$\underbrace{x_{1:n} \le x_{2:n} \le \cdots \le x_{t_1:n}}_{t_1 \text{ (trimmed)}} \le \underbrace{x_{t_1+1:n} \le x_{t_1+2:n} \le \cdots \le x_{m:n}}_{\text{m (observed)}} \le T \le \underbrace{x_{m+1:n} \le \cdots \le x_{n-1:n} \le x_{n:n}}_{n-t_1-m \text{ (censored)}}$$

2.1.1 TL-moments for Right Censoring (Type-AT)

The quantile function of Type-AT TL-moments is

$$y^A(u) = q(uc), \quad 0 < u < 1. \tag{2.3}$$

substitution into equation (2.2) leads to the Type-AT TL-moments where:

$$\begin{aligned}\mu_r^{A(t_1,0)} &= \frac{(r+t_1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} \int_0^1 u^{r+t_1-k-1}(1-u)^k y^A(u)du \\ &= \frac{(r+t_1)!}{rc^{r+t_1}} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} \int_0^c u^{r+t_1-k-1}(c-u)^k q(u)du. \end{aligned} \tag{2.4}$$

When we suppose the value of smallest trim is equal to one, i.e. $t_1 = 1$, from (2.4), we get

$$\mu_r^{A(1,0)} = \frac{(r+1)!}{rc^{r+1}} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k)!k!} \binom{r-1}{k} \int_0^c u^{r-k}(c-u)^k q(u)du. \tag{2.5}$$

In this case, the first four Type-AT TL-moments are given by the follows:

$$\mu_1^{A(1,0)} = \frac{2}{c^2} \int_0^c uq(u)du, \tag{2.6a}$$

$$\mu_2^{A(1,0)} = \frac{3}{c^3} \left[\frac{1}{2} \int_0^c u^2 q(u)du - \int_0^c u(c-u)q(u)du \right], \tag{2.6b}$$

$$\begin{aligned}\mu_3^{A(1,0)} &= \frac{4}{c^4} \left[\frac{1}{3} \int_0^c u^3 q(u)du - 2 \int_0^c u^2(c-u)q(u)du \right. \\ &\quad \left. + \int_0^c u(c-u)^2 q(u)du \right], \end{aligned} \tag{2.6c}$$

$$\begin{aligned}\mu_4^{A(1,0)} &= \frac{5}{c^5} \left[\frac{1}{4} \int_0^c u^4 q(u)du - 3 \int_0^c u^3(c-u)q(u)du \right. \\ &\quad \left. + \frac{9}{2} \int_0^c u^2(c-u)^2 q(u)du - \int_0^c u(c-u)^3 q(u)du \right]. \end{aligned} \tag{2.6d}$$

When we suppose the value of smallest trim is equal to two, i.e. $t_1 = 2$, from (2.4), we get

$$\mu_r^{A(2,0)} = \frac{(r+2)!}{rc^{r+2}} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k+1)!k!} \binom{r-1}{k} \int_0^c u^{r-k+1}(c-u)^k q(u)du. \tag{2.7}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

Substituting $r = 1, 2, 3, 4$ in Eq. (2.7), we get the first four Type-AT TL-moments:

$$\mu_1^{A(2,0)} = \frac{3}{c^3} \int_0^c u^2 q(u) du, \tag{2.8a}$$

$$\mu_2^{A(2,0)} = \frac{4}{2c^4} \left[\int_0^c u^3 q(u) du - 3 \int_0^c u^2 (c-u) q(u) du \right], \tag{2.8b}$$

$$\mu_3^{A(2,0)} = \frac{5}{3c^5} \left[\int_0^c u^4 q(u) du - 8 \int_0^c u^3 (c-u) q(u) du + 6 \int_0^c u^2 (c-u)^2 q(u) du \right], \tag{2.8c}$$

$$\mu_4^{A(2,0)} = \frac{6}{4c^6} \left[\int_0^c u^5 q(u) du - 15 \int_0^c u^4 (c-u) q(u) du + 30 \int_0^c u^3 (c-u)^2 q(u) du - 10 \int_0^c u^2 (c-u)^3 q(u) du \right]. \tag{2.8d}$$

Using the method of expectations, Type-AT TL-moments estimators is given by:

$$M_r^{A(t_1,0)} = \frac{1}{r\binom{m}{r+t_1}} \sum_{i=t_1+1}^{m} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t_1-k-1} \binom{m-i}{k} X_{i:n} \tag{2.9}$$

2.1.2 TL-moments for Right Censoring (Type-BT)

The quantile function of Type-BT TL-moments is

$$y^B(u) = \begin{cases} q(u), & 0 < u < c \\ q(c), & c \le u < 1 \end{cases}$$

substitution into the formula of left trimming in (2.2), the Type-BT TL-moments are given by

$$\begin{aligned} \mu_r^{B(t_1,0)} &= \frac{(r+t_1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} \int_0^1 u^{r+t_1-k-1} (1-u)^k y^B(u) du \\ &= \frac{(r+t_1)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} * \\ &\quad \left[\int_0^c u^{r+t_1-k-1} (1-u)^k q(u) du + q(c) \int_c^1 u^{r+t_1-k-1} (1-u)^k du \right]. \end{aligned}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

Using the results in appendix, the second integration can be written as

$$\mu_r^{B(t_1,0)} = \frac{(r+t_1)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r+t_1-k-1)!k!}\binom{r-1}{k}* \left[\beta^c(r+t_1-k,k+1)q(c)+\int_0^c u^{r+t_1-k-1}(1-u)^k q(u)du\right], \tag{2.10}$$

where $\beta^c(a,b)$ is the upper incomplete beta function.

When we suppose the value of smallest trim is equal to one, i.e. $t_1 = 1$, from (2.10), we get

$$\mu_r^{B(1,0)} = \frac{(r+1)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k)!k!}\binom{r-1}{k}* \left[\beta^c(r-k+1,k+1)q(c)+\int_0^c u^{r-k}(1-u)^k q(u)du\right] \tag{2.11}$$

In this case, the first four TL-moments for Type-BT right censoring are calculated as follows:

$$\mu_1^{B(1,0)} = (1-c^2)q(c)+2\int_0^c uq(u)du, \tag{2.12a}$$

$$\mu_2^{B(1,0)} = \left(3\beta_c(2,2)-\frac{c^3}{2}\right)q(c)+\frac{3}{2}\int_0^c u^2q(u)du-3\int_0^c u(1-u)q(u)du, \tag{2.12b}$$

$$\mu_3^{B(1,0)} = \left(8\beta_c(3,2)-4\beta_c(2,3)-\frac{c^4}{3}\right)q(c)+\frac{4}{3}\int_0^c u^3q(u)du - 8\int_0^c u^2(1-u)q(u)du+4\int_0^c u(1-u)^2q(u)du, \tag{2.12c}$$

$$\mu_4^{B(1,0)} = \left(15\beta_c(4,2)-\frac{45}{2}\beta_c(3,3)+5\beta_c(2,4)-\frac{c^5}{4}\right)q(c) + \frac{5}{4}\int_0^c u^4q(u)du-15\int_0^c u^3(1-u)q(u)du + \frac{45}{2}\int_0^c u^2(1-u)^2q(u)du-5\int_0^c u(1-u)^3q(u)du. \tag{2.12d}$$

When we suppose the value of smallest trim is equal to two, i.e. $t_1 = 2$, from (2.10), we get

$$\mu_r^{B(2,0)} = \frac{(r+2)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k+1)!k!}\binom{r-1}{k}* \left[\beta^c(r-k+2,k+1)q(c)+\int_0^c q(u)u^{r-k+1}(1-u)^k du\right] \tag{2.13}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

In this case, the first four TL-moments for Type-BT right censoring are calculated as follows:

$$\mu_1^{B(2,0)} = (1-c^3)q(c) + 3\int_0^c u^2 q(u)du, \tag{2.14a}$$

$$\mu_2^{B(2,0)} = \left(6\beta_c(3,2) - \frac{c^4}{2}\right)q(c) + 2\left[\int_0^c u^3 q(u)du - 3\int_0^c u^2(1-u)q(u)du\right], \tag{2.14b}$$

$$\begin{aligned}\mu_3^{B(2,0)} &= \left(\frac{40}{3}\beta_c(4,2) - 10\beta_c(3,3) - \frac{c^5}{3}\right)q(c) \\ &+ \frac{5}{3}\left[\int_0^c u^4 q(u)du - 8\int_0^c u^3(1-u)q(u)du + 6\int_0^c u^2(1-u)^2 q(u)du\right],\end{aligned} \tag{2.14c}$$

$$\begin{aligned}\mu_4^{B(2,0)} &= \left(\frac{45}{2}\beta_c(5,2) - 45\beta_c(4,3) + 15\beta_c(3,4) - \frac{c^6}{4}\right)q(c) \\ &+ \frac{3}{2}\left[\int_0^c u^5 q(u)du - 15\int_0^c u^4(1-u)q(u)du \right.\\ &\left. + 30\int_0^c u^3(1-u)^2 q(u)du - 10\int_0^c u^2(1-u)^3 q(u)du.\right.\end{aligned} \tag{2.14d}$$

Using the method of expectations, Type-BT TL-moments estimators is given by:

$$\begin{aligned}M_r^{B(t_1,0)} =& \frac{1}{r\binom{n}{r+t_1}}\left[\sum_{i=t_1+1}^{m}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r+t_1-k-1}\binom{n-i}{k}X_{i:n}\right. \\ &\left. + \left(\sum_{i=m+1}^{n}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r+t_1-k-1}\binom{n-i}{k}\right)T\right].\end{aligned} \tag{2.15}$$

2.2 Left Censoring for Right Trim

Let $x_1, x_2, ..., x_n$ be a random sample of size n. When we suppose right trim t_2, i.e. $t_1 = 0$. From formula (2.1) we get

$$\lambda_r^{(0,t_2)} = \frac{(r+t_2)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+t_2)!}\binom{r-1}{k}\int_0^1 q(u)u^{r-k-1}(1-u)^{k+t_2}du. \tag{2.16}$$

In this case, The random sample become on the form of $x_1, x_2, ..., x_{n-t_2}$. Type-I left censoring occurs when the observations below censoring time T are censored:

$$\underbrace{x_{1:n} \le x_{2:n} \le \cdots \le x_{m-1:n}}_{\text{s (censored)}} \le T \le \underbrace{x_{m:n} \le x_{m+1:n} \cdots \le x_{n-t_2:n}}_{\text{n-}t_2\text{-s (observed)}} \le \underbrace{x_{t_2:n} \le x_{t_2+1:n} \cdots \le x_{n:n}}_{t_2\text{(trimmed)}}$$

Let censoring time T satisfy $F(T) = h$ and h is the fraction of censored data.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

2.2.1 TL-moments for Left Censoring (Type-A'T)

The quantile function of Type-A'T TL-moments is

$$y^{A'}(u) = q((1-h)u+h) \quad 0<u<1$$

substitution into (2.16) leads to the Type-A'T TL-moments where:

$$\mu_r^{A'(0,t_2)} = \frac{(r+t_2)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+t_2)!}\binom{r-1}{k}\int_0^1 y^{A'}(u)u^{r-k-1}(1-u)^{k+t_2}du$$
$$= \frac{(r+t_2)!}{r(1-h)^{r+t_2}}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+t_2)!}\binom{r-1}{k}\int_h^1 q(u)(u-h)^{r-k-1}(1-u)^{k+t_2}du \quad (2.17)$$

When we suppose the value of largest trim is equal to one, i.e. $t_2=1$, from (2.17), we get

$$\mu_r^{A'(0,1)} = \frac{(r+1)!}{r(1-h)^{r+1}}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+1)!}\binom{r-1}{k}\int_h^1 q(u)(u-h)^{r-k-1}(1-u)^{k+1}du \quad (2.18)$$

In this case, the first four TL-moments for Type-A'T left censoring are calculated as follows:

$$\mu_1^{A'(0,1)} = \frac{2}{(1-h)^2}\int_h^1 (1-u)q(u)du, \quad (2.19a)$$
$$\mu_2^{A'(0,1)} = \frac{3}{(1-h)^3}\left[\int_h^1 (u-h)(1-u)q(u)du - \frac{1}{2}\int_h^1 (1-u)^2 q(u)du\right], \quad (2.19b)$$
$$\mu_3^{A'(0,1)} = \frac{4}{(1-h)^4}\left[\int_h^1 (u-h)^2(1-u)q(u)du - 2\int_h^1 (u-h)(1-u)^2 q(u)du + \frac{1}{3}\int_h^1 (1-u)^3 q(u)du\right], \quad (2.19c)$$
$$\mu_4^{A'(0,1)} = \frac{5}{(1-h)^5}\left[\int_h^1 (u-h)^3(1-u)q(u)du - \frac{9}{2}\int_h^1 (u-h)^2(1-u)^2 q(u)du + 3\int_h^1 (u-h)(1-u)^3 q(u)du - \frac{1}{4}\int_h^1 (1-u)^4 q(u)du\right]. \quad (2.19d)$$

When we suppose the value of largest trim is equal to two, i.e. $t_2=2$, from (2.17), we get

$$\mu_r^{A'(0,2)} = \frac{(r+2)!}{r(1-h)^{r+2}}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+2)!}\binom{r-1}{k}\int_h^1 q(u)(u-h)^{r-k-1}(1-u)^{k+2}du \quad (2.20)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

In this case, the first four TL-moments for Type-A'T left censoring are calculated as follows:

$$\mu_1^{A'(0,2)} = \frac{3}{(1-h)^3} \int_h^1 (1-u)^2 q(u)du, \tag{2.21a}$$

$$\mu_2^{A'(0,2)} = \frac{4}{2(1-h)^4} \left[3\int_h^1 (u-h)(1-u)^2 q(u)du - \int_h^1 (1-u)^3 q(u)du \right], \tag{2.21b}$$

$$\mu_3^{A'(0,2)} = \frac{5}{3(1-h)^5} \left[6\int_h^1 (u-h)^2(1-u)^2 q(u)du \right.$$
$$\left. -8\int_h^1 (u-h)(1-u)^3 q(u)du + \int_h^1 (1-u)^4 q(u)du \right], \tag{2.21c}$$

$$\mu_4^{A'(0,2)} = \frac{6}{4(1-h)^6} \left[10\int_h^1 (u-h)^3(1-u)^2 q(u)du - 30\int_h^1 (u-h)^2(1-u)^3 q(u)du \right.$$
$$\left. +15\int_h^1 (u-h)(1-u)^4 q(u)du - \int_h^1 (1-u)^5 q(u)du \right]. \tag{2.21d}$$

Using the method of expectations, Type-A'T TL-moments estimators is given by:

$$M_r^{A'(0,t_2)} = \frac{1}{r\binom{n-t_2-s}{r+t_2}} \sum_{i=1}^{n-t_2-s} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r-k-1} \binom{n-t_2-s-i}{k+t_2} X_{s+i:n} \tag{2.22}$$

2.2.2 TL-moments for Left Censoring (Type-B'T)

The quantile function of Type-B'T TL-moments is

$$y^{B'}(u) = \begin{cases} q(h), & 0 < u \le h \\ q(u), & h < u < 1 \end{cases}$$

substitution into equation (2.16) leads to the Type-B'T TL-moments where:

$$\mu_r^{B'(0,t_2)} = \frac{(r+t_2)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k-1)!(k+t_2)!} \binom{r-1}{k} \int_0^1 y^{B'}(u) u^{r-k-1}(1-u)^{k+t_2} du$$
$$= \frac{(r+t_2)!}{r} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k-1)!(k+t_2)!} \binom{r-1}{k} *$$
$$\left[\int_0^h q(h) u^{r-k-1}(1-u)^{k+t_2} du + \int_h^1 q(u) u^{r-k-1}(1-u)^{k+t_2} du \right].$$

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

Using the results in appendix, the first integration can be written as

$$\mu_r^{B'(0,t_2)} = \frac{(r+t_2)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+t_2)!}\binom{r-1}{k} \\ *\left[\beta_h(r-k,k+t_2+1)q(h)+\int_h^1 u^{r-k-1}(1-u)^{k+t_2}q(u)du\right], \tag{2.23}$$

where $\beta_z(a,b)$ is the lower incomplete beta function.

When we suppose the value of largest trim is equal to one, i.e. $t_2 = 1$, from (2.23), we get

$$\mu_r^{B'(0,1)} = \frac{(r+1)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+1)!}\binom{r-1}{k} \\ *\left[\beta_h(r-k,k+2)q(h)+\int_h^1 u^{r-k-1}(1-u)^{k+1}q(u)du\right]. \tag{2.24}$$

The first four TL-moments for Type-B'T left censoring are calculated as follows:

$$\mu_1^{B'(0,1)} = \left[1-(1-h)^2\right]q(h)+2\int_h^1(1-u)q(u)du, \tag{2.25a}$$

$$\mu_2^{B'(0,1)} = \left[\frac{1}{2}(-1+(1-h)^3)+3\beta_h(2,2)\right]q(h) \\ +3\int_h^1 u(1-u)q(u)du-\frac{3}{2}\int_h^1(1-u)^2q(u)du, \tag{2.25b}$$

$$\mu_3^{B'(0,1)} = \left[\frac{1}{3}(1-(1-h)^4)-8\beta_h(2,3)+4\beta_h(3,2)\right]q(h) \\ +4\int_h^1 u^2(1-u)q(u)du-8\int_h^1 u(1-u)^2q(u)du \\ +\frac{4}{3}\int_h^1(1-u)^3q(u)du, \tag{2.25c}$$

$$\mu_4^{B'(0,1)} = \left[\frac{1}{4}(-1+(1-h)^5)-15\beta_h(2,4)+\frac{45}{2}\beta_h(3,3)+5\beta_h(4,2)\right]q(h) \\ +5\int_h^1 u^3(1-u)q(u)du-\frac{45}{2}\int_h^1 u^2(1-u)^2q(u)du \\ +15\int_h^1 u(1-u)^3q(u)du-\frac{5}{4}\int_h^1(1-u)^4q(u)du. \tag{2.25d}$$

When we suppose the value of largest trim is equal to two, i.e. $t_2 = 2$, from (2.23), we get

$$\mu_r^{B'(0,2)} = \frac{(r+2)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+2)!}\binom{r-1}{k} \\ *\left[\beta_h(r-k,k+3)q(h)+\int_h^1 q(u)u^{r-k-1}(1-u)^{k+2}q(u)du\right]. \tag{2.26}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

The first four TL-moments for Type-B'T left censoring are calculated as follows:

$$\mu_1^{B'(0,2)} = \left[1-(1-h)^3\right] q(h) + 3\int_h^1 (1-u)^2 q(u)du, \tag{2.27a}$$

$$\mu_2^{B'(0,2)} = \left[\frac{1}{2}(-1+(1-h)^4) + 6\beta_h(2,3)\right] q(h) + 6\int_h^1 u(1-u)^2 q(u)du - 2\int_h^1 (1-u)^3 q(u)du, \tag{2.27b}$$

$$\mu_3^{B'(0,2)} = \left[\frac{1}{3}(1-(1-h)^5) - \frac{40}{3}\beta_h(2,4) + 10\beta_h(3,3)\right] q(h) + 10\int_h^1 u^2(1-u)^2 q(u)du - \frac{40}{3}\int_h^1 u(1-u)^3 q(u)du + \frac{5}{3}\int_h^1 (1-u)^4 q(u)du, \tag{2.27c}$$

$$\mu_4^{B'(0,2)} \left[\frac{1}{4}(-1+(1-h)^6) + \frac{45}{2}\beta_h(3,5) - 45\beta_h(3,4) + 15\beta_h(4,3)\right] q(h) + 15\int_h^1 u^3(1-u)^2 q(u)du - 45\int_h^1 u^2(1-u)^3 q(u)du + \frac{45}{2}\int_h^1 u(1-u)^4 q(u)du - \frac{3}{2}\int_h^1 (1-u)^4 q(u)du. \tag{2.27d}$$

Using the method of expectations, Type-B'T TL-moments estimators is given by:

$$M_r^{B'(0,t_2)} = \frac{1}{r\binom{n}{r+t_2}} \left[\left(\sum_{i=1}^{s}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k+t_2}T\right) + \left(\sum_{i=s+1}^{n-t_2}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}\binom{i-1}{r-k-1}\binom{n-i}{k+t_2}\right)X_{i:n}\right]. \tag{2.28}$$

3 TL-moments for Censored Data for Weibull distribution

In this section, the r^{th} population TL-moments for the Weibull distribution is introduced.

3.1 Right censoring with left trim

- **Type-AT;** $t_1 = 1$

  From equation (2.4), the r^{th} population Type-AT TL-moments for Type-I right censoring for the

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at *Math. Comput. Appl.* **2018**, *23*, 47; doi:10.3390/mca23030047

Weibull distribution is:

$$\mu_r^{A(t_1,0)} = \frac{a(r+t_1)!}{rc^{r+t_1}} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r+t_1-k-1)!k!} \binom{r-1}{k} \int_0^c u^{r+t_1-k-1}(c-u)^k[-\log(1-u)]^{\frac{1}{b}} du. \tag{3.1}$$

By taking that the value of smallest trim is equal to one $t_1 = 1$, from (2.5) we get

$$\mu_r^{A(1,0)} = \frac{a(r+1)!}{rc^{r+1}} \sum_{k=0}^{r-1} \frac{(-1)^k}{(r-k)!k!} \binom{r-1}{k} \int_0^c u^{r-k}(c-u)^k[-\log(1-u)]^{\frac{1}{b}} du. \tag{3.2}$$

Substituting $r = 1, 2$ in equation (2.6a); the first two Type-AT TL-moments for Type-I right censoring with left trim for Weibull distribution will be:

$$\mu_1^{A(1,0)} = \frac{2a}{c^2} \int_0^c u[-\log(1-u)]^{\frac{1}{b}} du.$$

Putting $z = -\log(1-u)$ this equation becomes:

$$\begin{aligned}\mu_1^{A(1,0)} &= \frac{2a}{c^2} \int_0^{-\log(1-c)} (1-e^{-z}) z^{\frac{1}{b}} e^{-z} dz \\ &= \frac{2a}{c^2} \left[\int_0^{-\log(1-c)} z^{\frac{1}{b}} e^{-z} dz - \int_0^{-\log(1-c)} z^{\frac{1}{b}} e^{-2z} dz \right].\end{aligned}$$

Using the results in the appendix this equation can be written as

$$\mu_1^{A(1,0)} = \frac{2a}{c^2} \left[\gamma(-\log(1-c), \frac{1}{b}+1) - 2^{-(\frac{1}{b}+1)} \gamma(-2\log(1-c), \frac{1}{b}+1) \right], \tag{3.3}$$

where $\gamma(c, b)$ is the lower incomplete gamma function.

Similarly, from equation (2.6b), we can also obtain the second Type-AT TL-moments, in case

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at *Math. Comput. Appl.* **2018**, *23*, 47; doi:10.3390/mca23030047

$t_1 = 1$, for Type-I right censoring for the Weibull distribution as follows:

$$\begin{aligned}
\mu_2^{A(1,0)} &= \frac{3a}{c^3}\left[\frac{1}{2}\int_0^c u^2[-\log(1-u)]^{\frac{1}{b}}du - \int_0^c u(c-u)[-\log(1-u)]^{\frac{1}{b}}du\right]\\
&= \frac{3a}{c^3}\left[\frac{3}{2}\int_0^c u^2[-\log(1-u)]^{\frac{1}{b}}du - c\int_0^c u[-\log(1-u)]^{\frac{1}{b}}du\right]\\
&= \frac{3a}{c^3}\left[\frac{3}{2}\int_0^{-\log(1-c)} (1-e^{-z})^2 z^{\frac{1}{b}}e^{-z}dz - c\int_0^{-\log(1-c)} (1-e^{-z})z^{\frac{1}{b}}e^{-z}dz\right]\\
&= \frac{3a}{c^3}\left[\frac{3}{2}\int_0^{-\log(1-c)} (1-2e^{-z}+e^{-2z})z^{\frac{1}{b}}e^{-z}dz - c\int_0^{-\log(1-c)} (1-e^{-z})z^{\frac{1}{b}}e^{-z}dz\right]\\
&= \frac{3a}{c^3}\left[(\frac{3}{2}-c)\int_0^{-\log(1-c)} z^{\frac{1}{b}}e^{-z}dz - (3-c)\int_0^{-\log(1-c)} z^{\frac{1}{b}}e^{-2z}dz\right.\\
&\qquad \left. +\frac{3}{2}\int_0^{-\log(1-c)} z^{\frac{1}{b}}e^{-3z}dz\right]\\
&= \frac{3a}{c^3}\left[(\frac{3}{2}-c)\gamma(-\log(1-c),\frac{1}{b}+1) - (3-c)2^{-(\frac{1}{b}+1)}\gamma(-2\log(1-c),\frac{1}{b}+1)\right.\\
&\qquad \left. +\frac{3^{-\frac{1}{b}}}{2}\gamma(-3\log(1-c),\frac{1}{b}+1)\right].
\end{aligned} \tag{3.4}$$

- **Type-AT;** $t_1 = 2$

When we suppose the value of smallest trim is equal to two, i.e. $t_1 = 2$, from (2.7), we get

$$\mu_r^{A(2,0)} = \frac{a(r+2)!}{rc^{r+2}}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k+1)!k!}\binom{r-1}{k}\int_0^c u^{r-k+1}(c-u)^k[-\log(1-u)]^{\frac{1}{b}}du. \tag{3.5}$$

Substituting $r = 1, 2$ in equation (3.5); the first two Type-AT TL-moments for Type-I right censoring with left trim for Weibull distribution will be:

$$\begin{aligned}
\mu_1^{A(2,0)} = \frac{3a}{c^3}&\left[\gamma(-\log(1-c),\frac{1}{b}+1) - 2^{-\frac{1}{b}}\gamma(-2\log(1-c),\frac{1}{b}+1)\right.\\
&\left. +3^{-(\frac{1}{b}+1)}\gamma(-3\log(1-c),\frac{1}{b}+1)\right].
\end{aligned} \tag{3.6}$$

And,

$$\begin{aligned}
\mu_2^{A(2,0)} = \frac{2a}{c^4}&\left[(4-3c)\gamma(-\log(1-c),\frac{1}{b}+1) - 2^{-\frac{1}{b}}(6-3c)\gamma(-2\log(1-c),\frac{1}{b}+1)\right.\\
&\left. +3^{-\frac{1}{b}}(4-c)\gamma(-3\log(1-c),\frac{1}{b}+1) - 4^{-\frac{1}{b}}\gamma(-4\log(1-c),\frac{1}{b}+1)\right].
\end{aligned} \tag{3.7}$$

- **Type-BT;** $t_1 = 1$

From equation (2.10), the r^{th} population Type-BT TL-moments for Type-I right censoring for

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at *Math. Comput. Appl.* **2018**, *23*, 47; doi:10.3390/mca23030047

the Weibull distribution is:

$$\mu_r^{B(t_1,0)} = \frac{a(r+t_1)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r+t_1-k-1)!k!}\binom{r-1}{k}* \left[\beta^c(r+t_1-k,k+1)[-\log(1-c)]^{\frac{1}{b}} + \int_0^c u^{r+t_1-k-1}(1-u)^k[-\log(1-u)]^{\frac{1}{b}}du\right]. \tag{3.8}$$

By taking that the value of smallest trim is equal to one $t_1 = 1$, from (2.11) we get

$$\mu_r^{B(1,0)} = \frac{a(r+1)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k)!k!}\binom{r-1}{k}* \left[\beta^c(r-k+1,k+1)[-\log(1-c)]^{\frac{1}{b}} + \int_0^c u^{r-k}(1-u)^k[-\log(1-u)]^{\frac{1}{b}}du\right]. \tag{3.9}$$

Substituting $r = 1,2$ in equation (3.9); the first two Type-BT TL-moments for Type-I right censoring with left trim for Weibull distribution will be:

$$\mu_1^{B(1,0)} = 2a\left[\frac{1}{2}(1-c^2)[-\log(1-c)]^{\frac{1}{b}} + \gamma(-\log(1-c),\frac{1}{b}+1) -2^{-(\frac{1}{b}+1)}\gamma(-2\log(1-c),\frac{1}{b}+1)\right]. \tag{3.10}$$

And,

$$\mu_2^{B(1,0)} = 3a\left[\frac{1}{2}(c^2-c^3)[-\log(1-c)]^{\frac{1}{b}} + \frac{1}{2}\gamma(-\log(1-c),\frac{1}{b}+1) -2^{-\frac{1}{b}}\gamma(-2\log(1-c),\frac{1}{b}+1) + \frac{3^{-\frac{1}{b}}}{2}\gamma(-3\log(1-c),\frac{1}{b}+1)\right]. \tag{3.11}$$

- **Type-BT;** $t_1 = 2$

When we suppose the value of smallest trim is equal to two, i.e. $t_1 = 2$, from (2.13), we get

$$\mu_r^{B(2,0)} = \frac{a(r+2)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k+1)!k!}\binom{r-1}{k} \left[\beta^c(r-k+2,k+1)[-\log(1-c)]^{\frac{1}{b}} + \int_0^c u^{r-k+1}(1-u)^k[-\log(1-u)]^{\frac{1}{b}}du\right]. \tag{3.12}$$

Substituting $r = 1,2$ in equation (3.12); the first two Type-BT TL-moments for Type-I right

 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

censoring with left trim for Weibull distribution will be:

$$\begin{aligned}\mu_1^{B(2,0)} = 3a\Bigg[&\frac{1-c^3}{3}[-\log(1-c)]^{\frac{1}{b}} + \gamma(-\log(1-c), \frac{1}{b}+1) \\ &-2^{-\frac{1}{b}}\gamma(-2\log(1-c), \frac{1}{b}+1) + 3^{-(\frac{1}{b}+1)}\gamma(-3\log(1-c), \frac{1}{b}+1)\Bigg].\end{aligned} \tag{3.13}$$

And,

$$\begin{aligned}\mu_2^{B(2,0)} = 2a\Bigg[&(3\beta_c(3,2) - \frac{c^4}{4})[-\log(1-c)]^{\frac{1}{b}} \\ &+\gamma(-\log(1-c), \frac{1}{b}+1) - 3*2^{-\frac{1}{b}}\gamma(-2\log(1-c), \frac{1}{b}+1) \\ &+3^{1-\frac{1}{b}}\gamma(-3\log(1-c), \frac{1}{b}+1) - 4^{-\frac{1}{b}}\gamma(-4\log(1-c), \frac{1}{b}+1)\Bigg].\end{aligned} \tag{3.14}$$

3.2 Left censoring with right trim

- **Type-A'T;** $t_2 = 1$

From equation (2.17), the r^{th} population Type-A'T TL-moments for Type-I left censoring for the Weibull distribution is:

$$\begin{aligned}\mu_r^{A'(0,t_2)} = &\frac{a(r+t_2)!}{r(1-h)^{r+t_2}}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+t_2)!}\binom{r-1}{k}* \\ &\int_h^1 [-\log(1-u)]^{\frac{1}{b}}(u-h)^{r-k-1}(1-u)^{k+t_2}du.\end{aligned} \tag{3.15}$$

When we suppose the value of largest trim is equal to one, i.e. $t_2 = 1$, from (2.18), we get

$$\begin{aligned}\mu_1^{A'(0,1)} = &\frac{a(r+1)!}{r(1-h)^{r+1}}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+1)!}\binom{r-1}{k}* \\ &\int_h^1 (u-h)^{r-k-1}(1-u)^{k+1}[-\log(1-u)]^{\frac{1}{b}}du.\end{aligned} \tag{3.16}$$

Substituting $r = 1, 2$ in equation (3.16); the first two Type-A'T TL-moments for Type-I left censoring with right trim for Weibull distribution will be:

$$\mu_1^{A'(0,1)} = \frac{2^{-\frac{1}{b}}a}{(1-h)^2}\Gamma(-2\log(1-h), \frac{1}{b}+1). \tag{3.17}$$

 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

And,

$$\mu_2^{A'(0,1)} = \frac{3a}{2(1-h)^3}\left[(1-h)2^{-\frac{1}{b}}\Gamma(-2\log(1-h),\frac{1}{b}+1) - 3^{-\frac{1}{b}}\Gamma(-3\log(1-h),\frac{1}{b}+1)\right]. \tag{3.18}$$

- **Type-A'T;** $t_2 = 2$

When we suppose the value of largest trim is equal to one, i.e. $t_2 = 2$, from (2.20), we get

$$\mu_r^{A'(0,2)} = \frac{a(r+2)!}{r(1-h)^{r+2}}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+2)!}\binom{r-1}{k}\int_h^1 (u-h)^{r-k-1}(1-u)^{k+2}[-\log(1-u)]^{\frac{1}{b}}du. \tag{3.19}$$

Substituting $r = 1, 2$ in equation (3.19); the first two Type-A'T TL-moments for Type-I left censoring with right trim for Weibull distribution will be:

$$\mu_1^{A'(0,2)} = \frac{3a}{(1-h)^3}\Gamma(-3\log(1-h),\frac{1}{b}+1). \tag{3.20}$$

And,

$$\mu_2^{A'(0,1)} = \frac{2a}{(1-h)^4}\left[3^{-\frac{1}{b}}(1-h)\Gamma(-3\log(1-h),\frac{1}{b}+1) - 4^{-\frac{1}{b}}\Gamma(-4\log(1-h),\frac{1}{b}+1)\right]. \tag{3.21}$$

- **Type-B'T;** $t_2 = 1$

From equation (2.23), the r^{th} population Type-B'T TL-moments for Type-I left censoring for the Weibull distribution is:

$$\mu_r^{B'(0,t_2)} = \frac{a(r+t_2)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+t_2)!}\binom{r-1}{k} * \left[\beta_h(r-k,k+t_2+1)[-\log(1-h)]^{\frac{1}{b}} + \int_h^1 u^{r-k-1}(1-u)^{k+t_2}[-\log(1-u)]^{\frac{1}{b}}du\right]. \tag{3.22}$$

When we suppose the value of largest trim is equal to one, i.e. $t_2 = 1$, from (2.24), we get

$$\mu_r^{B'(0,1)} = \frac{a(r+1)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+1)!}\binom{r-1}{k} * \left[\beta_h(r-k,k+2)[-\log(1-h)]^{\frac{1}{b}} + \int_h^1 u^{r-k-1}(1-u)^{k+1}[-\log(1-u)]^{\frac{1}{b}}du\right]. \tag{3.23}$$

Substituting $r = 1, 2$ in equation (3.23); the first two Type-B'T TL-moments for Type-I left censoring with right trim for Weibull distribution will be:

$$\mu_1^{B(0,1)} = a\left[1-(1-h)^2\right]\left[-\log(1-h)\right]^{\frac{1}{b}} + 2^{-\frac{1}{b}} a\Gamma(-2\log(1-c), \frac{1}{b}+1). \tag{3.24}$$

And,

$$\begin{aligned}\mu_2^{B'(0,1)} =& 3a\left[\left[\frac{1}{6}(-1+(1-h)^3)+\beta_h(2,2)\right]\left[-\log(1-h)\right]^{\frac{1}{b}}\right.\\ &\left.+2^{-(\frac{1}{b}+1)}\Gamma(-2\log(1-h), \frac{1}{b}+1) - \frac{3^{-\frac{1}{b}}}{2}\Gamma(-3\log(1-h), \frac{1}{b}+1)\right].\end{aligned} \tag{3.25}$$

- **Type-B'T;** $t_2 = 2$

When we suppose the value of largest trim is equal to one, i.e. $t_2 = 2$, from (2.26), we get

$$\begin{aligned}\mu_r^{B'(0,2)} =& \frac{a(r+2)!}{r}\sum_{k=0}^{r-1}\frac{(-1)^k}{(r-k-1)!(k+2)!}\binom{r-1}{k} *\\ &[\beta_h(r-k,k+3)[-\log(1-h)]^{\frac{1}{b}} + \int_h^1 [-\log(1-u)]^{\frac{1}{b}} u^{r-k-1}(1-u)^{k+2}du].\end{aligned} \tag{3.26}$$

Substituting $r = 1, 2$ in equation (3.26); the first two Type-B'T TL-moments for Type-I left censoring with right trim for Weibull distribution will be:

$$\mu_1^{B(2,0)} = 3a\left[\frac{1}{3}\left[1-(1-h)^3\right]\left[-\log(1-h)\right]^{\frac{1}{b}} + 3^{-(\frac{1}{b}+1)}\Gamma(-3\log(1-h), \frac{1}{b}+1)\right]. \tag{3.27}$$

And,

$$\begin{aligned}\mu_2^{B'(0,1)} =& 2a\left[\left[\frac{1}{4}(-1+(1-h)^4)+3\beta_h(2,3)\right]\left[-\log(1-h)\right]^{\frac{1}{b}}\right.\\ &\left.+3^{-\frac{1}{b}}\Gamma(-3\log(1-h), \frac{1}{b}+1) - 4^{-\frac{1}{b}}\Gamma(-4\log(1-h), \frac{1}{b}+1)\right].\end{aligned} \tag{3.28}$$

4 Simulation Study

This section is devoted to illustrate the effect of an adaptation of the TL-moments method to censored data in estimation process using a comparative numerical study. We will estimate the two unknown parameters of the Weibull distribution using TL-moments, Direct L-moments and ML methods given both right and left Type-I censored data. In this study we used the TL-moments by trimming one and two data from the right and also from the left. A comparative numerical study was carried out among the three methods based on estimate average, root of

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

mean square error (RMSE) and relative absolute biases (RAB). Following are the steps of this numerical study:

1. Generate random sample size n (60, 100 and 200) from the Weibull distribution with parameters (a, b) take these initial values $(0.5, 5)$, $(2, 4)$ and $(0.2, 0.8)$.
2. The generated data is ordered.
3. Determine the level of censoring, take $c = 20\%$ and $c = 50\%$.
4. Use TL-moments, Direct L-moments and ML estimators formulas mentioned in (2.9), (2.15), (2.22) and (2.28) respectively and equate them with the corresponding theoretical moments to get a and b estimates.
5. The simulation process to be repeated 5000 times.
6. Calculate means, root of mean square error (RMSE) and relative absolute biases (RAB) for each sample size used and parameter values considered.
7. The simulation results are reported in Table (1) to Table (6).

5 RESULTS AND CONCLUSION

The simulations show, when the data contained outliers the TL-moments gave better estimates comparing by Direct L-moments and ML methods. The tables show the results of various simulation studies to assess the effect of the adaptation of the TL-moments method to censored data. It should be noticed that; the RMSE and RAB of are fluctuate with small variations because the estimates of the parameter a and b have negative but small covariance. In all cases, results berform better when n gets larger.

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

Table 1

The estimates, root of mean square error (RMSE) and relative absolute biases (RAB) for two parameters of Wiebull distribution using TL-moments, L-moments and ML method based on left censoring (a=0.5 and b=5) in the presence of outliers.

Type (A)

n	Meth.	Estimation of a						Estimation of b					
		10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	0.5613	0.5620	0.0075	0.5075	0.5100	0.0001	1.4030	1.4047	2.5876	1.2473	1.2542	2.8165
	AD	0.5603	0.5610	0.0291	0.4916	0.4940	0.0001	1.8638	1.8772	0.0786	1.4723	3.2285	2.4888
	AT1	0.5036	0.5037	0.0000	0.5038	0.5040	0.0000	5.0780	5.1329	0.0012	4.9823	5.0541	0.0000
	AT2	0.5003	0.5005	0.0000	0.5017	0.5019	0.0000	5.2738	5.3525	0.0149	5.2834	5.3844	0.0160
50	ML	0.5364	0.5367	0.0026	0.4914	0.4920	0.0001	1.5442	1.5454	2.3884	1.3901	1.3920	2.6062
	AD	0.5399	0.5402	0.0031	0.4939	0.4947	0.0000	2.3484	2.3592	1.4061	1.7971	1.8096	2.0517
	AT1	0.5019	0.5020	0.0000	0.5020	0.5022	0.0000	5.0142	5.0442	0.0000	4.9952	5.0375	0.0000
	AT2	0.4999	0.5000	0.0000	0.5008	0.5009	0.0000	5.1309	5.1714	0.0034	5.1746	5.2336	0.0060
100	ML	0.5038	0.5171	0.0000	0.4799	0.4802	0.0008	1.7578	1.7588	2.1022	1.5982	1.5992	2.3143
	AD	0.5223	0.5224	0.0010	0.4966	0.4968	0.0000	3.0862	3.0935	0.7325	2.4767	2.4848	1.2733
	AT1	0.5013	0.5013	0.0000	0.5010	0.5011	0.0000	5.0041	5.0203	0.0000	4.9551	4.9740	0.0004
	AT2	0.5003	0.5004	0.0000	0.5004	0.5005	0.0000	5.0571	5.0788	0.0006	5.0339	5.0576	0.0002

Type (B)

n	Meth.	Estimation of a						Estimation of b					
		10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	0.5613	0.5620	0.0075	0.5075	0.5100	0.0001	1.4030	1.4047	2.5876	1.2473	1.2542	2.8165
	BD	0.5908	0.5912	0.0165	0.5633	0.5641	0.0080	2.2917	2.3030	1.4668	1.9279	1.9444	1.8874
	BT1	0.5102	0.5104	0.0002	0.5093	0.5095	0.0001	4.7625	4.8242	0.0112	4.5723	4.6449	0.0365
	BT2	0.5056	0.5059	0.0000	0.5044	0.5047	0.0000	5.0295	5.1229	0.0001	5.0961	5.2076	0.0018
50	ML	0.5364	0.5367	0.0026	0.4914	0.4920	0.0001	1.5442	1.5454	2.3884	1.3901	1.3920	2.6062
	BD	0.5585	0.5587	0.0068	0.5405	0.5409	0.0032	2.8124	2.8219	0.9571	2.4155	2.4289	1.3358
	BT1	0.5050	0.5051	0.0000	0.5044	0.5046	0.0000	4.8992	4.9339	0.0020	4.8112	4.8607	0.0071
	BT2	0.5027	0.5029	0.0000	0.5023	0.5025	0.0000	5.0393	5.0897	0.0003	5.0666	5.1393	0.0008
100	ML	0.5038	0.5171	0.0000	0.4799	0.4802	0.0008	1.7578	1.7588	2.1022	1.5982	1.5992	2.3143
	BD	0.5315	0.5315	0.0019	0.5217	0.5218	0.0009	3.5139	3.5209	0.4416	3.1174	3.1255	0.7088
	BT1	0.5025	0.5026	0.0000	0.5019	0.5020	0.0000	4.9548	4.9739	0.0004	4.8774	4.8987	0.0030
	BT2	0.5018	0.5019	0.0000	0.5012	0.5013	0.0000	5.0034	5.0303	0.0000	4.9661	4.9962	0.0002

Peer-reviewed version available at *Math. Comput. Appl.* **2018**, *23*, 47; doi:10.3390/mca23030047

Table 2

The estimates, root of mean square error (RMSE) and relative absolute biases (RAB) for two parameters of Wiebull distribution using TL-moments, L-moments and ML method based on left censoring (a=2 and b=4) in the presence of outliers.

Type (A)

n	Meth.	Estimation of a						Estimation of b					
		10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	2.273	2.279	0.037	1.928	1.936	0.002	1.351	1.355	1.753	1.125	1.128	2.065
	AD	2.281	2.284	0.039	1.794	1.808	0.021	1.722	1.744	1.297	0.910	2.104	2.386
	AT1	2.018	2.020	0.000	2.017	2.018	0.000	4.046	4.091	0.000	3.980	4.041	0.000
	AT2	2.001	2.003	0.000	2.010	2.012	0.000	4.186	4.250	0.008	4.289	4.379	0.021
50	ML	2.167	2.168	0.014	1.748	5.529	1.528	1.483	1.484	1.583	1.005	1.336	2.242
	AD	2.180	2.181	0.016	1.846	1.852	0.011	2.016	2.168	0.983	1.091	2.015	2.114
	AT1	2.008	2.009	0.000	2.009	2.010	0.000	4.016	4.039	0.000	3.975	4.014	0.000
	AT2	1.998	1.999	0.000	2.005	2.006	0.000	4.095	4.128	0.002	4.141	4.194	0.004
100	ML	2.083	2.084	0.003	1.853	1.854	0.010	1.676	1.677	1.349	1.449	1.450	1.626
	AD	2.104	2.104	0.005	1.913	1.914	0.003	2.693	2.699	0.426	1.966	1.974	1.033
	AT1	2.005	2.005	0.000	2.004	2.004	0.000	4.018	4.029	0.000	3.986	4.004	0.000
	AT2	2.000	2.001	0.000	2.002	2.002	0.000	4.057	4.072	0.000	4.060	4.084	0.000

Type (B)

n	Meth.	Estimation of a						Estimation of b					
		10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	2.273	2.279	0.037	1.928	1.936	0.002	1.351	1.355	1.753	1.125	1.128	2.065
	BD	2.387	2.389	0.074	2.177	2.183	0.015	2.030	2.052	0.969	2.704	8.551	1.828
	BT1	2.050	2.051	0.001	2.035	2.037	0.000	3.836	3.884	0.006	3.580	3.661	0.044
	BT2	2.030	2.032	0.000	2.018	2.019	0.000	4.017	4.090	0.000	4.079	4.187	0.001
50	ML	2.167	2.168	0.014	1.748	5.529	1.528	1.483	1.484	1.583	1.005	1.336	2.242
	BD	2.248	2.249	0.030	2.113	2.116	0.006	2.447	2.455	0.602	1.896	2.379	1.105
	BT1	2.023	2.024	0.000	2.017	2.018	0.000	3.917	3.944	0.001	3.800	3.847	0.009
	BT2	2.015	2.016	0.000	2.009	2.010	0.000	4.002	4.041	0.000	4.043	4.113	0.000
100	ML	2.083	2.084	0.003	1.853	1.854	0.010	1.676	1.677	1.349	1.449	1.450	1.626
	BD	2.137	2.137	0.009	2.061	2.062	0.001	2.973	2.979	0.263	2.513	2.521	0.552
	BT1	2.011	2.012	0.000	2.007	2.008	0.000	3.984	3.998	0.000	3.916	3.938	0.001
	BT2	2.008	2.008	0.000	2.004	2.005	0.000	4.019	4.038	0.000	4.007	4.038	0.000

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

Table 3

The estimates, root of mean square error (RMSE) and relative absolute biases (RAB) for two parameters of Wiebull distribution using TL-moments, L-moments and ML method based on left censoring (a=0.2 and b=0.8) in the presence of outliers.

Type (A)

n	Meth.	Estimation of a						Estimation of b					
		20%			50%			20%			50%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
60	ML	0.244	0.248	0.009	0.225	0.230	0.003	0.614	0.617	0.043	0.578	0.582	0.061
	AD	0.205	0.210	0.000	0.177	0.184	0.002	0.523	0.527	0.095	0.487	0.492	0.122
	AT1	0.211	0.214	0.000	0.209	0.213	0.000	0.796	0.803	0.000	0.788	0.798	0.000
	AT2	0.202	0.205	0.000	0.204	0.207	0.000	0.836	0.845	0.001	0.844	0.859	0.002
100	ML	0.225	0.227	0.003	0.212	0.215	0.000	0.640	0.642	-0.159	0.609	0.612	0.045
	AD	0.199	0.202	0.000	0.177	0.181	0.002	0.559	0.563	0.072	0.524	0.528	0.095
	AT1	0.205	0.207	0.000	0.206	0.208	0.000	0.797	0.801	0.000	0.796	0.801	0.000
	AT2	0.200	0.202	0.000	0.202	0.204	0.000	0.820	0.826	0.000	0.831	0.839	0.001
200	ML	0.211	0.212	0.000	0.203	0.204	0.000	0.676	0.677	0.019	0.648	0.650	0.028
	AD	0.199	0.201	0.000	0.182	0.183	0.001	0.616	0.618	0.042	0.579	0.582	0.060
	AT1	0.202	0.203	0.000	0.203	0.204	0.000	0.796	0.798	0.000	0.792	0.795	0.000
	AT2	0.200	0.201	0.000	0.201	0.202	0.000	0.806	0.809	0.000	0.807	0.811	0.000

Type (B)

n	Meth.	Estimation of a						Estimation of b					
		20%			50%			20%			50%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
60	ML	0.244	0.248	0.009	0.225	0.230	0.003	0.614	0.617	0.043	0.578	0.582	0.061
	BD	0.225	0.229	0.003	0.227	0.231	0.003	0.548	0.553	0.078	0.548	0.553	0.079
	BT1	0.224	0.228	0.003	0.220	0.224	0.002	0.753	0.759	0.002	0.743	0.753	0.003
	BT2	0.216	0.219	0.001	0.212	0.216	0.000	0.797	0.806	0.000	0.803	0.818	0.000
100	ML	0.225	0.227	0.003	0.212	0.215	0.000	0.640	0.642	-0.159	0.609	0.612	0.045
	BD	0.217	0.219	0.001	0.219	0.222	0.001	0.586	0.589	0.057	0.587	0.590	0.056
	BT1	0.212	0.214	0.000	0.211	0.213	0.000	0.774	0.778	0.000	0.772	0.778	0.000
	BT2	0.208	0.210	0.000	0.207	0.209	0.000	0.799	0.805	0.000	0.807	0.816	0.000
200	ML	0.211	0.212	0.000	0.203	0.204	0.000	0.676	0.677	0.019	0.648	0.650	0.028
	BD	0.212	0.213	0.000	0.212	0.213	0.000	0.640	0.643	0.031	0.636	0.638	0.033
	BT1	0.205	0.206	0.000	0.205	0.206	0.000	0.784	0.786	0.000	0.782	0.785	0.000
	BT2	0.204	0.205	0.000	0.203	0.204	0.000	0.794	0.797	0.000	0.796	0.801	0.000

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

Table 4

The estimates, root of mean square error (RMSE) and relative absolute biases (RAB) for two parameters of Wiebull distribution using TL-moments, L-moments and ML method based on right censoring (a=0.5 and b=5) in the presence of outliers.

Type (A)													
		Estimation of a						Estimation of b					
n	Meth.	10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	0.500	0.500	0.000	0.547	0.549	0.004	3.889	3.911	0.246	3.183	3.215	0.660
	AD	0.502	0.503	0.000	0.552	0.554	0.005	4.061	4.085	0.176	3.336	3.373	0.553
	AT1	0.497	0.497	0.000	0.511	0.512	0.000	4.753	4.790	0.012	4.350	4.425	0.084
	AT2	0.497	0.497	0.000	0.504	0.505	0.000	4.886	4.933	0.002	4.643	4.743	0.025
50	ML	0.500	0.501	0.000	0.536	0.537	0.002	4.051	4.069	0.179	3.464	3.493	0.471
	AD	0.503	0.503	0.000	0.540	0.541	0.003	4.210	4.231	0.124	3.627	3.660	0.376
	AT1	0.499	0.499	0.000	0.507	0.508	0.000	4.787	4.817	0.009	4.554	4.611	0.039
	AT2	0.499	0.499	0.000	0.502	0.503	0.000	4.884	4.918	0.002	4.782	4.854	0.009
100	ML	0.499	0.499	0.000	0.518	0.518	0.000	4.459	4.469	0.058	4.041	4.059	0.183
	AD	0.500	0.500	0.000	0.518	0.519	0.000	4.584	4.595	0.034	4.211	4.232	0.124
	AT1	0.498	0.498	0.000	0.503	0.503	0.000	4.893	4.907	0.002	4.779	4.808	0.009
	AT2	0.498	0.498	0.000	0.501	0.501	0.000	4.932	4.948	0.000	4.875	4.910	0.003

Type (B)													
		Estimation of a						Estimation of b					
n	Meth.	10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	0.500	0.500	0.000	0.547	0.549	0.004	3.889	3.911	0.246	3.183	3.215	0.660
	BD	0.500	0.500	0.000	0.532	0.533	0.002	4.124	4.146	0.153	3.582	3.620	0.401
	BT1	0.500	0.500	0.000	0.519	0.521	0.000	4.543	4.578	0.041	4.133	4.205	0.150
	BT2	0.500	0.500	0.000	0.519	0.520	0.000	4.518	4.566	0.046	4.212	4.309	0.123
50	ML	0.500	0.501	0.000	0.536	0.537	0.002	4.051	4.069	0.179	3.464	3.493	0.471
	BD	0.501	0.501	0.000	0.523	0.524	0.001	4.267	4.286	0.107	3.862	3.893	0.259
	BT1	0.501	0.501	0.000	0.514	0.514	0.000	4.623	4.651	0.028	4.369	4.427	0.079
	BT2	0.501	0.501	0.000	0.513	0.514	0.000	4.602	4.637	0.031	4.432	4.512	0.064
100	ML	0.499	0.499	0.000	0.518	0.518	0.000	4.459	4.469	0.058	4.041	4.059	0.183
	BD	0.499	0.499	0.000	0.511	0.512	0.000	4.606	4.616	0.030	4.347	4.366	0.085
	BT1	0.500	0.500	0.000	0.507	0.507	0.000	4.791	4.804	0.008	4.637	4.668	0.026
	BT2	0.499	0.499	0.000	0.507	0.507	0.000	4.766	4.783	0.010	4.644	4.688	0.025

Peer-reviewed version available at *Math. Comput. Appl.* **2018**, *23*, 47; doi:10.3390/mca23030047

Table 5

The estimates, root of mean square error (RMSE) and relative absolute biases (RAB) for two parameters of Wiebull distribution using TL-moments, L-moments and ML method based on right censoring (a=2 and b=4) in the presence of outliers.

Type (A)													
		Estimation of a						Estimation of b					
n	Meth.	10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	2.007	2.009	0.000	2.167	2.174	0.014	3.148	3.168	0.181	2.769	2.797	0.378
	AD	2.015	2.016	0.000	2.172	2.179	0.014	3.313	3.335	0.117	2.919	2.950	0.291
	AT1	1.990	1.991	0.000	2.039	2.043	0.000	3.785	3.818	0.011	3.578	3.628	0.044
	AT2	1.988	1.989	0.000	2.016	2.020	0.000	3.895	3.937	0.002	3.772	3.836	0.012
50	ML	2.008	2.009	0.000	2.136	2.141	0.009	3.290	3.306	0.125	2.917	2.940	0.292
	AD	2.014	2.016	0.000	2.138	2.143	0.009	3.443	3.461	0.077	3.077	3.104	0.212
	AT1	1.995	1.996	0.000	2.032	2.036	0.000	3.840	3.866	0.006	3.647	3.690	0.031
	AT2	1.994	1.995	0.000	2.016	2.019	0.000	3.923	3.956	0.001	3.795	3.852	0.010
100	ML	2.001	2.002	0.000	2.071	2.072	0.002	3.565	3.574	0.047	3.341	3.355	0.108
	AD	2.005	2.005	0.000	2.069	2.071	0.002	3.671	3.680	0.027	3.478	3.495	0.067
	AT1	1.995	1.996	0.000	2.017	2.018	0.000	3.885	3.897	0.003	3.828	3.851	0.007
	AT2	1.995	1.995	0.000	2.010	2.011	0.000	3.925	3.940	0.001	3.900	3.928	0.002
Type (B)													
		Estimation of a						Estimation of b					
n	Meth.	10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	2.007	2.009	0.000	2.167	2.174	0.014	3.148	3.168	0.181	2.769	2.797	0.378
	BD	2.005	2.007	0.000	2.116	2.122	0.006	3.346	3.366	0.106	3.054	3.084	0.223
	BT1	2.005	2.006	0.000	2.081	2.086	0.003	3.604	3.635	0.039	3.392	3.446	0.092
	BT2	2.005	2.006	0.000	2.081	2.087	0.003	3.583	3.625	0.043	3.442	3.519	0.077
50	ML	2.008	2.009	0.000	2.136	2.141	0.009	3.290	3.306	0.125	2.917	2.940	0.292
	BD	2.007	2.008	0.000	2.093	2.096	0.004	3.472	3.489	0.069	3.195	3.221	0.161
	BT1	2.007	2.008	0.000	2.065	2.068	0.002	3.693	3.719	0.023	3.491	3.537	0.064
	BT2	2.007	2.008	0.000	2.065	2.069	0.002	3.675	3.709	0.026	3.529	3.595	0.055
100	ML	2.001	2.002	0.000	2.071	2.072	0.002	3.565	3.574	0.047	3.341	3.355	0.108
	BD	2.001	2.002	0.000	2.045	2.047	0.001	3.687	3.695	0.024	3.560	3.575	0.048
	BT1	2.001	2.002	0.000	2.031	2.032	0.000	3.809	3.821	0.009	3.743	3.766	0.016
	BT2	2.001	2.002	0.000	2.031	2.033	0.000	3.798	3.815	0.010	3.757	3.789	0.014

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1
Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

Table 6

The estimates, root of mean square error (RMSE) and relative absolute biases (RAB) for two parameters of Wiebull distribution using TL-moments, L-moments and ML method based on right censoring (a=0.2 and b=0.8) in the presence of outliers.

Type (A)

n	Meth.	Estimation of a						Estimation of b					
		10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	0.207	0.210	0.000	0.269	0.289	0.024	0.691	0.696	0.014	0.645	0.653	0.029
	AD	0.197	0.200	0.000	0.233	0.253	0.005	0.751	0.758	0.002	0.710	0.720	0.010
	AT1	0.196	0.199	0.000	0.224	0.244	0.002	0.769	0.777	0.001	0.740	0.754	0.004
	AT2	0.196	0.199	0.000	0.221	0.243	0.002	0.778	0.789	0.000	0.758	0.774	0.002
50	ML	0.205	0.208	0.000	0.254	0.267	0.014	0.711	0.715	0.009	0.666	0.673	0.022
	AD	0.197	0.199	0.000	0.224	0.235	0.003	0.767	0.773	0.001	0.728	0.736	0.006
	AT1	0.196	0.199	0.000	0.217	0.228	0.001	0.780	0.787	0.000	0.753	0.764	0.002
	AT2	0.196	0.199	0.000	0.214	0.226	0.001	0.786	0.795	0.000	0.767	0.782	0.001
100	ML	0.203	0.204	0.000	0.225	0.230	0.003	0.745	0.747	0.003	0.718	0.722	0.008
	AD	0.199	0.200	0.000	0.210	0.214	0.000	0.778	0.780	0.000	0.762	0.767	0.001
	AT1	0.199	0.200	0.000	0.207	0.211	0.000	0.785	0.788	0.000	0.775	0.780	0.000
	AT2	0.199	0.200	0.000	0.206	0.210	0.000	0.789	0.793	0.000	0.781	0.787	0.000

Type (B)

n	Meth.	Estimation of a						Estimation of b					
		10%			30%			10%			30%		
		Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB	Est.	RMSE	RAB
25	ML	0.207	0.210	0.000	0.269	0.289	0.024	0.691	0.696	0.014	0.645	0.653	0.029
	BD	0.206	0.209	0.000	0.281	0.317	0.033	0.700	0.707	0.012	0.645	0.657	0.029
	BT1	0.206	0.209	0.000	0.288	0.332	0.039	0.690	0.700	0.015	0.646	0.662	0.029
	BT2	0.204	0.208	0.000	0.290	0.339	0.040	0.678	0.690	0.018	0.651	0.672	0.027
50	ML	0.205	0.208	0.000	0.254	0.267	0.014	0.711	0.715	0.009	0.666	0.673	0.022
	BD	0.204	0.207	0.000	0.254	0.274	0.015	0.724	0.730	0.007	0.679	0.690	0.018
	BT1	0.204	0.206	0.000	0.258	0.283	0.017	0.714	0.722	0.009	0.681	0.695	0.017
	BT2	0.203	0.205	0.000	0.260	0.287	0.018	0.702	0.713	0.011	0.686	0.704	0.016
100	ML	0.203	0.204	0.000	0.225	0.230	0.003	0.745	0.747	0.003	0.718	0.722	0.008
	BD	0.202	0.203	0.000	0.221	0.226	0.002	0.759	0.762	0.002	0.736	0.741	0.005
	BT1	0.202	0.203	0.000	0.223	0.228	0.002	0.755	0.759	0.002	0.735	0.742	0.005
	BT2	0.202	0.203	0.000	0.224	0.229	0.003	0.750	0.755	0.003	0.736	0.745	0.005

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 August 2018 doi:10.20944/preprints201808.0039.v1

Peer-reviewed version available at Math. Comput. Appl. 2018, 23, 47; doi:10.3390/mca23030047

APPENDIX

The formal definition of the gamma function take the following form:

$$\alpha^{-b}\Gamma(b) = \int_0^\infty x^{b-1}e^{-\alpha x}dx; \qquad \alpha > 0, \tag{A. 1}$$

putting $y = e^{-\alpha x}$ this form becomes:

$$\int_0^1 (-\log(y))^{b-1}dy = \Gamma(b). \tag{A. 2}$$

Also; the lower incomplete gamma function is:

$$\alpha^{-b}\gamma(\alpha c, b) = \int_0^c x^{b-1}e^{-\alpha x}dx, \tag{A. 3}$$

it can be shown that,

$$\int_0^c (-\log(y))^{b-1}dy = \gamma(c, b). \tag{A. 4}$$

And; the upper incomplete gamma function is

$$\alpha^{-b}\Gamma(\alpha c, b) = \int_c^\infty x^{b-1}e^{-\alpha x}dx, \tag{A. 5}$$

it can be shown that,

$$\int_c^\infty (-\log(y))^{b-1}dy = \Gamma(c, b). \tag{A. 6}$$

The formal definition of the beta function take the following form:

$$\beta(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt. \tag{A. 7}$$

Lower incomplete beta function:

$$\beta_c(a, b) = \int_0^c t^{a-1}(1-t)^{b-1}dt. \tag{A. 8}$$

Upper incomplete beta function:

$$\beta^c(a, b) = \int_c^1 t^{a-1}(1-t)^{b-1}dt. \tag{A. 9}$$

Peer-reviewed version available at *Math. Comput. Appl.* **2018**, *23*, 47; doi:10.3390/mca23030047

References

Barnett, V., & Lewis, T. (1994). *Outliers in statistical data* (Vol. 3) (No. 1). Wiley New York.

Elamir, E. A., & Seheult, A. H. (2003). Trimmed l-moments. *Computational Statistics & Data Analysis*, *43*(3), 299–314.

Hosking, J. (1990). L-moments: analysis and estimation of distributions using linear combinations of statistics. *J. Roy. Statist. Soc. B*, *52*, 105–124.

Hosking, J. (1995). The use of L-moments in the analysis of censored data. In N. Balakrishnan (Ed.), *Recent Advances in Life-Testing and Reliability* (pp. 546–560). CRC Press, Boca Raton, Fla.

Mahmoud, M. R., Khalil, F. A., El-Kelany, G. A., & Ibrahim, H. A. (2017). Direct L-moments for Type-I censored data with application to the Kumaraswamy distribution. *Cogent Mathematics*, *4*(1), 1357236.

Wang, Q. (1990a). Estimation of the GEV distribution from censored samples by method of partial probability weighted moments. *Journal of Hydrology*, *120*(1), 103–114.

Wang, Q. (1990b). Unbiased estimation of probability weighted moments and partial probability weighted moments from systematic and historical flood information and their application to estimating the GEV distribution. *Journal of hydrology*, *120*(1-4), 115–124.

Wang, Q. (1996a). Using partial probability weighted moments to fit the extreme value distributions to censored samples. *Water Resources Research*, *32*(6), 1767–1771.

Weibull, W. (1951). A Statistical Distribution Function of Wide Applicability. *Journal of applied mechanics*, *103*(730), 293–297.

Zafirakou-Koulouris, A., Vogel, R. M., Craig, S. M., & Habermeier, J. (1998). L moment diagrams for censored observations. *Water resources research*, *34*(5), 1241–1249.