Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Limits of *it*-soft sets and their applications for rough sets ☆

Xiaoliang Xie

(*Mobile E-business Collaborative Innovation Center of Hunan Province, Hunan University of Commerce, Changsha, Hunan 410205, P.R.China.*)

Jiali He*

(*Key Laboratory of Complex System Optimization and Big Data Processing in Department of Guangxi Education, Yulin Normal University, Yulin, Guangxi 537000, P.R.China*)

Abstract

Soft set theory is a mathematical tool for dealing with uncertainty. This paper investigates limits of interval type of soft sets (for short, *it*-soft sets). The concept of *it*-soft sets is first introduced. Then, limits of *it*-soft sets are proposed and their properties are obtained. Next, point-wise continuity of *it*-soft sets and continuous *it*-soft sets are discussed. Finally, an application for rough sets is given.

Key words: Soft set; *it*-soft set; Limit; Continuity; Rough set.

1. Introduction

To solve complicated problems in economics, engineering, environmental science and social science, methods in classical mathematics are not always successful because of various types of uncertainties present in these problems. There are several theories: probability theory, fuzzy set theory [22],

☆This work is supported by National Natural Science Foundation of China (11461005), Key Laboratory of Hunan Province for New Retail Virtual Reality Technology and Key Laboratory of Hunan Province for Mobile Business Intelligence.

*Corresponding author: Jiali He

Email addresses: 1941@hnuc.edu.cn (Xiaoliang Xie), 20110019@gxun.edu.cn (Jiali He)

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

rough set theory [18] and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. For example, probability theory can deal only with stochastically stable phenomena (see [17]). To overcome these kinds of difficulties, Molodtsov [17] proposed a completely new approach, which is called soft set theory, for modeling uncertainty.

Presently, works on soft sets theory are progressing rapidly. Maji et al. [14, 15] further studied soft sets theory and used this theory to solve some decision making problems. Aktas et al. [1] defined soft groups. Jiang et al. [7] extended soft sets with description logics. Feng et al. [4] investigated the relationship among soft sets, rough sets and fuzzy sets. Ge et al. [8] discussed the relationship between soft sets and topological spaces. Li et al. [12] obtained the relationship among soft sets, soft rough sets and topologies. Li et al. [13] studied parameter reductions of soft coverings.

Rough set theory, proposed by Pawlak [18], is an important tool for dealing with fuzzyness and uncertainty of knowledge. After thirty years development, this theory has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [18, 19, 20, 21]. The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Through these rough approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules [19, 20, 21]. Pawlak's rough set model is based on the completeness of available information, and ignores the incompleteness of available information and the possible existence of statistical information. This model for extracting rules in uncoordinate decision information systems often seems incapable. These have motivated many researchers to investigate probabilistic generalization of rough set theory and provide new rough set model for the study of uncertain information system.

Probabilistic rough set model is probabilistic generalization of rough set theory. In probabilistic rough set model, probabilistic rough approximations are dependent on parameters. Researching the infinite change trend or the limit state of these approximations accordance with parameters is helpful for the study of probabilistic rough sets.

It is well-known that calculus theory is the foundation of modern science. Limits of functions are its basic concepts, which play an important role in the process of development [10]. Since probabilistic rough approximations and

 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

level sets of a fuzzy set are both it-soft sets (i.e., interval type of soft sets or soft sets whose parameter sets are the intervals in R), we may attempt to study the infinite change trend or the limit state of it-soft sets. It is worth mentioning that there is no systematic research and summary for limits of it-soft sets although the limit though of it-soft sets has formed in [24, 25].

In general, most of uncertain mathematical theories can only deal with uncertainty problems of discreteness. If limit theory of it-soft sets is established, then these theories may be used to solve uncertainty problems of continuity The purpose of this paper is to establish preliminarily limit theory of interval type soft set so that some uncertain mathematical theories such as rough set theory may be used to solve uncertainty problems of continuity.

The remaining part of this paper is organized as follows. In Section 2, we recall some basic concepts about limits of set sequences and rough sets. In Section 3, we introduce it-soft sets and related notions. In Sections 4, we propose the concept of limits of it-soft sets and obtain their properties. In Sections 5, we discuss the continuity of it-soft sets including point-wise continuity of it-soft sets and continuous it-soft sets. In Sections 6, we give an application for rough sets. Sections 7 summarizes this paper.

2. Preliminaries

In this section, we recall some basic concepts about limits of s-sequences, rough sets and it-soft sets.

Throughout this paper, U denotes the universe which may be an infinite set, 2^U denotes the family of all subsets of U, E denotes a set of all possible parameters, R denotes the set of all real numbers, N denotes the set of all natural numbers and I denotes the interval in R.

2.1. Limits of set sequences

Definition 2.1 ([3, 9]). *Let U be the universe. If for each $n \in N$, $E_n \in 2^U$, then $\{E_n\}$ is called a set sequence in U. Define*

$$\overline{\lim_{n\to\infty}} E_n = \{x \in U : \{n \in N : x \in E_n\} \text{ is infinite}\},$$

$$\underline{\lim_{n\to\infty}} E_n = \{x \in U : \{n \in N : x \notin E_n\} \text{ is finite}\}.$$

If $\underline{\lim_{n\to\infty}} E_n = \overline{\lim_{n\to\infty}} E_n = E$, then $\{E_n : n \in N\}$ is called to has the limit E, which is denoted by $\lim_{n\to\infty} E_n$, i.e., $\lim_{n\to\infty} E_n = E$; If $\underline{\lim_{n\to\infty}} E_n \neq \overline{\lim_{n\to\infty}} E_n$, then $\{E_n : n \in N\}$ is called to has no the limit.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Obviously, $\underline{\lim}_{n\to\infty} E_n \subseteq \overline{\lim}_{n\to\infty} E_n$.

Proposition 2.2 ([3, 9]). *Let $\{E_n : n \in N\}$ be a set sequence in U.*

(1) $\overline{\lim}_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$.

(2) $\underline{\lim}_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k$.

Proposition 2.3 ([3, 9]). *Let $\{E_n : n \in N\}$ be a set sequence in U.*

(1) *If $\{E_n\}\uparrow$, then $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$.*

(2) *If $\{E_n\}\downarrow$, then $\lim_{n\to\infty} E_n = \bigcap_{n=1}^{\infty} E_n$.*

2.2. Rough sets

Let R be an equivalence relation on the universe U. Then the pair (U, R) is called a Pawlak approximation space. Based on (U, R), one can define the following two rough approximations:

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\},\ \overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

Then $\underline{R}(X)$ and $\overline{R}(X)$ are called the Pawlak lower approximation and the Pawlak upper approximation of X, respectively.

The boundary region of X, defined by the difference between these rough approximations, that is $Bnd_R(X) = \overline{R}(X) - \underline{R}(X)$.

A set is rough if its boundary region is not empty; otherwise, it is crisp. Thus, X is rough if $\underline{R}(X) \neq \overline{R}(X)$.

Definition 2.4 ([24, 25]). *Let U be a finite universe. Then a function $P : 2^U \to [0, 1]$ is called a probability measure over U, if $P(U) = 1$ and $P(A \cup B) = P(A) + P(B)$ whenever $A \cap B = \emptyset$.*

If P is a probability measure over U, $A, B \in 2^U$ and $P(B) > 0$, then $P(A|B) = \frac{P(A\cap B)}{P(B)}$ is called the conditional probability of the event A when the event B occurs.

Definition 2.5 ([24, 25]). *Let U be a finite universe, R an equivalence relation over U and P a probability measure over U. Then the pair (U, R, P) is called a probabilistic approximate space. Based on (U, R, P), the lower and*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

upper approximation of X*, are respectively denoted by* $\underline{PI}_\alpha(X)$ *and* $\overline{PI}_\beta(X)$*, are defined as follows:*

$$\underline{PI}_\alpha(X) = \{x \in U : P(X|[x]) \geq \alpha\},\ \overline{PI}_\beta(X) = \{x \in U : P(X|[x]) > \beta),$$

where $0 \leq \beta < \alpha \leq 1$*.*

Theorem 2.6 ([24, 25])**.** *Let* (U, R, P) *be a probabilistic approximate space. Then the following properties hold.*

(1) $\underline{PI}_\alpha(\emptyset) = \overline{PI}_\alpha(\emptyset) = \emptyset$, $\underline{PI}_\alpha(U) = \overline{PI}_\alpha(U) = U$.
(2) $\underline{PI}_\alpha(X) \subseteq \overline{PI}_\alpha(X)$.
(3) $\underline{PI}_\alpha(U - X) = U - \overline{PI}_{1-\alpha}(X)$, $\overline{PI}_\alpha(U - X) = U - \underline{PI}_{1-\alpha}(X)$.
(4) *If* $X \subseteq Y$*, then* $\underline{PI}_\alpha(X) \subseteq \underline{PI}_\alpha(Y)$, $\overline{PI}_\alpha(X) \subseteq \overline{PI}_\alpha(Y)$.
(5) *If* $0 < \alpha_1 < \alpha_2 \leq 1$, $0 \leq \beta_1 < \beta_2 < 1$ *then*

$$\underline{PI}_{\alpha_2}(X) \subseteq \underline{PI}_{\alpha_1}(X),\ \overline{PI}_{\beta_2}(X) \subseteq \overline{PI}_{\beta_2}(X).$$

Theorem 2.7 ([24, 25])**.** *Let* (U, R, P) *be a probabilistic approximate space. Then for* $0 < \gamma < 1$, $X \in 2^U$,

(1) $\lim\limits_{\alpha\uparrow\gamma} \underline{PI}_\alpha(X) = \bigcap\limits_{\alpha\in(0,\gamma)} \underline{PI}_\alpha(X) = \underline{PI}_\gamma(X)$,
$\lim\limits_{\alpha\downarrow\gamma} \underline{PI}_\alpha(X) = \bigcup\limits_{\alpha\in(\gamma,1]} \underline{PI}_\alpha(X) = \overline{PI}_\gamma(X)$;
(2) $\lim\limits_{\alpha\uparrow\gamma} \overline{PI}_\alpha(X) = \bigcap\limits_{\alpha\in[0,\gamma)} \overline{PI}_\alpha(X) = \underline{PI}_\gamma(X)$,
$\lim\limits_{\alpha\downarrow\gamma} \overline{PI}_\alpha(X) = \bigcup\limits_{\alpha\in(\gamma,1)} \overline{PI}_\alpha(X) = \overline{PI}_\gamma(X)$.

Although the limit though of *it*-soft sets has formed in Theorem 2.6, there is no systematic research and summary for limits of *it*-soft sets. Thus, limit theory of interval type soft set deserves deeply study so that rough set theory can be used to solve uncertainty problems of continuity.

3. Soft sets

Definition 3.1 ([17])**.** *Let* $A \subseteq E$*. A pair* (f, A) *is called a soft set over* U*, if* f *is a mapping given by* $f : A \to 2^U$*. We also denote* (f, A) *by* f_A*.*

In other words, a soft set f_A over U is a parametrized family of subsets of the universe U. For $e \in A$, $f(e)$ may be considered as the set of e-approximate elements of the soft set f_A. Clearly, a soft set is not a set.

 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Definition 3.2 ([14]). *Let f_A and g_B be two soft sets over U.*

(1) f_A is called a soft subset of g_B, if $A \subseteq B$ and $f(e) = g(e)$ for each $e \in A$. We denote it by $f_A \widetilde{\subset} g_B$.

(2) f_A is called a soft super set of g_B, if $g_B \widetilde{\subset} f_A$. We denote it by $f_A \widetilde{\supset} g_B$.

Definition 3.3 ([14]). *Let f_A and g_B be two soft sets over U.*

f_A and g_B are called soft equal, if $A \subseteq B$ and $f(e) = g(e)$ for each $e \in A$. We denote it by $f_A = g_B$.

Obviously, $f_A = g_B$ if and only if $f_A \widetilde{\subset} g_B$ and $f_A \widetilde{\supset} g_B$.

Definition 3.4 ([14]). *Let f_A be a soft set over U.*

(1) f_A is called null, if $f(e) = \emptyset$ for each $e \in A$. We denote it by $\widetilde{\emptyset}$.

(2) f_A is called absolute, if $f(e) = U$ for each $e \in A$. We denote it by $\widetilde{U}$.

(3) f_A is called constant, if there exists $X \in 2^U$ such that $f(e) = X$ for each $e \in A$. We denote it by $\widetilde{X}$ or X_A.

Definition 3.5 ([14]). *Let f_A and g_B be two soft sets over U.*

(1) h_C is called the intersection of f_A and g_B, if $C = A \cap B$ and $h(e) = f(e) \cap g(e)$ for each $e \in C$. We denote it by $f_A \widetilde{\cap} g_B = h_C$.

(2) h_C is called the union of f_A and g_B, if $C = A \cup B$ and

$$h(e) = \begin{cases} f(e), & \text{if } \ e \in A - B, \\ g(e), & \text{if } \ e \in B - A, \\ f(e) \cup g(e), & \text{if } \ e \in A \cap B. \end{cases}$$

We denote it by $f_A \widetilde{\cup} g_B = h_C$.

(3) h_C is called the bi-intersection of f_A and g_B, if $C = A \times B$ and $h(a, b) = f(a) \cap g(b)$ for each $a \in A$ and $b \in B$. We denote it by $f_A \bigwedge g_B = h_C$.

(4) h_C is called the bi-union of f_A and g_B, if $C = A \times B$ and $h(a, b) = f(a) \cup g(b)$ for each $a \in A$ and $b \in B$. We denote it by $f_A \bigvee g_B = h_C$.

Definition 3.6 ([16]). *The relative complement of a soft set f_A is denoted by ${f^c}_A$ and is defined by ${f^c}_A = (f^c, A)$, where $f^c : A \to 2^U$ is a mapping given by $f^c(e) = U - f(e)$ for each $e \in A$.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1
Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

Definition 3.7 ([4]). *Let f_A be a soft set over U.*

(1) *f_A is called full, if $\bigcup_{e\in A} f(e) = U$.*

(2) *f_A is called partition, if $\{f(e) : e \in A\}$ forms a partition of U.*

Definition 3.8 ([12]). *Let f_A be a soft set over U.*

(1) *f_A is called topological, if $\{f(e) : e \in A\}$ is a topology on U.*

(2) *f_A is called keeping intersection, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \cap f(b) = f(c)$.*

(2) *f_A is called keeping union, if for any $a, b \in A$, there exists $c \in A$ such that $f(a) \cup f(b) = f(c)$.*

(3) *f_A is called perfect, if $f : A \to 2^U$ is onto.*

(4) *f_A is called having no kernel, if $\cap\{f(e) : e \in A\} = \emptyset$.*

Definition 3.9. *Let f_A be a soft set over U.*

(1) *f_A is called strong keeping intersection, if for each $B \subseteq A$, there exists $b \in A$ such that $\bigcap_{a\in A} f(a) = f(b)$.*

(2) *f_A is called strong keeping union, if for each $B \subseteq A$, there exists $b \in A$ such that $\bigcup_{a\in A} f(a) = f(b)$.*

Obviously, f_A is strong keeping intersection $\Rightarrow$ f_A is keeping intersection, f_A is strong keeping union $\Rightarrow$ f_A is keep union.

Proposition 3.10 ([12]). *Let f_A be a soft set over U. Then the following properties hold.*

(1) *If f_A is topological, then f_A is full, keeping intersection and strong keep union.*

(2) *f_A is perfect if and only if $\{f(e) : e \in A\}$ is a discrete topology over U.*

(3) *If f_A is perfect, then f_A is topological.*

(4) *f_A is having no kernel if and only if (f^c, A) is full.*

Example 3.11. *Let $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$. Define f_A as follows:*

$$f(e) = \begin{cases} \{x_1, x_2, x_5\}, & \text{if } \alpha \in [0, \frac{1}{4}), \\ \emptyset, & \text{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_1, x_2\}, & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ U, & \text{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

Then f_A is topological. But f_A is neither perfect nor partition.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Example 3.12. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$. *Define* f_A *as follows:*

$$f(e) = \begin{cases} \{x_1, x_2, x_5\}, & \text{if } \alpha \in [0, \frac{1}{4}), \\ \{x_1, x_2\}, & \text{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_3\}, & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ \{x_3, x_4\}, & \text{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

Note that $\{x_1, x_2, x_5\} \cap \{x_3\} = \emptyset \neq f(\alpha)$ $(\forall\ \alpha \in I)$. *Then* f_A *is not keeping intersection.*

Example 3.13. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$. *Define* f_A *as follows:*

$$f(e) = \begin{cases} \{x_1\}, & \text{if } \alpha \in [0, \frac{1}{4}), \\ \{x_1, x_4\}, & \text{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_1, x_3, x_4\}, & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ U, & \text{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

Then f_A *is full, keeping intersection and strong keeping union. But* f_A *is not topological.*

Example 3.14. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$. *Define* f_A *as follows:*

$$f(e) = \begin{cases} \{x_1, x_2\}, & \text{if } \alpha \in [0, \frac{1}{4}), \\ \{x_5\}, & \text{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_3\}, & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ \{x_4\}, & \text{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

Then f_A *is partition. But* f_A *is neither topological nor perfect.*

Example 3.15. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$. *Define* f_A *as follows:*

$$f(e) = \begin{cases} \{x_1, x_2, x_5\}, & \text{if } \alpha \in [0, \frac{1}{4}), \\ \emptyset, & \text{if } \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_3\}, & \text{if } \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ \{x_3, x_4\}, & \text{if } \alpha \in [\frac{3}{4}, 1). \end{cases}$$

Then f_A *is full and strong keeping intersection. But*

$$\{x_1, x_2, x_5\} \cup \{x_3\} = \{x_1, x_2, x_3, x_5\} \neq f(\alpha)\ (\forall\ \alpha \in I).$$

Thus f_A *is not keeping union.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Example 3.16. *Let* $U = \{x_1, x_2, x_3, x_4, x_5\}$, $A = [0, 1)$. *Define* f_A *as follows:*

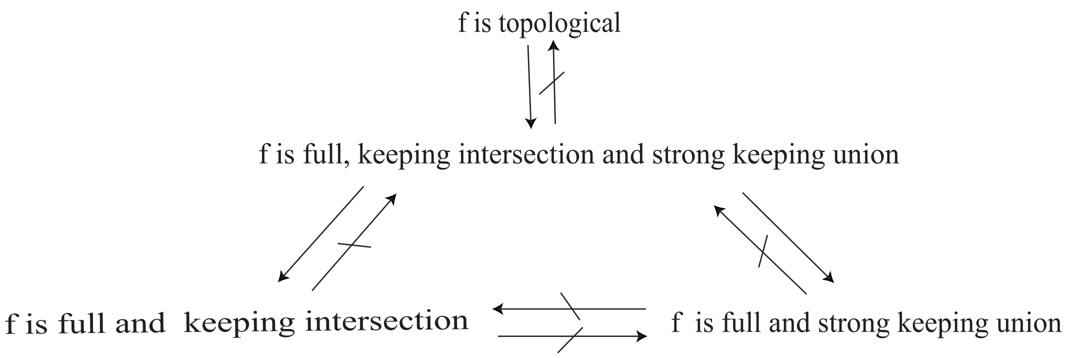
$$f(e) = \begin{cases} \{x_1\}, & if \ \ \alpha \in [0, \frac{1}{4}), \\ \{x_2\}, & if \ \ \alpha \in [\frac{1}{4}, \frac{1}{2}), \\ \{x_1, x_2\}, & if \ \ \alpha \in [\frac{1}{2}, \frac{3}{4}), \\ U, & if \ \ \alpha \in [\frac{3}{4}, 1). \end{cases}$$

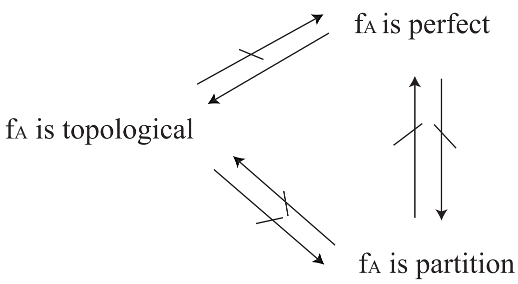
Then f_A *is full and strong keeping union. But*

$$\{x_1\} \cap \{x_2\} = \emptyset \neq f(\alpha) \ (\forall \ \alpha \in I).$$

Thus f_A *is not keeping intersection.*

From Examples 3.11, 3.12, 3.13, 3.14, 3.15 and 3.16, we have the following relationships:





4. Limit theory of *it*-soft sets

4.1. The concept of *it*-soft sets

Definition 4.1. *Let* f_A *be a soft set over* U. *If there exists the interval* I *in* R *such that* $A = I$. *Then* f_A *is called an it-soft set over* U. *Denote it with* f_I.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

It is worth mentioning that the *it*-soft sets are different from interval soft sets in [23].

Definition 4.2. *Let f_I be an it-soft set over U.*

(1) *If for any $e_1, e_2 \in I, e_1 < e_2$ implies $f(e_1) \subset f(e_2)$(resp., $f(e_1) \supset f(e_2)$), then f_I is called strictly increasing (resp., strictly decreasing) on I.*

(2) *If for any $e_1, e_2 \in I, e_1 < e_2$ implies $f(e_1) \subseteq f(e_2)$(resp., $f(e_1) \supseteq f(e_2)$), then f_I is called increasing (resp., decreasing) on I.*

Definition 4.3. *Let f_I be an it-soft set over U.*

(1) *If for any $e \in I$, $f(e) \subseteq f(e_0)$ $(e_0 \in I)$, then $f(e_0)$ is called the maximum value of f_I.*

(2) *If for any $e \in I$, $f(e) \supseteq f(e_0)$ $(e_0 \in I)$, then $f(e_0)$ is called the minimum value of f_I.*

4.2. Limits of *it*-soft sets

Let $e_0 \in R$, $\delta > 0$. Denote

$$U(e_0, \delta) = \{e : |e - e_0| < \delta\},\ U^0(e_0, \delta) = \{e : 0 < |e - e_0| < \delta\}.$$

Then $U(e_0, \delta)$ is called δ neighborhood of e_0, $U^0(e_0, \delta)$ is called δ neighborhood of e_0 having no the heart, e_0 is the center of the neighborhood, δ is the radius of the neighborhood.

$U^+(e_0, \delta) = [e_0, e_0 + \delta)$ is called the δ right neighborhood of e_0,

$U^-(e_0, \delta) = (e_0 - \delta, e_0]$ is called the δ left neighborhood of e_0.

Obviously, $U(e_0, \delta) = (e_0 - \delta, e_0 + \delta) = U^+(e_0, \delta) \cup U^-(e_0, \delta)$.

Let f_I be an *it*-soft set over U. For $e_0 \in I$, $x \in U$, denote

$$[x]_{f_I} = \{e \in I - \{e_0\} : x \in f(e)\},$$

$$(x)_{f_I} = \{e \in I - \{e_0\} : x \notin f(e)\}.$$

Remark 4.4. (1) $[x]_{f_I} \cup (x)_{f_I} = I - \{e_0\}$, $[x]_{f_I} \widetilde{\cap} (x)_{f_I} = \emptyset$.

(2) $[x]_{f_I} \cap [x]_{g_I} = [x]_{f_I \widetilde{\cap} g_I}$, $[x]_{f_I} \cup [x]_{g_I} = [x]_{f_I \widetilde{\cup} g_I}$.

(3) $(x)_{f_I} \cap (x)_{g_I} = (x)_{f_I \widetilde{\cup} g_I}$, $(x)_{f_I} \cup (x)_{g_I} = (x)_{f_I \widetilde{\cap} g_I}$.

(4) $[x]_{f_I^c} = (x)_{f_I}$, $(x)_{f_I^c} = [x]_{f_I}$.

Definition 4.5. *Let f_I be an it-soft set over U. For $e_0 \in I$, define*

(1) $\overline{\lim\limits_{e \to e_0^+}} f(e) = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta)$ *is infinite*$\}$, *which is called the over-right limit of f_I as $e \to e_0$ (or the over limit of f_I as $e \to e_0^+$);*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

(2) $\underline{\lim}_{e\to e_0^+} f(e) = \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0,\delta)\ is\ finite\}$, *which is called the under-right limit of* f_I *as* $e \to e_0$(*or the under limit of* f_I *as* $e \to e_0^+$).

(3) $\overline{\lim}_{e\to e_0^-} f(e) = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^-(e_0,\delta)\ is\ infinite\}$, *which is called the over-left limit of* f_I *as* $e \to e_0$(*or the over limit of* f_I *as* $e \to e_0^-$).

(4) $\underline{\lim}_{e\to e_0^-} f(e) = \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^-(e_0,\delta)\ is\ finite\}$, *which is called the under-left limit of* f_I *as* $e \to e_0$(*or the under limit of* f_I *as* $e \to e_0^-$).

Theorem 4.6. *Let* f_I *be an it-soft set over* U. *Then for* $e_0 \in I$,

(1) $\overline{\lim}_{e\to e_0^+} f(e) = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^+(e_0,\delta) \neq \emptyset\}$

$= \{x \in U : \forall\ n \in N,\ [x]_{f_I} \cap U^+(e_0,\frac{1}{n}) \neq \emptyset\}$.

(2) $\underline{\lim}_{e\to e_0^+} f(e) = \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0,\delta) = \emptyset\}$

$= \{x \in U : \exists\ n \in N,\ (x)_{f_I} \cap U^+(e_0,\frac{1}{n}) = \emptyset\}$.

(3) $\overline{\lim}_{e\to e_0^-} f(e) = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^-(e_0,\delta) \neq \emptyset\}$

$= \{x \in U : \forall\ n \in N,\ [x]_{f_I} \cap U^-(e_0,\frac{1}{n}) \neq \emptyset\}$.

(4) $\underline{\lim}_{e\to e_0^-} f(e) = \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0,\delta) = \emptyset\}$

$= \{x \in U : \exists\ n \in N,\ (x)_{f_I} \cap U^-(e_0,\frac{1}{n}) = \emptyset\}$.

Proof. (1) Put

$$S = \overline{\lim}_{e\to e_0^+} f(e),\ T = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^+(e_0,\delta) \neq \emptyset\},$$
$$L = \{x \in U : \forall\ n \in N,\ [x]_{f_I} \cap U^+(e_0,\tfrac{1}{n}) \neq \emptyset\}.$$

Obviously, $S \subseteq T \subseteq L$. We only need to prove $L \subseteq S$. Suppose $L \nsubseteq S$. Then $L - S \neq \emptyset$. Pick $x \in L - S$. We have $x \notin S$. So $\exists\ \delta_0 > 0$, $[x]_{f_I} \cap U^+(e_0,\delta_0)$ is finite. Denote

$$[x]_{f_I} \cap U^+(e_0,\delta_0) = \{e_1, e_2, \dots, e_n\}.$$

Put $e^* = \min\{e_1, e_2, \dots, e_n\},\ 0 < \frac{1}{n_0} < e^* - e_0$. Then

$$0 < \frac{1}{n_0} < \delta_0,\ [x]_{f_I} \cap U^+(e_0,\frac{1}{n_0}) = \emptyset.$$

So $x \notin L$. But $x \in L$. This is a contradiction. Thus $L \subseteq S$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

(2) Put

$$P = \underline{\lim}_{e \to e_0^+} f(e),\ Q = \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\},$$
$$K = \{x \in U : \exists\ n \in N,\ (x)_{f_I} \cap U^+(e_0, \tfrac{1}{n}) = \emptyset\}.$$

Obviously, $K \subseteq Q \subseteq P$. We only need to prove $P \subseteq K$. Suppose $P \nsubseteq K$. Then $P - K \neq \emptyset$. Pick $x \in P - K$. Then $x \notin K$.

Claim $\forall\ \delta,\ (x)_{f_I} \cap U^+(e_0, \delta)$ is infinite.

In fact, suppose that $\exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta)$ is finite. Put

$$(x)_{f_I} \cap U^+(e_0, \delta) = \{e_1, e_2, \ldots, e_n\},\ \ e^* = \min\{e_1, e_2, \ldots, e_n\},\ 0 < \frac{1}{n_0} < e^* - e_0.$$

Then $0 < \frac{1}{n_0} < \delta$, $(x)_{f_I} \cap U^+(e_0, \frac{1}{n_0}) = \emptyset$. So $x \in K$, But $x \notin K$. This is a contradiction.

Since $\forall\ \delta > 0$, $(x)_{f_I} \cap U^+(e_0, \delta)$ is infinite, we have $x \notin P$. But $x \in P$. This is a contradiction. Thus $P \subseteq K$.

(3) The proof is similar to (1).

(4) The proof is similar to (2). □

Example 4.7. *Consider Example 3.12, pick* $e_0 = \frac{1}{4}$*, we have*

$$[x_1]_f = [x_2]_f = [0, \frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2}),\ \ [x_3]_f = [\frac{1}{2}, 1),\ \ [x_4]_f = [\frac{3}{4}, 1),\ \ [x_5]_f = [0, \frac{1}{4}).$$

$$(x_1)_f = (x_2)_f = [\frac{1}{2}, 1),\ (x_3)_f = [0, \frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2}),\ (x_4)_f = [0, \frac{1}{4}) \cup [\frac{1}{4}, \frac{3}{4}),\ (x_5)_f = (\frac{1}{4}, 1).$$

By Theorem 4.6,

$$\overline{\lim}_{e \to e_0^+} f(e) = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta) \neq \emptyset\} = \{x_1, x_2\};$$

$$\underline{\lim}_{e \to e_0^+} f(e) = \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\} = \{x_1, x_2\};$$

$$\overline{\lim}_{e \to e_0^-} f(e) = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^-(e_0, \delta) \neq \emptyset\} = \{x_1, x_2, x_5\};$$

$$\underline{\lim}_{e \to e_0^-} f(e) = \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^-(e_0, \delta) = \emptyset\} = \{x_1, x_2, x_5\}.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Lemma 4.8. *Let f_I be an it-soft set over U. Then for $e_0 \in I$,*

(1) $\overline{\lim\limits_{e\to e_0^+}} f(e) = \bigcap\limits_{n=1}^{\infty} \bigcap\limits_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup\limits_{\beta\in(e_0,e]} f(\beta)$.

(2) $\underline{\lim\limits_{e\to e_0^+}} f(e) = \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap\limits_{\beta\in(e_0,e]} f(\beta)$.

(3) $\overline{\lim\limits_{e\to e_0^-}} f(e) = \bigcap\limits_{n=1}^{\infty} \bigcap\limits_{e\in(e_0-\frac{1}{n},e_0)\cap I} \bigcup\limits_{\beta\in[e,e_0)} f(\beta)$.

(4) $\underline{\lim\limits_{e\to e_0^-}} f(e) = \bigcup\limits_{n=1}^{\infty} \bigcup\limits_{e\in(e_0-\frac{1}{n},e_0)\cap I} \bigcap\limits_{\beta\in[e,e_0)} f(\beta)$.

Proof. (1) Denote

$$S = \overline{\lim_{e\to e_0^+}} f(e), \quad T = \bigcap_{n=1}^{\infty} \bigcap_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta).$$

To prove $S = T$, it suffices to show that

$$x \in S \ \Leftrightarrow\ \forall\, n \in N,\ \forall\, e \in (e_0, e_0 + \frac{1}{n}) \cap I,\ \exists\, \beta \in (e_0, e],\ x \in f(\beta).$$

" $\Rightarrow$ ". Let $x \in S$, $\forall\ n \in N$, $\forall\ e \in (e_0, e_0 + \frac{1}{n}) \cap I$. Put $\delta = e - e_0$. Then $0 < \delta < \frac{1}{n}$.

Since $x \in S$, by Theorem 4.6(1), we have $[x]_{f_I} \cap U^+(e_0,\ \delta) \neq \emptyset$. Pick $\beta \in\ [x]_{f_I} \cap U^+(e_0, \delta)$. Then $\beta \in\ [x]_{f_I},\ \ \beta \in U^+(e_0, \delta)$.

This implies $x \in f(\beta)$, $e_0 < \beta < e_0 + \delta = e$. Thus $\beta \in (e_0, e]$.

" $\Leftarrow$ ". $\forall\ n \in N$, pick $e \in (e_0, e_0 + \frac{1}{n}) \cap I$.

By the condition, $\exists\ \beta \in (e_0, e]$, $x \in f(\beta)$. Then $\beta \in U^+(e_0, \frac{1}{n}), \beta \in\ [x]_{f_I}$. Thus $\forall\ n \in N$, $[x]_{f_I} \cap U^+(e_0, \frac{1}{n}) \neq \emptyset$.

By Theorem 4.6(1), $x \in S$.

(2) By (1) and Theorem 4.6(2),

$$x \notin \underline{\lim_{e\to e_0^+}} f(e)$$
$$\Longleftrightarrow \forall\ n \in N,\ (x)_{f_I} \cap U^+(e_0, \tfrac{1}{n}) \neq \emptyset$$
$$\Longleftrightarrow \forall\ n \in N, \{e \in I - e_0 : x \in U - f(e)\} \cap U^+(e_0, \tfrac{1}{n}) \neq \emptyset$$
$$\Longleftrightarrow x \in \bigcap_{n=1}^{\infty} \bigcap_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup_{\beta\in(e_0,e]} (U - f(\beta))$$
$$\Longleftrightarrow x \in U - \bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

$$\Longleftrightarrow x \notin \bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta).$$

Hence $\underline{\lim}_{e\to e_0^+} f(e) = \bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$.

(3) The proof is similar to (1).

(4) The proof is similar to (2). □

Lemma 4.9. *Let f_I be an it-soft set over U. Then for $e_0 \in I$,*

(1) $\bigcap_{n=1}^{\infty} \bigcap_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta) = \bigcap_{e\in(e_0,e_0+1)\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta)$.

(2) $\bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta) = \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$.

(3) $\bigcap_{n=1}^{\infty} \bigcap_{e\in(e_0-\frac{1}{n},e_0)\cap I} \bigcup_{\beta\in[e,e_0)} f(\beta) = \bigcap_{e\in(e_0-1,e_0)\cap I} \bigcup_{\beta\in[e,e_0)} f(\beta)$.

(4) $\bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0-\frac{1}{n},e_0)\cap I} \bigcap_{\beta\in[e,e_0)} f(\beta) = \bigcup_{e\in(e_0-1,e_0)\cap I} \bigcap_{\beta\in[e,e_0)} f(\beta)$.

Proof. (1) Put $E_n = \bigcap_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta)$. Then $\{E_n\}\uparrow$. So $\bigcap_{n=1}^{\infty} E_n = E_1$.

Thus

$$\bigcap_{n=1}^{\infty} \bigcap_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta) = \bigcap_{e\in(e_0,e_0+1)\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta).$$

(2) Put $F_n = \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta)$. Then $\{F_n\}\downarrow$. So $\bigcup_{n=1}^{\infty} F_n = F_1$.

Thus

$$\bigcup_{n=1}^{\infty} \bigcup_{e\in(e_0,e_0+\frac{1}{n})\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta) = \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta).$$

(3) The proof is similar to (1).

(4) The proof is similar to (2). □

Theorem 4.10. *Let f_I be an it-soft set over U. Then for $e_0 \in I$,*

(1) $\overline{\lim}_{e\to e_0^+} f(e) = \bigcap_{e\in(e_0,e_0+1)\cap I} \bigcup_{\beta\in(e_0,e]} f(\beta)$; *If f_I increasing, then*

$$\overline{\lim}_{e\to e_0^+} f(e) = \bigcap_{e\in(e_0,e_0+1)\cap I} f(e).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1
Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

(2) $\underline{\lim}_{e\to e_0^+} f(e) = \bigcup_{e\in(e_0,e_0+1)\cap I}\ \bigcap_{\beta\in(e_0,e]} f(\beta)$; *If* f_I *decreasing, then*

$$\underline{\lim}_{e\to e_0^+} f(e) = \bigcup_{e\in(e_0,e_0+1)\cap I} f(e).$$

(3) $\overline{\lim}_{e\to e_0^-} f(e) = \bigcap_{e\in(e_0-1,e_0)\cap I}\ \bigcup_{\beta\in[e,e_0)} f(\beta)$; *If* f_I *decreasing, then*

$$\overline{\lim}_{e\to e_0^-} f(e) = \bigcap_{e\in(e_0-1,e_0)\cap I} f(e).$$

(4) $\underline{\lim}_{e\to e_0^-} f(e) = \bigcup_{e\in(e_0-1,e_0)\cap I}\ \bigcap_{\beta\in[e,e_0)} f(\beta)$; *If* f_I *increasing, then*

$$\underline{\lim}_{e\to e_0^-} f(e) = \bigcup_{e\in(e_0-1,e_0)\cap I} f(e).$$

Proof. This holds by Lemmas 4.8 and 4.9. □

Definition 4.11. *Let* f_I *be an it-soft set over* U. *Then for* $e_0 \in I$,

(1) *If* $\underline{\lim}_{e\to e_0^+} f(e) = \overline{\lim}_{e\to e_0^+} f(e) = S$, *then* f_I *is called to has the limit* S *as* $e \to e_0^+$ (*or has the right-limit* S *as* $e \to e_0$), *which is denoted by* $\lim_{e\to e_0^+} f(e)$, *i.e.*, $\lim_{e\to e_0^+} f(e) = S$;

If $\underline{\lim}_{e\to e_0^+} f(e) \neq \overline{\lim}_{e\to e_0^+} f(e)$, *then* f_I *is called to has no the limit as* $e \to e_0^+$ (*or has no the right-limit as* $e \to e_0$).

(2) *If* $\underline{\lim}_{e\to e_0^-} f(e) = \overline{\lim}_{e\to e_0^-} f(e) = S$, *then* f_I *is called to has the limit* S *as* $e \to e_0^-$ (*or has the left-limit* S *as* $e \to e_0$), *which is denoted by* $\lim_{e\to e_0^-} f(e)$, *i.e.*, $\lim_{e\to e_0^-} f(e) = S$;

If $\underline{\lim}_{e\to e_0^+} f(e) \neq \overline{\lim}_{e\to e_0^+} f(e)$, *then* f_I *is called to has no the limit as* $e \to e_0^+$ (*or has no the left-limit as* $e \to e_0$).

(3) *If* $\lim_{e\to e_0^-} f(e) = \lim_{e\to e_0^+} f(e) = S$, *then* f_I *is called to has the limit* S *as* $e \to e_0$, *which is denoted by* $\lim_{e\to e_0} f(e)$, *i.e.*, $\lim_{e\to e_0} f(e) = S$;

If $\lim_{e\to e_0^-} f(e) \neq \lim_{e\to e_0^+} f(e))$, *then* f_I *is called to has no the limit as* $e \to e_0$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

Definition 4.12. *Let f_I be an it-soft set over U. Then for $e_0 \in I$,*

(1) *If* $\overline{\lim\limits_{e\to e_0^-}} f(e) = \overline{\lim\limits_{e\to e_0^+}} f(e) = S$, *then f_I is called to has the over-limit S as $e \to e_0$, which is denoted by* $\overline{\lim\limits_{e\to e_0}} f(e)$, *i.e.,* $\overline{\lim\limits_{e\to e_0}} f(e) = S$;

If $\overline{\lim\limits_{e\to e_0^-}} f(e) \neq \overline{\lim\limits_{e\to e_0^+}} f(e)$, *then f_I is called to has no the over-limit as $e \to e_0^+$.*

(2) *If* $\underline{\lim\limits_{e\to e_0^-}} f(e) = \underline{\lim\limits_{e\to e_0^+}} f(e) = S$, *then f_I is called to has the under-limit S as $e \to e_0$, which is denoted by* $\underline{\lim\limits_{e\to e_0}} f(e)$, *i.e.,* $\underline{\lim\limits_{e\to e_0}} f(e) = S$;

If $\underline{\lim\limits_{e\to e_0^-}} f(e) \neq \underline{\lim\limits_{e\to e_0^+}} f(e)$, *then f_I is called to has no the under-limit as $e \to e_0$.*

(3) *If* $\underline{\lim\limits_{e\to e_0}} f(e) = \overline{\lim\limits_{e\to e_0}} f(e) = S$, *then f_I is called to has the limit as $e \to e_0$, which is denoted by* $\lim\limits_{e\to e_0} f(e)$, *i.e.,* $\lim\limits_{e\to e_0} f(e) = S$;

If $\underline{\lim\limits_{e\to e_0}} f(e) \neq \overline{\lim\limits_{e\to e_0}} f(e)$, *then f_I is called to has no the limit as $e \to e_0$.*

Remark 4.13. *The limit in Definition 4.11(3) and the limit in Definition 4.12(3) is consistent.*

Example 4.14. *Let X_I be a constant it-soft set over U where $X \in 2^U$. Then for $e_0 \in I$,* $\lim\limits_{e\to e_0} X(e) = X$.

Obviously, $[x]_{X_I} = \begin{cases} I - \{e_0\}, & x \in X \\ \emptyset, & x \notin X \end{cases}$, $(x)_{X_I} = \begin{cases} I - \{e_0\}, & x \notin X \\ \emptyset, & x \in X \end{cases}$.

By Theorem 4.6,

$$\overline{\lim_{e\to e_0^+}} X(e) = \{x \in U : \forall\ \delta > 0,\ [x]_{\widetilde{A}} \cap U^+(e_0, \delta) \neq \emptyset\},$$

$$\underline{\lim_{e\to e_0^+}} X(e) = \{x \in U : \exists\ \delta > 0,\ (x)_{\widetilde{A}} \cap U^+(e_0, \delta) = \emptyset\}.$$

Then $\overline{\lim\limits_{e\to e_0^+}} X(e) = X$, $\underline{\lim\limits_{e\to e_0^+}} X(e) = X$.

Similarly, $\overline{\lim\limits_{e\to e_0^-}} X(e) = X$, $\underline{\lim\limits_{e\to e_0^-}} X(e) = X$.

Thus $\lim\limits_{e\to e_0} X(e) = X$.

Other types of limits of *it*-soft sets are proposed by the following definition and these limits can be discussed in a similar way.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

Definition 4.15. *Let* $(f, (-\infty, +\infty))$ *be an it-soft set over* U. *Define*

$$(1)\quad \overline{\lim_{e\to+\infty}} f(e) = \overline{\lim_{e\to 0^+}} f(\frac{1}{e}), \qquad \overline{\lim_{e\to-\infty}} f(e) = \overline{\lim_{e\to 0^-}} f(\frac{1}{e}),$$

$$\overline{\lim_{e\to\infty}} f(e) = \overline{\lim_{e\to 0}} f(\frac{1}{e}).$$

$$(2)\quad \underline{\lim_{e\to+\infty}} f(e) = \underline{\lim_{e\to 0^+}} f(\frac{1}{e}), \qquad \underline{\lim_{e\to-\infty}} f(e) = \underline{\lim_{e\to 0^-}} f(\frac{1}{e}),$$

$$\underline{\lim_{e\to\infty}} f(e) = \underline{\lim_{e\to 0}} f(\frac{1}{e}).$$

$$(3)\quad \lim_{e\to+\infty} f(e) = \lim_{e\to 0^+} f(\frac{1}{e}), \qquad \lim_{e\to-\infty} f(e) = \lim_{e\to 0^-} f(\frac{1}{e}),$$

$$\lim_{e\to\infty} f(e) = \lim_{e\to 0} f(\frac{1}{e}).$$

4.3. Properties of limits of *it*-soft sets

Proposition 4.16. *For the over-right limit, the following properties hold:*

(1) *If* $f(e) \subseteq g(e)(\forall\ e \in (e_0, e_0 + \delta_0))$, *then* $\overline{\lim_{e\to e_0^+}} f(e) \subseteq \overline{\lim_{e\to e_0^+}} g(e)$.

(2) $\overline{\lim_{e\to e_0^+}} (f(e) \cup g(e)) = \overline{\lim_{e\to e_0^+}} f(e) \cup \overline{\lim_{e\to e_0^+}} g(e)$.

(3) $\overline{\lim_{e\to e_0^+}} (U - f(e)) = U - \underline{\lim_{e\to e_0^+}} f(e)$.

(4) *If* $\overline{\lim_{e\to e_0^+}} f(e) = \triangle \subset B$, *then* $\exists\ \delta > 0, \forall\ e \in (e_0, e_0 + \delta)$, $f(e) \subset B$.

(5) 1) $\overline{\lim_{e\to e_0^+}} (f(e) \times g(e)) \subseteq \overline{\lim_{e\to e_0^+}} f(e) \times \overline{\lim_{e\to e_0^+}} g(e)$;

2) $\overline{\lim_{e\to e_0^+}} f(e) \times \overline{\lim_{e\to e_0^+}} g(e) = \bigcap_{e\in(e_0,e_0+1)\cap I} \bigcup_{\beta,\gamma\in(e_0,e]} (f(\beta) \times g(\gamma))$.

Proof. (1) Denote

$$[x]_{f_I} = \{e \in I - \{e_0\} : x \in f(e)\},\ [x]_{g_I} = \{e \in I - \{e_0\} : x \in g(e)\}.$$

$\forall\ x \in \overline{\lim_{e\to e_0^+}} f(e)$, by Theorem 4.6(1), $\forall\ \delta > 0$, $[x]_{f_I} \cap U^+(e_0, \delta) \neq \emptyset$. Pick $e_\delta \in [x]_{f_I} \cap U^+(e_0, \delta)$. Then $x \in f(e_\delta)$, $e_\delta \in U^+(e_0, \delta)$.

1) If $\delta \leq \delta_0$, then $e_\delta \in U^+(e_0, \delta_0)$. By the condition, $f(e_\delta) \subseteq g(e_\delta)$. Then $x \in g(e_\delta)$. This implies $e_\delta \in (x)_{f_I} \cap U^+(e_0, \delta)$. So $(X)_{f_I} \cap U^+(e_0, \delta) \neq \emptyset$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

2) If $\delta > \delta_0$, then $U^+(e_0, \delta_0) \subseteq U^+(e_0, \delta)$. So $(x)_{f_I} \cap U^+(e_0, \delta_0) \subseteq (X)_{f_I} \cap U^+(e_0, \delta)$. Since $e_{\delta_0} \in (X)_{f_I} \cap U^+(e_0, \delta_0)$, we have $(x)_{f_I} \cap U^+(e_0, \delta) \neq \emptyset$.

By 1) and 2), $\forall\ \delta > 0$, $(x)_{f_I} \cap U^+(e_0, \delta) \neq \emptyset$. By Theorem 4.6(1), $x \in \overline{\lim_{e\to e_0^+}}\ g(e)$.

Thus

$$\overline{\lim_{e\to e_0^+}}\ f(e) \subseteq \overline{\lim_{e\to e_0^+}}\ g(e).$$

(2) " $\supseteq$ ". This holds by (1).

" $\subseteq$ ". Suppose $\overline{\lim_{e\to e_0^+}}(f(e) \cup g(e)) \nsubseteq \overline{\lim_{e\to e_0^+}}\ f(e) \cup \overline{\lim_{e\to e_0^+}}\ g(e)$. Then

$$\overline{\lim_{e\to e_0^+}}(f(e) \cup g(e)) - \overline{\lim_{e\to e_0^+}}\ f(e) \cup \overline{\lim_{e\to e_0^+}}\ g(e) \neq \emptyset.$$

Pick $x \in \overline{\lim_{e\to e_0^+}}(f(e) \cup g(e)) - \overline{\lim_{e\to e_0^+}}\ f(e) \cup \overline{\lim_{e\to e_0^+}}\ g(e)$. We have

$$x \in \overline{\lim_{e\to e_0^+}}(f(e) \cup g(e)), \quad x \notin \overline{\lim_{e\to e_0^+}}\ f(e) \text{ and } x \notin \overline{\lim_{e\to e_0^+}}\ g(e).$$

By Theorem 4.6, $\exists\ \delta_1, \delta_2 > 0$, $[x]_f \cap U^+(e_0, \delta_1) = \emptyset$, $[x]_g \cap U^+(e_0, \delta_2) = \emptyset$.

Pick $\delta_3 = \min\{\delta_1, \delta_2\}$. Then $[x]_f \cap U^+(e_0, \delta_3) = \emptyset$ and $[x]_g \cap U^+(e_0, \delta_3) = \emptyset$. It follows

$$([x]_f \cup [x]_{g_I}) \cap U^+(e_0, \delta_3) = ([x]_f \cap U^+(e_0, \delta_3)) \cup ([x]_g \cap U^+(e_0, \delta_3)) = \emptyset.$$

By Remark 4.4, $[x]_{f\cup g} \cap U^+(e_0, \delta_3) = \emptyset$.

Thus

$x \notin \overline{\lim_{e\to e_0^+}}(f \cup g)(e) = \overline{\lim_{e\to e_0^+}}(f(e) \cup g(e))$. This is a contradiction.

(3) $\forall\ x \in \overline{\lim_{e\to e_0^+}}(U - f(e))$. Then $x \in \overline{\lim_{e\to e_0^+}}\ f^c(e)$. By Theorem 4.6, $\forall\ \delta > 0$, $[x]_{f^c} \cap U^+(e_0, \delta) \neq \emptyset$. By Remark 4.4, $(x)_f \cap U^+(e_0, \delta) \neq \emptyset$. Thus

$$x \in U - \underline{\lim_{e\to e_0^+}}\ f(e).$$

Conversely, the proof is similar.

(4) Suppose that $\forall\ \delta > 0$, $\exists\ e \in (e_0, e_0 + \delta)$, $f(e) \nsubseteq B$ or $f(e) = B$.

1) If $f(e) \nsubseteq B$, then $f(e) - B \neq \emptyset$. Pick $x \in f(e) - B$.

We have

$$x \in f(e), x \notin B, e \in\ [x]_{f_I}.$$

Since $e \in (e_0, e_0 + \delta)$. Then $[x]_{f_I} \cap (e_0, e_0 + \delta) \neq \emptyset$. So $x \in \overline{\lim_{e\to e_0^+}}\ f(e)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

Thus $x \in B$. This is a contradiction.

2) If $f(e) = B$, then $\triangle - B = \emptyset$. So $\exists\, x \in B, x \notin \triangle$.

Since $x \in f(e)$, we have $x \in [x]_{f_I}$, $[x]_{f_I} \cap (e_0, e_0 + \delta) \neq \emptyset$. So

$$x \in \overline{\lim_{e \to e_0^+}} f(e) = \triangle.$$

This is a contradiction.

(5) 1) Put

$$H_{f\times g}(e) = \bigcup_{\beta \in (e_0, e]} (f(\beta) \times g(\beta)).$$

By Theorem 4.10(1),

$$\overline{\lim_{e \to e_0^+}} (f(e) \times g(e)) = \bigcap_{e \in (e_0, e_0+1) \cap I} H_{f\times g}(e).$$

$\forall\, (x, y) \in \overline{\lim_{e \to e_0^+}} (f(e) \times g(e))$, we have $(x, y) \in \bigcap_{e \in (e_0, e_0+1) \cap I} H_{f\times g}(e)$. Since

$$H_{f\times g}(e) = \bigcup_{\beta \in (e_0, e]} (f(\beta) \times g(\beta)),$$

we have $\forall\, e \in (e_0, e_0 + 1) \cap I$, $\exists\, \beta_e \in (e_0, e]$, $(x, y) \in f(\beta_e) \times g(\beta_e)$. It follows $x \in f(\beta_e)$, $y \in g(\beta_e)$. Then $x \in H_f(e)$ and $y \in H_g(e)$. So

$$x \in \bigcap_{e \in (e_0, e_0+1) \cap I} H_f(e) = \overline{\lim_{e \to e_0^+}} f(e), \quad y \in \bigcap_{e \in (e_0, e_0+1) \cap I} H_g(e) = \overline{\lim_{e \to e_0^+}} g(e).$$

Thus $(x, y) \in \overline{\lim_{e \to e_0^+}} f(e) \times \overline{\lim_{e \to e_0^+}} g(e)$.

Thus

$$\overline{\lim_{e \to e_0^+}} (f(e) \times g(e)) \subseteq \overline{\lim_{e \to e_0^+}} f(e) \times \overline{\lim_{e \to e_0^+}} g(e).$$

2) $\forall\, (x, y) \in \overline{\lim_{e \to e_0^+}} f(e) \times \overline{\lim_{e \to e_0^+}} g(e)$, we have

$$x \in \overline{\lim_{e \to e_0^+}} f(e) = \bigcap_{e \in (e_0, e_0+1) \cap I} \bigcup_{\beta \in (e_0, e]} f(\beta), \; y \in \overline{\lim_{e \to e_0^+}} g(e) = \bigcap_{e \in (e_0, e_0+1) \cap I} \bigcup_{\beta \in (e_0, e]} g(\beta).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

Then $\forall\ e \in (e_0, e_0+1)\cap I,\ \exists\ \beta_e, \gamma_e \in (e_0, e],\ x \in f(\beta_e),\ y \in g(\gamma_e)$. Then $(x, y) \in f(\beta_e) \times g(\gamma_e)$. So

$$(x, y) \in \bigcap_{e\in(e_0,e_0+1)\cap I} \bigcup_{\beta,\gamma\in(e_0,e]} (f(\beta) \times g(\gamma)).$$

Conversely, the proof is similar.

Thus

$$\overline{\lim_{e\to e_0^+}} f(e) \times \overline{\lim_{e\to e_0^+}} g(e) = \bigcap_{e\in(e_0,e_0+1)\cap I} \bigcup_{\beta,\gamma\in(e_0,e]} (f(\beta) \times g(\gamma)).$$

□

Proposition 4.17. *For the under-right limit, the following properties hold.*

(1) *If* $f(e) \subseteq g(e)$ $(\forall\ e \in (e_0, e_0+\delta_0))$, *then* $\underline{\lim}_{e\to e_0^+} f(e) \subseteq \underline{\lim}_{e\to e_0^+} g(e)$.

(2) $\underline{\lim}_{e\to e_0^+} (f(e) \cap g(e)) = \underline{\lim}_{e\to e_0^+} f(e) \cap \underline{\lim}_{e\to e_0^+} g(e)$.

(3) $\underline{\lim}_{e\to e_0^+} (U - f(e)) = U - \overline{\lim}_{e\to e_0^+} f(e)$.

(4) *If* $\underline{\lim}_{e\to e_0^+} f(e) = \triangle \supset A$, *then* $\exists\ \delta > 0,\ \forall\ e \in (e_0, e_0+\delta),\ f(e) \supset A$.

(5) $\underline{\lim}_{e\to e_0^+} (f(e) \times g(e)) = \underline{\lim}_{e\to e_0^+} f(e) \times \underline{\lim}_{e\to e_0^+} g(e)$.

Proof. (1) The proof is similar to Proposition 4.16(1).

(2) " $\subseteq$ ". This holds by (1).

" $\supseteq$ ". Suppose $\underline{\lim}_{e\to e_0^+} f(e)\cap \underline{\lim}_{e\to e_0^+} g(e) \not\subseteq \underline{\lim}_{e\to e_0^+} (f(e)\cap g(e))$. Then $\underline{\lim}_{e\to e_0^+} f(e)\cap \underline{\lim}_{e\to e_0^+} g(e) - \underline{\lim}_{e\to e_0^+} (f(e)\cap g(e)) \neq \emptyset$. Pick $x \in \underline{\lim}_{e\to e_0^+} f(e)\cap \underline{\lim}_{e\to e_0^+} g(e) - \underline{\lim}_{e\to e_0^+} (f(e)\cap g(e))$. We have

$$x \in \underline{\lim}_{e\to e_0^+} f(e),\ x \in \underline{\lim}_{e\to e_0^+} g(e) \text{ and } x \notin \underline{\lim}_{e\to e_0^+} (f(e) \cap g(e)).$$

By Theorem 4.6,

$$\exists\ \delta_1, \delta_2 > 0,\ (x)_f \cap U^+(e_0, \delta_1) = \emptyset,\ (x)_g \cap U^+(e_0, \delta_2) = \emptyset.$$

Pick $\delta_3=\min\{\delta_1, \delta_2\}$. Then $(x)_f \cap U^+(e_0, \delta_3) = \emptyset,\ (x)_g \cap U^+(e_0, \delta_3) = \emptyset$. It follows

$$((x)_f \cup (x)_{g_I}) \cap U^+(e_0, \delta_3) = ((x)_f \cap U^+(e_0, \delta_3)) \cup ((x)_g \cap U^+(e_0, \delta_3)) = \emptyset.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

By Remark 4.4 , $(x)_{f\cap g} \cap U^+(e_0, \delta_3) = \emptyset$.

Thus

$x \in \underline{\lim}_{e\to e_0^+} (f \cap g)(e) = \underline{\lim}_{e\to e_0^+} (f(e) \cap g(e))$. This is a contradiction.

(3) $\forall\, x \in \underline{\lim}_{e\to e_0^+} (U - f(e))$. Then $x \in \underline{\lim}_{e\to e_0^+} f^c(e)$. By Theorem 4.6, $\exists\, \delta > 0$, $(x)_{f^c} \cap U^+(e_0, \delta) = \emptyset$. By Remark 4.4, $[x]_f \cap U^+(e_0, \delta) = \emptyset$.

Thus $x \in U - \overline{\lim}_{e\to e_0^+} f(e)$.

Conversely, the proof is similar.

(4) By Proposition 4.16(3),

$$\overline{\lim}_{e\to e_0^+} (U - f(e)) = U - \underline{\lim}_{e\to e_0^+} f(e).$$

Since $\underline{\lim}_{e\to e_0^+} f(e) = \triangle \supset A$, we have $\overline{\lim}_{e\to e_0^+} (U - f(e)) \subset U - A$.

By Proposition 4.16(4), $\exists\, \delta > 0, \forall\, e \in (e_0, e_0 + \delta)$, $U - f(e) \subset U - A$.

Thus

$$\exists\, \delta > 0, \forall\, e \in (e_0, e_0 + \delta),\ f(e) \supset A.$$

(5) $\forall\, (x, y) \in \underline{\lim}_{e\to e_0^+} (f(e) \times g(e))$, by Theorem 4.10(2),

$$(x, y) \in \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} (f(\beta) \times g(\beta)).$$

Then $\exists\, e \in (e_0, e_0 + 1) \cap I$, $\forall\, \beta \in (e_0, e]$, $(x, y) \in f(\beta) \times g(\beta)$. It follows $x \in f(\beta)$, $y \in g(\beta)$. Then

$$x \in \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta), \quad y \in \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} g(\beta).$$

By Theorem 4.10(2), $x \in \underline{\lim}_{e\to e_0^+} f(e)$, $y \in \underline{\lim}_{e\to e_0^+} g(e)$. Thus $(x, y) \in \underline{\lim}_{e\to e_0^+} f(e) \times \underline{\lim}_{e\to e_0^+} g(e)$.

$\forall\, (x, y) \in \underline{\lim}_{e\to e_0^+} f(e) \times \underline{\lim}_{e\to e_0^+} g(e)$, By Theorem 4.10(2),

$$x \in \underline{\lim}_{e\to e_0^+} f(e) = \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} f(\beta), \quad y \in \underline{\lim}_{e\to e_0^+} g(e) = \bigcup_{e\in(e_0,e_0+1)\cap I} \bigcap_{\beta\in(e_0,e]} g(\beta).$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

Then $\exists\ e_1, e_2 \in (e_0, e_0+1)\cap I$, $\forall\ \beta \in (e_0, e_1]$, $\forall\ \gamma \in (e_0, e_2]$, $x \in f(\beta)$, $y \in g(\gamma)$.

Put $e^* = \min\{e_1, e_2\}$. Then $e^* \in (e_0, e_0+1)\cap I$, $(e_0, e^*] \subseteq (e_0, e_1]\cap(e_0, e_2]$. Then $\forall\ \beta \in (e_0, e^*]$, $x \in f(\beta)$, $y \in g(\beta)$. It follows $(x, y) \in f(\beta)\times g(\beta)$. So

$$(x,y) \in \bigcup_{e\in(e_0,e_0+1)\cap I}\ \bigcap_{\beta,\gamma\in(e_0,e]} (f(\beta)\times g(\beta)).$$

By Theorem 4.10(2), $(x,y) \in \underline{\lim}_{e\to e_0^+}(f(e)\times g(e))$. So

Thus

$$\underline{\lim}_{e\to e_0^+}(f(e)\times g(e)) = \underline{\lim}_{e\to e_0^+} f(e) \times \underline{\lim}_{e\to e_0^+} g(e).$$

□

Proposition 4.18. *For the over-left limit, the following properties hold:*

(1) *If* $f(e) \subseteq g(e)$ $(\forall\ e \in (e_0-\delta_0, e_0))$, *then* $\overline{\lim}_{e\to e_0^-} f(e) \subseteq \overline{\lim}_{e\to e_0^-} g(e)$.

(2) $\overline{\lim}_{e\to e_0^-}(f(e)\cup g(e)) = \overline{\lim}_{e\to e_0^-} f(e) \cup \overline{\lim}_{e\to e_0^-} g(e)$.

(3) $\overline{\lim}_{e\to e_0^-}(U - f(e)) = U - \underline{\lim}_{e\to e_0^-} f(e)$.

(4) *If* $\overline{\lim}_{e\to e_0^-} f(e) = \triangle \subset B$, *then* $\exists\ \delta > 0, \forall\ e \in (e_0-\delta, e_0)$, $f(e) \subset B$.

(5) 1) $\overline{\lim}_{e\to e_0^-}(f(e)\times g(e)) \subseteq \overline{\lim}_{e\to e_0^-} f(e) \times \overline{\lim}_{e\to e_0^-} g(e)$.

2) $\overline{\lim}_{e\to e_0^-} f(e) \times \overline{\lim}_{e\to e_0^-} g(e) = \bigcap_{e\in(e_0-1,e_0)\cap I}\ \bigcup_{\beta,\gamma\in[e,e_0)} (f(\beta)\times g(\gamma))$.

Proof. The proof is similar to Proposition 4.16. □

Proposition 4.19. *For the under-left limit, the following properties hold:*

(1) *If* $f(e) \subseteq g(e)$ $(\forall\ e \in (e_0-\delta_0, e_0))$, *then* $\underline{\lim}_{e\to e_0^-} f(e) \subseteq \underline{\lim}_{e\to e_0^-} g(e)$.

(2) $\underline{\lim}_{e\to e_0^-}(f(e)\cap g(e)) = \underline{\lim}_{e\to e_0^-} f(e) \cap \underline{\lim}_{e\to e_0^-} g(e)$.

(3) $\underline{\lim}_{e\to e_0^-}(U - f(e)) = U - \overline{\lim}_{e\to e_0^-} f(e)$.

(4) *If* $\underline{\lim}_{e\to e_0^-} f(e) = \triangle \supset A$, *then* $\exists\ \delta > 0, \forall\ e \in (e_0-\delta, e_0)$, $f(e) \supset A$.

(5) $\underline{\lim}_{e\to e_0^-}(f(e)\times g(e)) = \underline{\lim}_{e\to e_0^-} f(e) \times \underline{\lim}_{e\to e_0^-} g(e)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

Proof. The proof is similar to Proposition 4.17. □

Corollary 4.20. *Let f_I be an it-soft set over U and $A \in 2^U$. For $e_0 \in I$,*

(1) *If $f(e) \subseteq A$ or $f(e) \subset A$ $(\forall\ e \in (e_0, e_0 + \delta_0))$, then*

$$\overline{\lim_{e \to e_0^+}} f(e) \subseteq A, \quad \underline{\lim_{e \to e_0^+}} f(e) \subseteq A.$$

(2) *If $f(e) \subseteq A$ or $f(e) \subset A$ $(\forall\ e \in (e_0 - \delta_0, e_0))$, then*

$$\overline{\lim_{e \to e_0^-}} f(e) \subseteq A, \quad \underline{\lim_{e \to e_0^-}} f(e) \subseteq A.$$

Proof. This holds by Propositions 4.16, 4.17, 4.18, 4.19. □

Corollary 4.21. *Let f_I be an it-soft set over U and $A \in 2^U$. For $e_0 \in I$,*

(1) *If $f(e) \supseteq A$ or $f(e) \supset A$ $(\forall\ e \in (e_0, e_0 + \delta_0))$, then*

$$\overline{\lim_{e \to e_0^+}} f(e) \supseteq A, \quad \underline{\lim_{e \to e_0^+}} f(e) \supseteq A.$$

(2) *If $f(e) \supseteq A$ or $f(e) \supset A$ $(\forall\ e \in (e_0 - \delta_0, e_0))$, then*

$$\overline{\lim_{e \to e_0^-}} f(e) \supseteq A, \quad \underline{\lim_{e \to e_0^-}} f(e) \supseteq A.$$

Proof. This holds by Propositions 4.16, 4.17, 4.18, 4.19. □

Theorem 4.22. *For the over limit, the following properties hold:*

(1) *If $f(e) \subseteq g(e)$ $(\forall\ e \in U^0(e_0, \delta_0))$, then $\overline{\lim_{e \to e_0}} f(e) \subseteq \overline{\lim_{e \to e_0}} g(e)$.*

(2) $\overline{\lim_{e \to e_0}} (f(e) \cup g(e)) = \overline{\lim_{e \to e_0}} f(e) \cup \overline{\lim_{e \to e_0}} g(e)$.

(3) $\overline{\lim_{e \to e_0}} (U - f(e)) = U - \underline{\lim_{e \to e_0}} f(e)$.

(4) *If $\overline{\lim_{e \to e_0}} f(e) = \triangle \subset B$, then $\exists\ \delta > 0, \forall\ e \in U^0(e_0, \delta)$, $f(e) \subset B$.*

(5) $\overline{\lim_{e \to e_0}} (f(e) \times g(e)) \subseteq \overline{\lim_{e \to e_0}} f(e) \times \overline{\lim_{e \to e_0}} g(e)$.

Proof. This holds by Propositions 4.16 and 4.18. □

Theorem 4.23. *For the under limit, the following properties hold:*

(1) *If $f(e) \subseteq g(e)$ $(\forall\ e \in U^0(e_0, \delta_0))$, then $\underline{\lim_{e \to e_0}} f(e) \subseteq \underline{\lim_{e \to e_0}} g(e)$.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

(2) $\underline{\lim}_{e \to e_0} (f(e) \cap g(e)) = \underline{\lim}_{e \to e_0} f(e) \cap \underline{\lim}_{e \to e_0} g(e)$.

(3) $\underline{\lim}_{e \to e_0} (U - f(e)) = U - \overline{\lim}_{e \to e_0} f(e)$.

(4) *If* $\underline{\lim}_{e \to e_0} f(e) = \triangle \supset A$, *then* $\exists\ \delta > 0, \forall\ e \in U^0(e_0, \delta),\ f(e) \supset A$.

(5) $\underline{\lim}_{e \to e_0} (f(e) \times g(e)) = \underline{\lim}_{e \to e_0} f(e) \times \underline{\lim}_{e \to e_0} g(e)$.

Proof. This holds by Propositions 4.17 and 4.19. □

Lemma 4.24. *Let f_I be an it-soft set over U . For $e_0 \in I$, denote*
$W = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U(e_0, \delta) \neq \emptyset\}$,
$S = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^+(e_0, \delta) \neq \emptyset\}$,
$T = \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^-(e_0, \delta) \neq \emptyset\}$.

Then

$$W = S \cup T.$$

Proof. Suppose $W \nsubseteq S \cup T$. Then $W - S \cup T \neq \emptyset$.

Pick $x \in W - S \cup T$. Then $x \notin S,\ x \notin T$. So $\exists\ \delta_1, \delta_2 > 0$,

$$[x]_{f_I} \cap U^+(e_0, \delta_1) = \emptyset,\ [x]_{f_I} \cap U^-(e_0, \delta_2) = \emptyset.$$

Put $\delta^* = \min\{\delta_1, \delta_2\}$. Then $\delta^* > 0$, $[x]_{f_I} \cap U^+(e_0, \delta^*) = \emptyset$, $[x]_{f_I} \cap U^-(e_0, \delta^*) = \emptyset$. It follows $[x]_{f_I} \cap U(e_0, \delta^*) = \emptyset$. Then $x \notin W$. This is a contradiction.

Thus $W \subseteq S \cup T$.

On the other hand, suppose $S \cup T \nsubseteq W$, we have $S \cup T - W \neq \emptyset$.

Pick $x \in S \cup T - W$. Then $x \notin W$. So $\exists\ \delta^* > 0$, $[x]_{f_I} \cap U(e_0, \delta^*) = \emptyset$. This implies $[x]_{f_I} \cap U^+(e_0, \delta^*) = \emptyset$, $[x]_{f_I} \cap U^-(e_0, \delta^*) = \emptyset$. Then $x \notin S$, $x \notin T$. So $x \notin S \cup T$. This is a contradiction.

Thus $S \cup T \subseteq W$.

Hence $W = S \cup T \nsubseteq W$.

□

Theorem 4.25. *Let f_I be an it-soft set over U. Then for $e_0 \in I$,*

(1) $\{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U(e_0, \delta)\ is\ infinite\}$
$= \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U(e_0, \delta) \neq \emptyset\}$
$= \overline{\lim}_{e \to e_0^+} f(e) \cup \overline{\lim}_{e \to e_0^-} f(e)$.

(2) $\{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U(e_0, \delta)\ is\ finite\}$
$= \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0, \delta) = \emptyset\}$
$= \underline{\lim}_{e \to e_0^+} f(e) \cap \underline{\lim}_{e \to e_0^-} f(e)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Proof. (1) Similar to the proof of Theorem 4.6(1), we have

$$\{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^+(e_0,\delta) \neq \emptyset\}$$
$$= \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U^+(e_0,\delta)\ is\ infinite\}.$$

By Lemma 4.24,

$$\{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U(e_0,\delta) \neq \emptyset\}$$
$$= \overline{\lim_{e\to e_0^+}} f(e) \cup \overline{\lim_{e\to e_0^-}} f(e).$$

(2) Similar to the proof of Theorem 4.6(2), we have

$$\{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0,\delta) = \emptyset\}$$
$$= \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0,\delta)\ is\ finite\}.$$

By Proposition 4.16(3), $\underline{\lim_{e\to e_0^+}} f(e) = U - \overline{\lim_{e\to e_0^+}} (U - f(e))$.

By Proposition 4.18(3), $\underline{\lim_{e\to e_0^-}} f(e) = U - \overline{\lim_{e\to e_0^-}} (U - f(e))$.

By (1),

$$\underline{\lim_{e\to e_0^+}} f(e) \cap \underline{\lim_{e\to e_0^-}} f(e)$$
$$= [U - \overline{\lim_{e\to e_0^+}} (U - f(e))] \cap [U - \overline{\lim_{e\to e_0^-}} (U - f(e))]$$
$$= U - [\overline{\lim_{e\to e_0^+}} (U - f(e)) \cup \overline{\lim_{e\to e_0^-}} (U - f(e))]$$
$$= U - \{x \in U : \forall\ \delta > 0,\ (x)_{f_I} \cap U(e_0,\delta) \neq \emptyset\}$$
$$= \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U(e_0,\delta) = \emptyset\}.$$

□

Theorem 4.26. *Let f_I be an it-soft set over U. Then for $e_0 \in I$,*

(1) $\{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U(e_0,\delta)\ is\ infinite\}$
$= \{x \in U : \forall\ \delta > 0,\ [x]_{f_I} \cap U(e_0,\delta) \neq \emptyset\}$
$= \overline{\lim_{e\to e_0}} f(e)$.

(2) $\{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0,\delta)\ is\ finite\}$
$= \{x \in U : \exists\ \delta > 0,\ (x)_{f_I} \cap U^+(e_0,\delta) = \emptyset\}$
$= \underline{\lim_{e\to e_0}} f(e)$.

Proof. This holds by Theorem 4.25. □

Theorem 4.27. *For the right limit, the following properties hold:*

(1) *If $f(e) \subseteq g(e)$ $(\forall\ e \in (e_0, e_0+\delta_0))$, then $\lim\limits_{e\to e_0^+} f(e) \subseteq \lim\limits_{e\to e_0^+} g(e)$.*

(2) *If $\lim\limits_{e\to e_0^+} f(e) = \triangle$, $A \subset \triangle \subset B$, then $\exists\ \delta > 0, \forall\ e \in (e_0, e_0+\delta)$, $A \subset f(e) \subset B$.*

 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

(3) $\lim_{e\to e_0^+}(f(e)\times g(e))\subseteq \lim_{e\to e_0^+} f(e)\times \lim_{e\to e_0^+} g(e)$.

Proof. This holds by Propositions 4.16 and 4.17. □

Theorem 4.28. *For the left limit, the following properties hold:*

(1) *If* $f(e)\subseteq g(e)$ $(\forall\ e\in(e_0-\delta_0,e_0))$, *then* $\lim_{e\to e_0^-} f(e)\subseteq \lim_{e\to e_0^-} g(e)$.

(2) *If* $\lim_{e\to e_0^-} f(e) = \triangle$, $A\subset\triangle\subset B$, *then* $\exists\ \delta>0, \forall\ e\in(e_0-\delta,e_0)$, $A\subset f(e)\subset B$.

(3) $\lim_{e\to e_0^-}(f(e)\times g(e))\subseteq \lim_{e\to e_0^-} f(e)\times \lim_{e\to e_0^-} g(e)$.

Proof. This holds by Propositions 4.18 and 4.19. □

Theorem 4.29. *For the limit, the following properties hold:*

(1) *If* $f(e)\subseteq g(e)$ $(\forall\ e\in U^0(e_0,\delta_0))$, *then* $\lim_{e\to e_0} f(e)\subseteq \lim_{e\to e_0} g(e)$.

(2) *If* $\lim_{e\to e_0^-} f(e) = \triangle$, $A\subset\triangle\subset B$, *then* $\exists\ \delta>0, \forall\ e\in U^0(e_0,\delta_0)$, $A\subset f(e)\subset B$.

(3) $\lim_{e\to e_0}(f(e)\times g(e))\subseteq \lim_{e\to e_0} f(e)\times \lim_{e\to e_0} g(e)$.

Proof. This holds by Theorems 4.27 and 4.28. □

5. Continuity of *it*-soft sets

5.1. Point-wise continuity of *it*-soft sets

Definition 5.1. *Let* f_I *be an it-soft set over* U. *Then for* $e_0\in I$,

(1) f_I *is called over-right continuous at* e_0, *if* $\overline{\lim}_{e\to e_0^+} f(e)=f(e_0)$.

(2) f_I *is called under-right continuous at* e_0, *if* $\underline{\lim}_{e\to e_0^+} f(e)=f(e_0)$.

(3) f_I *is called over-left continuous at* e_0, *if* $\overline{\lim}_{e\to e_0^-} f(e)=f(e_0)$.

(4) f_I *is called under-left continuous at* e_0, *if* $\underline{\lim}_{e\to e_0^-} f(e)=f(e_0)$.

Definition 5.2. *Let* f_I *be an it-soft set over* U. *Then for* $e_0\in I$,

(1) f_I *is called over-continuous at* e_0, *if* f_I *is both over-left and over-right continuous at* e_0.

(2) f_I *is called under-continuous at* e_0, *if* f_I *is both under-left and under-right continuous at* e_0.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

(3) *f_I is called continuous at e_0, if f_I is both over-continuous and under-continuous at e_0.*

Definition 5.3. *Let f_I be an it-soft set over U. Then for $e_0 \in I$,*

(1) *f_I is called right-continuous at e_0, if f_I is both over-right and under-right continuous at e_0.*

(2) *f_I is called left-continuous at e_0, if f_I is both over-left and under-left continuous at e_0.*

(3) *f_I is called continuous at e_0, if f_I is both left-continuous and right-continuous at e_0.*

Remark 5.4. *The point-wise continuity in Definition 5.2(3) and the point-wise continuity in Definition 5.3(3) is consistent.*

Denote

$$C^{or}(e_0) = \{f_I : f_I \text{ is over-right continuous at } e_0\},$$

$$C^{ur}(e_0) = \{f_I : f_I \text{ is under-right continuous at } e_0\},$$

$$C^{ol}(e_0) = \{f_I : f_I \text{ is over-left continuous at } e_0\},$$

$$C^{ul}(e_0) = \{f_I : f_I \text{ is under-left continuous at } e_0\};$$

$$C^{o}(e_0) = \{f_I : f_I \text{ is over-continuous at } e_0\},\ C^{u}(e_0) = \{f_I : f_I \text{ is under-continuous at } e_0\};$$

$$C^{l}(e_0) = \{f_I : f_I \text{ is left-continuous at } e_0\},\ C^{r}(e_0) = \{f_I : f_I \text{ is right-continuous at } e_0\};$$

$$C(e_0 = \{f_I : f_I \text{ is continuous at } e_0\}.$$

Proposition 5.5. (1) $C^o(e_0) = C^{ol}(e_0) \cap C^{or}(e_0)$.

(2) $C^u(e_0) = C^{ul}(e_0) \cap C^{ur}(e_0)$.

(3) $C^l(e_0) = C^{ol}(e_0) \cap C^{ul}(e_0)$.

(4) $C^r(e_0) = C^{or}(e_0) \cap C^{ur}(e_0)$.

(5) $C(e_0) = C^o(e_0) \cap C^u(e_0) = C^l(e_0) \cap C^r(e_0)$.

Proof. This is obvious. □

Proposition 5.6. *Let f_I and g_I be two it-soft sets over U. Then for $e_0 \in I$,*

(1) *If $f_I, g_I \in C^{or}(e_0)$, then $f_I \tilde{\cup} g_I \in C^{or}(e_0)$.*

(2) *If $f_I \in C^{or}(e_0)$, then $f_I^c \in C^{ur}(e_0)$.*

Proof. This holds by Proposition 4.16. □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Proposition 5.7. *Let f_I and g_I be two it-soft sets over U. Then for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^{ur}(e_0)$, then $f_I \widetilde{\cap} g_I \in C^{ur}(e_0)$.*
(2) *If $f_I \in C^{ur}(e_0)$, then $f_I^c \in C^{or}(e_0)$.*
(3) *If $f_I, g_I \in C^{ur}(e_0)$, then $f_I \widetilde{\times} g_I \in C^{ur}(e_0)$.*

Proof. This holds by Proposition 4.17. □

Proposition 5.8. *Let f_I and g_I be two it-soft sets over U. Then for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^{ol}(e_0)$, then $f_I \widetilde{\cup} g_I \in C^{ol}(e_0)$.*
(2) *If $f_I \in C^{ol}(e_0)$, then $f_I^c \in C^{ul}(e_0)$.*

Proof. This holds by Proposition 4.18. □

Proposition 5.9. *Let f_I and g_I be two it-soft sets over U. Then for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^{ul}(e_0)$, then $f_I \widetilde{\cap} g_I \in C^{ul}(e_0)$.*
(2) *If $f_I \in C^{ul}(e_0)$, then $f_I^c \in C^{ol}(e_0)$.*
(3) *If $f_I, g_I \in C^{ul}(e_0)$, then $f_I \widetilde{\times} g_I \in C^{ul}(e_0)$.*

Proof. This holds by Proposition 4.19. □

Theorem 5.10. *Let f_I and g_I be two it-soft sets over U. Then for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^{o}(e_0)$, then $f_I \widetilde{\cup} g_I \in C^{o}(e_0)$.*
(2) *If $f_I \in C^{o}(e_0)$, then $f_I^c \in C^{u}(e_0)$.*

Proof. This holds by Propositions 5.6 and 5.8. □

Theorem 5.11. *Let f_I and g_I be two it-soft sets over U. Then for $e_0 \in I$,*
(1) *If $f_I, g_I \in C^{u}(e_0)$, then $f_I \widetilde{\cap} g_I \in C^{u}(e_0)$.*
(2) *If $f_I \in C^{u}(e_0)$, then $f_I^c \in C^{o}(e_0)$.*
(3) *If $f_I, g_I \in C^{u}(e_0)$, then $f_I \widetilde{\times} g_I \in C^{u}(e_0)$.*

Proof. This holds by Propositions 5.7 and 5.9. □

5.2. Continuous *it*-soft sets

Definition 5.12. *Let f_I be an it-soft set over U.*
(1) *f_I is called over-continuous, if $\forall\ e_0 \in I$, f_I is over-continuous at e_0.*
(2) *f_I is called under-continuous, if $\forall\ e_0 \in I$, f_I under-continuous at e_0.*
(3) *f_I is called left-continuous, if $\forall\ e_0 \in I$, f_I is left-continuous at e_0.*
(4) *f_I is called right-continuous, if $\forall\ e_0 \in I$, f_I right-continuous at e_0.*
(5) *f_I is called continuous, if $\forall\ e_0 \in I$, f_I continuous at e_0.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Denote

$$C^{or}(e_0) = \{f_I : f_I \text{ is over-right continuous}\},$$

$$C^{ur}(e_0) = \{f_I : f_I \text{ is under-right continuous}\},$$

$$C^{ol}(e_0) = \{f_I : f_I \text{ is over-left continuous}\},$$

$$C^{ul}(e_0) = \{f_I : f_I \text{ is under-left continuous}\};$$

$$C^{o}(I) = \{f_I : f_I \text{ is over-continuous}\},\ C^{u}(I) = \{f_I : f_I \text{ is under-continuous}\};$$

$$C^{l}(I) = \{f_I : f_I \text{ is left-continuous}\},\ C^{r}(I) = \{f_I : f_I \text{ is right-continuous }\};$$

$$C(I) = \{f_I : f_I \text{ is continuous}\}.$$

Proposition 5.13. (1) $C^o(I) = C^{ol}(I) \cap C^{or}(I)$.
(2) $C^u(I) = C^{ul}(I) \cap C^{ur}(I)$.
(3) $C^l(I) = C^{ol}(I) \cap C^{ul}(I)$.
(4) $C^r(I) = C^{or}(I) \cap C^{ur}(I)$.
(5) $C(I) = C^o(I) \cap C^u(I) = C^l(I) \cap C^r(I)$.

Proof. This is obvious. □

Theorem 5.14. *Let f_I and g_J be two it-soft sets over U.*
(1) *If $f_I \in C^o(I)$, $g_J \in C^o(J)$, then $f_I \,\widetilde{\cup}\, g_I \in C^o(I \cup J)$.*
(2) *If $f_I \in C^o(I)$, then $f_I^c \in C^u(I)$.*

Proof. This holds by Theorem 5.10. □

Theorem 5.15. *Let f_I and g_J be two it-soft sets over U.*
(1) *If $f_I \in C^u(I)$, $g_J \in C^u(J)$ then $f_I \,\widetilde{\cap}\, g_J \in C^u(I \cap J)$.*
(2) *If $f_I \in C^u(I)$, then $f_I^c \in C^o(I)$.*

Proof. This holds by Theorem 5.11. □

Theorem 5.16. *Let $(f, [a, b])$ be an it-soft set over U.*

(1) If $(f, [a, b])$ is strong keeping union or increasing, then $(f, [a, b])$ has the maximum value.

(2) If $(f, [a, b])$ is strong keeping intersection or decreasing, then $(f, [a, b])$ has the minimum value.

Corollary 5.17. *If $(f, [a, b])$ is a perfect it-soft set over U, then $(f, [a, b])$ has the maximum and minimum value.*

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

Proof. This is obvious. □

Lemma 5.18. *Let* $f_I \in C^o(e_0)$*. If* $\lim_{n\to\infty} e_n = e_0$*, then* $\overline{\lim_{n\to\infty}} f(e_n) \subseteq f(e_0)$*.*

Proof. Since $\overline{\lim_{n\to\infty}} f(e_n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} f(e_k)$,
we only need to prove that

$$if\ \forall\ n \in N,\ \exists\ k \geq n,\ x \in f(e_k),\ \text{then}\ x \in f(e_0).$$

$\forall\ \delta,\ \exists\ n \in N_1,\ \frac{1}{n_1} < \delta$. It follows $U(e_0, \frac{1}{n_1}) \subset U(e_0, \delta)$.
Since $\lim_{n\to\infty} e_n = e_0$, $\exists\ n \in N_2$, when $n > n_2$ we have $e_n \in U(e_0, \frac{1}{n_1})$.
Put $n_3 = n_1 + n_2$. Then for n_3, $\exists\ k \geq n_3$, $x \in f(e_k)$. So $e_k \in\ [x]_{f_I}$.
$k \geq n_3 > n_2$ implies

$$e_k \in U(e_0, \frac{1}{n_1}) \subset U(e_0, \delta).$$

Then $e_k \in\ [x]_{f_I} \cap U(e_0, \delta)$. So $\forall\ \delta,\ \ [x]_{f_I} \cap U(e_0, \delta) \neq \emptyset$.
By Theorem 4.25, $x \in \underline{\lim_{e\to e_0}} f(e)$.

Since $f \in C^o(e_0)$, we have $f(e_0) = \overline{\lim_{e\to e_0}} f(e)$.

Hence $x \in f(e_0)$. □

Theorem 5.19. *Let* $(f, [a, b]) \in C([a, b])$*.*
(1) *Suppose* $f(a) \subset f(b)$*, then* $\forall\ \mu : f(a) \subseteq \mu \subseteq f(b)$*,* $\exists\ e_0 \in [a, b]$*,* $f(e_0) = \mu$*. Moreover, if* $f(a) \subset \mu \subset f(b)$*, then* $\exists\ e_0 \in (a, b)$*,* $f(e_0) = \mu$*.*
(2) *Suppose* $f(b) \subset f(a)$*, then* $\forall\ \mu : f(b) \subseteq \mu \subseteq f(a)$*,* $\exists\ e_0 \in [a, b]$*,* $f(e_0) = \mu$*. Moreover, if* $f(b) \subset \mu \subset f(a)$*, then* $\exists\ e_0 \in (a, b)$*,* $f(e_0) = \mu$*.*

Proof. (1) It suffices to show that

$$if\ \ f(a) \subset \mu \subset\ f(b),\ then\ \exists\ e_0 \in (a, b),\ f(e_0) = \mu.$$

Denote $E = \{e \in [a, b] : f(e) \supset \mu\}$. Put $e_0 = \inf E$. Then

$$\exists\ \{e_n : n \in N\} \subseteq E - \{e_0\},\ \ \lim_{n\to\infty} e_n = e_0.$$

Since $\forall\ n \in N,\ f(e_n) \supset \mu$, we have $\overline{\lim_{n\to\infty}} f(e_n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} f(e_k) \supseteq \mu$. Since $f \in C^o(e_0)$, by Lemma 5.18,

$$f(e_0) \supseteq \overline{\lim_{n\to\infty}} f(e_n) \supseteq \mu.$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Note that $f(a) \subset \mu$. Then $e_0 \neq a$.

We assert $e_0 \neq b$. Suppose $e_0 = b$. Since

$$\mu \subset f(b) = \lim_{e \to b^-} f(e) = \underline{\lim}_{e \to b^-} f(e),$$

by Proposition 4.19(4), then

$$\exists\, \delta,\ \forall\, e \in (b-\delta, b), f(e) \supset \mu.$$

Put $e_1 \in (b-\delta, b)$. Then $f(e_1) \supset \mu$. We have $e_1 \in E$. This implies $e_1 \geq e_0$. But $e_1 < b = e_0$. This is a contradiction.

Thus $e_0 \in (a, b)$.

We claim $f(e_0) \not\supset \mu$. Suppose $f(e_0) \supset \mu$. Since $f \in C^u(e_0)$, we have

$$\mu \subset f(e_0) = \lim_{e \to e_0} f(e) = \underline{\lim}_{e \to e_0} f(e).$$

By Theorem 4.23(4),

$$\exists\, \delta,\ \forall\, e \in U^0(e_0, \delta),\ f(e) \supset \mu.$$

Put $e_1 \in (e_0 - \delta, e_0)$. Then $f(e_1) \supset \mu$. We have $e_1 \in E$. This implies $e_1 \geq e_0$. This is a contradiction.

Note that $f(e_0) \supseteq \mu$. Thus $f(e_0) = \mu$.

(2) The proof is similar to (1).

□

6. An application for rough sets

Definition 6.1. *Let (U, R, P) be a probabilistic approximate space. For $e \in [0, 1]$, $X \in 2^U$, denote*

$$f_X(e) = \underline{PI}_e(X), \quad g_X(e) = \overline{PI}_e(X).$$

Then $(f_X, [0, 1])$ and $(g_X, [0, 1])$ are two it-soft sets over U, which are called the it-soft sets induced by the lower and upper approximations of X, respectively.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

Theorem 6.2. *Let (U, R, P) be a probabilistic approximate space. Then for $e_0 \in (0,1)$, $X \in 2^U$,*

(1) 1) $\overline{\lim\limits_{e\to e_0^+}} f_X(e) = \bigcap\limits_{e\in(e_0,1]} \bigcup\limits_{\beta\in(e_0,e]} f_X(\beta)$;

2) $\overline{\lim\limits_{e\to e_0^-}} f_X(e) = \bigcap\limits_{e\in[0,e_0)} f_X(e) = f_X(e_0)$;

3) $\underline{\lim\limits_{e\to e_0^+}} f_X(e) = \bigcup\limits_{e\in(e_0,1]} f_X(e) = g_X(e_0)$;

4) $\underline{\lim\limits_{e\to e_0^-}} f_X(e) = \bigcup\limits_{e\in[0,e_0)} \bigcap\limits_{\beta\in[e,e_0)} f_X(\beta)$.

(2) 1) $\overline{\lim\limits_{e\to e_0^+}} g_X(e) = \bigcap\limits_{e\in(e_0,1]} \bigcup\limits_{\beta\in(e_0,e]} g_X(\beta)$;

2) $\overline{\lim\limits_{e\to e_0^-}} g_X(e) = \bigcap\limits_{e\in[0,e_0)} g_X(e) = f_X(e_0)$;

3) $\underline{\lim\limits_{e\to e_0^+}} g_X(e) = \bigcup\limits_{e\in(e_0,1]} g_X(e) = g_X(e_0)$;

4) $\underline{\lim\limits_{e\to e_0^-}} g_X(e) = \bigcup\limits_{e\in[0,e_0)} \bigcap\limits_{\beta\in[e,e_0)} g_X(\beta)$.

(3) 1) $f_{U-X}(e) = U - g_X(1-e)$,

2) $g_{U-X}(e) = U - f_X(1-e)$.

Proof. This holds by Theorems 2.6, 2.7 and 4.10. □

Corollary 6.3. *Let (U, R, P) be a probabilistic approximate space. Then for $X \in 2^U$,*

$$(f_X, [0,1]) \in C^{ol}((0,1)), \ (g_X, [0,1]) \in C^{ur}((0,1)).$$

Proof. This holds by Theorems 6.2. □

Example 6.4. *Let $U = \{x_i : 1 \le i \le 20\}$, $P(X) = \frac{|X|}{|U|}$ $(X \in 2^U)$, $U/R = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ where*

$X_1 = \{x_1, x_2, x_3, x_4, x_5\}$, $X_2 = \{x_6, x_7, x_8\}$, $X_3 = \{x_9, x_{10}, x_{11}, x_{12}\}$,

$X_4 = \{x_{13}, x_{14}\}$, $X_5 = \{x_{15}, x_{16}, x_{17}, x_{18}\}$, $X_6 = \{x_{19}, x_{20}\}$.

Put

$$X^* = \{x_6, x_7, x_8, x_{13}, x_{17}\}.$$

By Example 4.9 in [24] or Example 8.1 in [25],

$$f_{X^*}(0.5) = X_2 \cup X_4, \quad g_{X^*}(0.5) = X_2.$$

 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

By Theorem 2.7,

$$\underline{\lim}_{e\to 0.5^+} f_{X^*}(e) = g_{X^*}(0.5) \neq f_{X^*}(0.5).$$

By Theorem 2.7,

$$\overline{\lim}_{e\to 0.5^-} g_{X^*}(e) = f_{X^*}(e_0) \neq g_{X^*}(0.5).$$

Thus

$$(f_{X^*}, [0,1]) \notin C^{ur}(0.5),\ (g_{X^*}, [0,1]) \notin C^{ol}(0.5).$$

This example illustrates that

$$(f_{X^*}, [0,1]) \notin C^{ur}((0,1)),\ (g_{X^*}, [0,1]) \notin C^{ol}((0,1)).$$

Example 6.5. *Let* $U = \{x_i : 1 \leq i \leq 10\}$, $P(X) = \frac{|X|}{|U|}$ $(X \in 2^U)$, $U/R = \{X_1, X_2, X_3, X_4\}$ *where*

$X_1 = \{x_1, x_3\}$, $X_2 = \{x_2, x_4, x_5, x_7\}$, $X_3 = \{x_6, x_8\}$, $X_4 = \{x_9, x_{10}\}$.
(1) *Put* $X^* = \{x_1, x_5, x_6, x_8\}$. *Then*

$$f_{X^*}(e) = \begin{cases} X_1 \cup X_2 \cup X_3, & \textit{if} \ \ e \in (0, \frac{1}{4}], \\ X_1 \cup X_3, & \textit{if} \ \ e \in (\frac{1}{4}, \frac{1}{2}], \\ X_3, & \textit{if} \ \ e \in (\frac{1}{2}, 1]; \end{cases}$$

$$g_{X^*}(e) = \begin{cases} X_1 \cup X_2 \cup X_3, & \textit{if} \ \ e \in [0, \frac{1}{4}), \\ X_1 \cup X_3, & \textit{if} \ \ e \in [\frac{1}{4}, \frac{1}{2}), \\ X_3, & \textit{if} \ \ e \in [\frac{1}{2}, 1). \end{cases}$$

So $\overline{\lim}_{e\to 0.5^+} f_{X^*}(e) = \bigcap_{e\in(0.5,1]} \bigcup_{\beta\in(0.5,e]} f_{X^*}(\beta) = X_3 \neq X_1 \cup X_3 = f_{X^*}(0.5)$,
$\underline{\lim}_{e\to 0.5^-} g_X(e) = \bigcup_{e\in[0,0.5)} \bigcap_{\beta\in[e,0.5)} g_X(\beta) = X_1 \cup X_3 \neq X_3 = g_{X^*}(0.5)$.
Thus

$$(f_{X^*}, [0,1]) \notin C^{or}(0.5),\ (g_{X^*}, [0,1]) \notin C^{ul}(0.5).$$

(2) *Put* $Y^* = \{x_2, x_9, x_{10}\}$. *Then*

$$f_{Y^*}(e) = \begin{cases} X_2 \cup X_4, & \textit{if} \ \ e \in (0, \frac{1}{4}], \\ X_4, & \textit{if} \ \ e \in (\frac{1}{4}, 1]. \end{cases}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at Symmetry 2018, 10, 406; doi:10.3390/sym10090406

So $\underline{\lim}_{e\to 0.5^-} f_{Y^*}(e) = \bigcup_{e\in[0,0.5)} \bigcap_{\beta\in[e,0.5)} f_{Y^*}(\beta) = X_2 \cup X_4 \neq X_4 = f_{Y^*}(0.5)$.
Thus

$$(f_{Y^*}, [0,1]) \notin C^{ul}(0.5).$$

(3) *Put*

$$Z^* = U - Y^*.$$

By Proposition 4.16(3) and Theorem 2.7,

$$\begin{aligned} \overline{\lim}_{e\to 0.5^+} g_{Z^*}(e) &= \overline{\lim}_{e\to 0.5^+} (U - f_{Y^*}(1-e)) \\ &= U - \underline{\lim}_{e\to 0.5^+} f_{Y^*}(1-e) \\ &= U - \underline{\lim}_{1-e\to 0.5^-} f_{Y^*}(1-e). \end{aligned}$$

Note that $\underline{\lim}_{e\to 0.5^-} f_{Y^*}(e) \neq f_{Y^*}(0.5)$. *Then by Theorem 2.7,*

$$\overline{\lim}_{e\to 0.5^+} g_{Z^*}(e) \neq U - f_{Y^*}(0.5) = g_{Z^*}(0.5).$$

Thus

$$(g_{Z^*}, [0,1]) \notin C^{or}(0.5).$$

This example illustrates that

$$(f_{X^*}, [0,1]) \notin C^{or}((0,1)),\ (g_{X^*}, [0,1]) \notin C^{ul}((0,1));$$

$$(f_{Y^*}, [0,1]) \notin C^{ul}((0,1));\ (g_{Z^*}, [0,1]) \notin C^{or}((0,1)).$$

7. Conclusions

In this paper, limits of *it*-soft sets have been proposed. Point-wise continuity of *it*-soft sets and continuous *it*-soft sets have been investigated. An application for rough sets has been given. These results will be helpful for the study of soft sets. In the future, we will further study applications of these limits in information science.

 doi:10.20944/preprints201808.0010.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

References

[1] H.Aktas, N.Cağman, Soft sets and soft groups, Information Sciences, 177(2007), 2726-2735.

[2] J.P.Aubin, H.Frankowska, Set-valued analysis, Birkhäuser, Berlin, 1990.

[3] M.Dai, W.Chen, G.Zhang, Real analysis and functional analysis, Chinese Scientific Publishers, Beijing, 2007.

[4] F.Feng, C.Li, B.Davvaz, M.Irfan Ali, Soft sets combined with fuzzy sets and rough sets: a tentative approach, Soft Computing, 14(2010), 899-911.

[5] B.Flagg, R.Kopperman, Continuity spaces: Reconciling domains and metric spaces, Theoretical Computer Science, 177(1997), 111-138.

[6] F.Feng, X.Liu, V.Leoreanu-Fotea, Y.Jun, Soft sets and soft rough sets, Information Sciences, 181(2011), 1125-1137.

[7] Y.Jiang, Y.Tang, Q.Chen, J.Wang, S.Tang, Extending soft sets with description logics, Computers and Mathematics with Applications, 59(2010), 2087-2096.

[8] X.Ge, Z.Li, Y.Ge, Topological spaces and soft sets, Journal of Computational Analysis and Applications, 13(2011), 881-885.

[9] P.R.Halmos, Measure theory, Van Nostrand, New York, 1950.

[10] A.N.Kolmogorov, S.V.Fomin, Introductory real analysis, Dover Publications, New York, 1975.

[11] E.Klein, A.C.Thompson, Theory of correspondences, John Wiley and Sons, New York, 1984.

[12] Z.Li, T.Xie, The relationship among soft sets, soft rough sets and topologies, Soft Computing, 18(2014), 717-728.

[13] Z.Li, N.Xie, G.Wen, Soft coverings and their parameter reductions, Applied Soft Computing, 31(2015), 48-60

[14] P.K.Maji, R.Biswas, A.R.Roy, Soft set theory, Computers and Mathematics with Applications, 45(2003), 555-562.

Peer-reviewed version available at *Symmetry* **2018**, *10*, 406; doi:10.3390/sym10090406

[15] P.K.Maji, A.R.Roy, An application of soft sets in a decision making problem, Computers and Mathematics with Applications, 44(2002), 1077-1083.

[16] P.Majumdar, S.K.Samanta, Similarity measure of soft sets, New Mathematics and Natural Computation, 4(2008), 1-12.

[17] D.Molodtsov, Soft set theory-first result, Computers and Mathematics with Applications, 37 (1999), 19-31.

[18] Z.Pawlak, Rough sets, International Journal of Computer and Information Science, 11(1982), 341-356.

[19] Z.Pawlak, Rough sets: Theoretical aspects of reasoning about data, Kluwer Academic Publishers, Boston, 1991.

[20] Z.Pawlak, A.Skowron, Rudiments of rough sets, Information Sciences, 177(2007), 3-27.

[21] Z.Pawlak, A.Skowron, Rough sets: some extensions, Information Sciences, 177(2007), 28-40.

[22] L.A.Zadeh, Fuzzy sets, Information and Control, 8(1965), 338-353.

[23] X.Zhang, On interval soft sets with applications, International Journal of Computational Intelligence Systems, 7(1)(2014), 186-196.

[24] W.Zhang, G.Qiu, Uncertain decision making based on rough sets, Tsinghua University Publishers, Beijing, 2005.

[25] W.Zhang, W.Wu, J.Liang, D.Li, Rough set theory and methods, Chinese Scientific Publishers, Beijing, 2001.