Peer-reviewed version available at *Symmetry* **2018**, *10*, 407; doi:10.3390/sym10090407

Article

Vertex labeling and routing for Farey-type symmetrically-structured graphs

Wenchao Jiang [1], Yinhu Zhai [2], Zhigang Zhuang [1], Paul Martin [3], Zhiming Zhao [3] and Jia-Bao Liu [4,*]

[1] School of Computer, Guangdong University of Technology, Guangzhou, 510006, China; jiangwenchao@gdut.edu.cn; mutingtao2014@gmail.com;
[2] School of Information Engineering, Guangdong University of Technology, Guangzhou, 510006, China; zhaiyh@gdut.edu.cn
[3] System and Network Engineering research group, Informatics Institute, University of Amsterdam, Science Park 904, 1098XH, Amsterdam, the Netherlands; p.w.martin@uva.nl; z.zhao@uva.nl
[4] School of Mathematics and Physics, Anhui Jianzhu University, Hefei 230601, PR China;
* Correspondence: liujiabaoad@163.com

Received: date; Accepted: date; Published: date

Abstract: The generalization of Farey graphs and extended Farey graphs are all originated from Farey graph and scale-free and small-world simultaneously. We propose a labeling of the vertices for it that allows determining all the shortest paths routing between any two vertices only based on their labels. The maximum number of shortest paths between any two vertices is huge as the product of two Fibonacci numbers, however, the label-based routing algorithm runs in linear time $O(n)$. The existence of an efficient routing protocol for Farey-type models should help the understanding of several physical dynamic processes on it.

Keywords: complex networks; deterministic models; Farey-type graphs; vertex labeling; shortest path routing;

---

1. Introduction

Deterministic models have unique advantages in improving our comprehension about some important physical mechanisms in complex networks. Especially, in comparison with the empirical and random graphs, the solutions of deterministic model can be obtained by rigorous derivation, and the computation can be ended only by a small amount of calculation. A lot of deterministic models are created imaginatively and studied carefully, which are inspired by simple recursive operation[1,2], or techniques of plane filling[3], or generating processes of fractal[4], or even the relationship between natural numbers[5]. Recently, on the basis of the classical Farey sequences, Zhang etc. introduced Farey graphs (FG) which are simultaneously minimally 3-colorable, uniquely Hamiltonian, maximally outer-planar and perfect[6,7]. The merger of three FG coincides with the network created by edge iterations[8], or evolving graphs with geographical attachment preference[9]; while the combination of six FG generates the graphs with multidimensional growth[10]. Moreover, two new kinds of Farey-type graphs, the generalization of Farey graphs (GFG) and the extended Farey graphs (EFG), are deduced by generalizing the construction mechanism of FG, and they all are scale-free and small-world[11–13].

Deterministic graphs also provide a new perspective and method on the classic research of physical processes. For example, some important dynamical processes on the basis of Apollonian models[14], a kind of deterministic graphs of scale-free, small-world, Euclidean, space-filling and matching graphs, are researched densely. The accurate analytical solutions are derived, including percolation[15], electrical conduction[14], Ising models[14], quantum transport[16], partially connected feedforward neural networks[17], traffic gridlock[18], Bose-Einstein condensation[19], Free-electron

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at Symmetry 2018, 10, 407; doi:10.3390/sym10090407

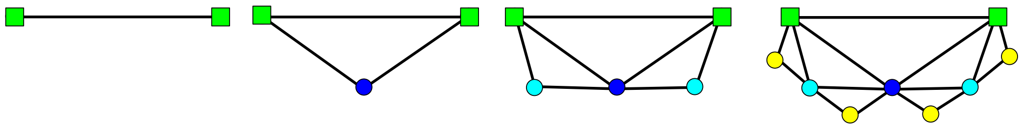


Figure 1. The Farey graphs $F(t)$ at steps $t = 0, 1, 2, 3$.

gas[20], and so on. However, the physical processes on FG are still lacking, for the recursive relationship in FG are more complex than Apollonian networks.

The routing protocol for the label-based all shortest paths may also bridge Farey-type graphs to the fields below. The all-pairs shortest paths problem is unquestionably one of the most well known problems in algorithm design, frequently studied in textbooks; yet, the complexity of the problem has remained open to this day. For arbitrary dense (directed and undirected) real-weighted graphs, the classical algorithms, such as Dijkstra, Bellman-Ford, A* and Floyd-Warshall, run in sub-cubic time $O(|V|^{3-\delta})$, where $\delta > 0$ [21–25].The K shortest path routing (KSPR) algorithm is an extension algorithm of the shortest path routing algorithm in a given network[26], in which K is the number of shortest paths to find. KSPR not only finds the shortest path, but also $K-1$ other paths in order of increasing cost. KSPR in Farey-type graphs will partly shrink to finding out all the shortest paths. The Graph Steiner tree problem (GSTR) is superficially similar to the minimum spanning tree problem: given a set V of vertices, interconnect them by a network of shortest length, where the length is the sum of the lengths of all edges[27]. GSTR has applications in circuit layout or network design, and most versions of GSTR are NP-complete. Moreover, for the bottleneck of many network analysis algorithms is the extortionate computational complexity of calculating the shortest paths, scientists have only to study the approximate algorithm of it[28].

Graphs are composed of vertices and edges and are very often studied considering branch of discrete mathematics known as graph theory. One active subject in graph theory is graph labeling. This is not only due to its theoretical importance but also because of the wide range of applications in many fields, such as crystallography, coding theory, circuit design and communication design[29]. Finding shortest paths in graphs is a well-studied and important problem with also many applications. On the relationship between vertices labels and the shortest paths, several deterministic models have been pioneered by Zhang and Camellas [29–32]. Only by their labels, one of shortest paths is determined just by simple rules and few computations.

Here we provide a labeling method for Farey-type graphs, so that queries for all the shortest paths between any pair of vertices can be efficiently answered thanks to it. In spite of the huge number of shortest paths between two vertices, the routing algorithm runs in linear time $O(n)$.

2. Generation of Farey-type graphs

For GFG and EFG are generalized from FG, we present the definitions of the three in turn.

Definition 2.1 (Generation of FG) Farey graph $F(t) = (V(t), E(t))$, in which the iteration step $0 \leq t$, with vertex set $V(t)$ and edge set $E(t)$ is constructed as follows[6]:

- For $t = 0$, $F(0)$ has two initial vertices and an edge joining them.
- For $t \geq 1$, $F(t)$ is obtained from $F(t-1)$ by adding to every edge introduced at step $t-1$ a new vertex adjacent to the end vertices of this edge (see Fig. 1).

Remark The order and size of FG are $|V(t)| = 2^t + 1$ and $|E(t)| = 2^{t+1} + 1$, respectively. The number of vertices adding at step t is $n_t = 2^{t-1}$. The cumulative degree distribution of $F(t)$ follows an exponential distribution $P_{cum}(\delta) = 2^{-\frac{\delta}{2}}$, and the degree correlations $k_m(\delta)$ is approximately a linear function of δ, which suggests that FG is assortative [6, 7].

If two initial vertices in $F(t)$ are named as X and Y, then all the vertices in $F(t)$ are divided into three groups by their distances to X and Y:$V^x(t)$, $V^{xy}(t)$ and $V^y(t)$.The vertices in the set

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at Symmetry 2018, 10, 407; doi:10.3390/sym10090407

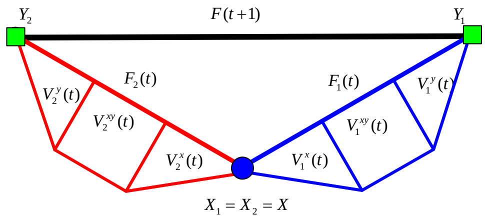


Figure 2. Schematic illustration of construction of $F(t+1)$.

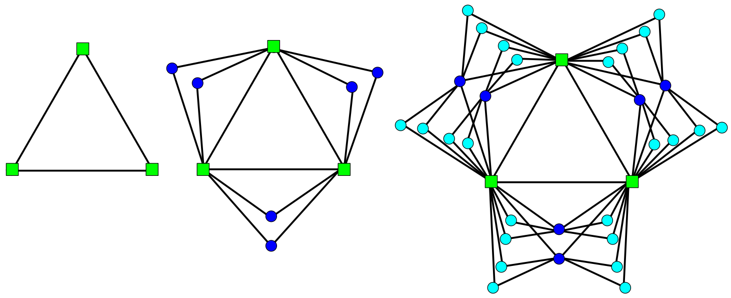


Figure 3. The generalization of Farey graphs $GF(t,k)$ at steps $t = 0, 1 and 2$ when $k = 2$.

$V^x(t)$(including X) have shorter distances from them to X than to Y, while the vertices in $V^y(t)$ (including Y) have shorter distances to Y than to X. If the distances are equal, vertices are all in $V^{xy}(t)$.That is to say, $V(t) = V^x(t) \cup V^{xy}(t) \cup V^y(t)$. Noticing that, the distances differences above are 0 or 1, for X and Y are neighbors. From Fig. 2, if two copies of $F(t)$ are named as $F_1(t)$ and $F_2(t)$, with initial vertices X_1, Y_1 and X_2, Y_2, then $F(t+1)$ is generated just by merging initial vertices X_1 and X_2 into X and linking Y_1 and Y_2 into Y directly, and the hub of $F(t+1)$ is X obviously.

Definition 2.2 (Generation of GFG) $GF(t,k)$ is deduced by the rules:

- For $t = 0$, $GF(0)$ is composed of three initial vertices which are linking each other.
- For $t \geq 1$, $GF(t)$ is constructed from $GF(t-1,k)$ by adding k new vertices adjacent to the two end vertices of every edge introduced at step $t-1$, then linking the k new vertices to the two end vertices (see Fig. 3).

Remark $GF(t,1)$ is exactly the graphs created by edge iterations[8], or evolving graphs with geographical attachment preference[9]. GFG can also be treated as a flower which has $3 \times k$ same pedals, which are noting as $P_i(t), i = 1, 2, \ldots, 3 \times k$. All pedals rooted from two initial vertices, for GFG is made up of three groups, so that each group contains k same pedals.

Definition 2.3 (Generation of EFG) The construction of $EF(t,k)$ is shown as below:

- For $t = 0$, $EF(0,k)$ holds three vertices which are linking each other.
- For $t \geq 1$, $EF(t,k)$ originates from $EF(t-1,k)$ by adding k new vertices to every edge introduced at step $t-1$ and three edges added at $t = 0$, then linking the k new vertices to the two end vertices (see Fig. 4).

Remark The generation method of EFG is slightly different from GFG, which lies in the three initial edges, not only edges merged at step $t-1$, are active edges which can generate new vertices. EFG has $3 \times t \times k$ pedals, naming as $P_i(t_j), i = 1, 2, \ldots, 3 \times k$ and $t_j = 0, 1, \ldots, t$, obviously, GFG is just parts of EFG.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 407; doi:10.3390/sym10090407

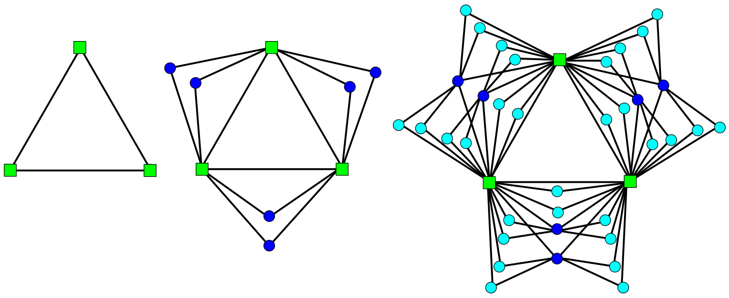


Figure 4. The extended Farey graphs $EF(t,k)$ at steps $t = 0, 1, and 2$ when $k = 2$.

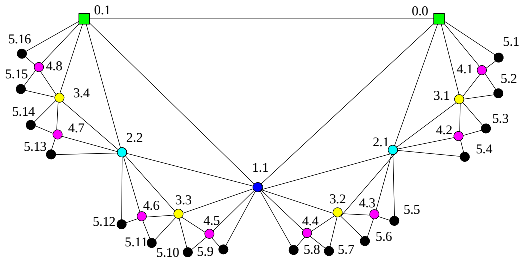


Figure 5. Labels of all vertices in $F(t)$ when $t = 5$.

3. Labeling and routing of $F(t)$

The labeling and routing protocol of GFG and EFG are extended from FG, here we firstly proposal the algorithms of it for completeness.

Definition 3.1 The labeling of any vertex in FG is performed according to the following rules:

- Label two initial vertices as 0.0 and 0.1.
- At step $t \geq 1$, the adding new vertices are marked with labels from $t.1$ to $t.2^{t-1}$ in clockwise direction (see Fig. 5).

Supposing any two vertices are labeled with $t_i.k$ and $t_j.l$ and $t_i \geq t_j$, then the mother vertex of $t_i.k$ joins in graph at step $t_i - 1$, while the father vertex adds to graphs at step $t_i - 2$ or earlier. Two vertices with same mother are brothers. The relationships between different vertices, as the following properties, are extracted by the help of their labels.

Property 3.1 *(The family of $t_i.k$)*

When $t_i \geq 1$, two children of $t_j.k$ are $(t_i + 1).2k$ and $(t_i + 1).(2k - 1)$.

When $t_i \geq 2$, the brother of $t_i.(k + 1)$ for k is odd, or $t_i.(k - 1)$ for k is even.

When $t_i \geq 2$, $t_i.k$ and its parents shape a triangle, the mother is $(t_i - 1).\left\lfloor \frac{K}{2} \right\rfloor$, the father is $(t_i - l).\left\lfloor \frac{k - rem(k,2)}{2^t} \right\rfloor$, in which $\lfloor x \rfloor = floor(x)$ is a function rounding the real number x toward negative infinity, $rem(k, 2)$ is the remainder of k divided by 2, the integer $l (\geq 2)$ denotes the sum of one and the number of the continuous zeros from right to left in the binary sequence which is converted by the decimal number $k - rem(k, 2)$.

If $t_i \geq t_j$, the $(t_i - t_j)th$ generation of maternal ancestor of $t_i.k$ is $t_j.\left\lfloor \frac{k}{2^{t_i - t_j}} \right\rfloor$.

Proof Several results are very obvious besides the father's label, here we only proof it. If $t_i \geq 2$ and $t = t_i - t_j$, thus, $t \in \{2, \ldots, t_i - 2\}$. When k is even, t is the sum of one and the number of the continuous zeros from right to left of the binary numbers of k, so that the father's label is $t - \Delta t.\left\lfloor \frac{k}{2^{\Delta t}} \right\rfloor$; When k is odd but excluding one, t is the sum of one and the number of the continuous zeros from

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at Symmetry 2018, 10, 407; doi:10.3390/sym10090407

right to left of the binary numbers of $k-1$, the father labels with $t-\Delta t.\left\lfloor\frac{k-1}{2^{\Delta t}}\right\rfloor$. When $k=1$, the father of $t_i.l$ is 0.0 for any t_i. In summary, when $t_i \geq 2$, the father of $t_i.k$ is marked with $(t_i-l).\left\lfloor\frac{k-rem(k,2)}{2^l}\right\rfloor$.

Property 3.2 (The neighbors of t_i, k)

When $t_i \geq 2$, he neighbors of $t_i.k$ is $\{(t_i-l).\left\lfloor\frac{k-rem(k,2)}{2^l}\right\rfloor,(t_i-1).\left\lfloor\frac{k}{2}\right\rfloor,(t_i+x).2^{x-1}(2k-1),(t_i+x).\left[2^{x-1}(2k-1)+1\right]\}, \quad x\in\{1,2,\ldots,t-t_i\}$.

When $t_i = 1$, the neighbors of 1.1 are $\{0.0, 0.1, (1+x).2^{x-1}(2k-1),(1+x).\left[2^{x-1}(2k-1)+1\right]\}, \quad t_i\in\{0,1,2,\ldots,t\}$.

When $t_i=0$, the neighbors of 0.0 and 0.1 are $\{t_i.1\}, \quad t_i\in\{0,1,2,\ldots,t\}$ and $\{0.0, t_i.2^{t_i-1}\}, \quad t_i\in\{1,2,\ldots,t\}$.

Proof This property can be proved obviously, for the neighbors of $t_i.k$ are all family members by property 3.1.

Property 3.3 When two vertices are located in different sub-graphs $F_1(t-1)$ and $F_2(t-1)$ of $F(t)$, the hub X of $F(t)$ is on the shortest paths, if

a) $t_i.k\in V_1^x(t-1)\cup V_1^{xy}(t-1)$ and $t_j.l\in V_2^x(t-1)\cup V_2^{xy}(t-1)$,

b) or $t_i.k\in V_2^x(t-1)\cup V_2^{xy}(t-1)$ and $t_j.l\in V_1^x(t-1)\cup V_1^{xy}(t-1)$,

c) or $t_i.k\in V_1^x(t-1)$ and $t_j.l\in V_2^y(t-1)/\{Y2\}$,

d) or $t_i.k\in V_2^x(t-1)$ and $t_j.l\in V_1^y(t-1)/\{Y1\}$.

Two initial vertices Y_1 and Y_2 of $F(t)$ are located on the shortest paths, if

a) $t_i.k\in V_1^y(t-1)$ and $t_j.l\in V_2^y(t-1)$,

b) or $t_i.k\in V_2^y(t-1)$ and $t_j.l\in V_1^y(t-1)$.

Vertices X, Y_1 and Y_2 lie on the shortest paths simultaneously, if

a) $t_i.k\in V_1^y(t-1)$ and $t_j.l\in V_2^{xy}(t-1)$,

b) or $t_i.k\in V_2^y(t-1)$ and $t_j.l\in V_1^{xy}(t-1)$.

Proof $F(t)$ is combined with two sub-graphs $F_1(t-1)$ and $F_2(t-1)$. Same as $F(t)$, all the vertices in $F_\eta(t-1) \quad (\eta=1,2)$ can be divided into three groups $V_\eta^x(t-1)$, $V_\eta^{xy}(t-1)$ and $V_\eta^y(t-1)$ similarly. If $t_i.k\in V_1^x(t-1)\cup V_1^{xy}(t-1)$ and $t_j.l\in V_2^x(t-1)\cup V_2^{xy}(t-1)$, the routes between $t_i.k$ and $t_j.l$ should go by X, or by Y_1 and Y_2;but the distance by X is shorter than by Y_1 and Y_2 so that the shortest paths in this condition pass X.The proof of the other seven conditions is similar.

Property 3.4 All the shortest paths between any pair of vertices are located in a minimum common sub-graph (MCSG) denoting as $F^{ncsg}(t_{min})$, moreover, one vertex is positioned in the outermost layer of $F_\eta^{mcsg}(t_{min}-1)$, the other is an initial vertex or a $(p+1)th$ layer vertex in $F_{3-\eta}^{mcsg}(t_{min}-1)$.

Proof By the construction algorithm of $F(t)$, all the shortest paths between $t_i.k$ and $t_j.l$ are irrelevant to vertices which are adding to $F(t)$ after step t_i. That is to say, all the shortest paths are located in an embedded sub-graph $F(t_i)$. However, the relationship of their positions may be more closely, the minimum embedded sub-graph, or $F^{mcsg}(t_{min})$, is obtained by decreasing t_i till the minimum t_{min}.

Case 1: If $l=\left\lfloor\frac{k}{2^{t_i-t_j}}\right\rfloor$

If $t_i.k$ is a neighbor of $t_j.l$, by property 3.2, MCSG is $F(0)$.

If $k=(l-1)\times 2^{t_i-t_j}+2^{t_i-t_j-1}+2$, or $k\in\{(l-1)\times 2^{t_i-t_j-1}+3,(l-1)\times 2^{t_i-t_j}+2^{t_i-t_j-1}+4\}$, or $\ldots$, or $k\in\{(l-1)\times 2^{t_i-t_j}+2^{t_i-t_j-1}+2^{t_i-t_j-2}+1, l\times 2^{t_i-t_j}\}$, MCSG is $F(2)$, or $F(3)$, or $\ldots$, or $F(t_i-t_j)$, respectively. In these conditions, $t_j.l$ is the initial vertex 0.0 of MCSG.

If $k\in\{(l-1)\times 2^{t_i-t_j}+2^{t_i-t_j-1}-2\}$, or $k\in\{(l-1)\times 2^{t_i-t_j}+2^{t_i-t_j-1}-3,(l-1)\times 2^{t_i-t_j}+2^{t_i-t_j-1}-4\}$, or $\ldots$, or $k\in\{(l-2)\times 2^{t_i-t_j}+1,(l-1)\times 2^{t_i-t_j}+2^{t_i-t_j-1}\}$, MCSG is $F(2)$, or $F(3)$, or $\ldots$, or $F(t_i-t_j)$ .and Vertex $t_j.l$ is the other initial vertex 0.1 in $F^{mcsg}(t_{min})$.

Case 2: If $l\neq m=\left\lfloor\frac{k}{2^{t_i-t_j}}\right\rfloor$

By $2^{p-1}\leq|m-l|\leq 2^p$, $F^{mcsg}(t_{min})=F(t_i-t_j+p+1)$, then $t_i.k$ is an outermost layer vertex in $F_\eta^{mcsg}(t_{min}-1)$ and $t_j.l$ is a $(p+1)th$ layer vertex in $F_{3-\eta}^{mcsg}(t_{min}-1)$.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 407; doi:10.3390/sym10090407

The detailed shortest routing protocol between $t_i.k$ and $t_j.l$ is shown as follows. Firstly, find out $F^{mcsg}(t_{min})$ for $t_i.k$ and $t_j.l$. Then, determine the hub X and two initial vertices Y_1 and Y_2 of $F^{mcsg}(t_{min})$ are whether on the shortest paths or not. Thirdly, form new pairs of vertices from vertices of $t_i.k$, $t_j.l$, x, Y_1 and Y_2, then go to the first step. Repeat three steps till all pairs of vertices are neighbors.

Property 3.5 (The shortest paths routing algorithm of Farey graphs)

1. Given a pair of vertices is labeled with $t_i.k$ and $t_j.l$.
2. Determine whether the two vertices are neighbors or not.

   If $t_i - t_j = 1$ and $l = \left\lfloor \frac{k}{2} \right\rfloor$, or $t_i - t_j = m$ and $l = \left\lfloor \frac{k-rem(k,2)}{2^m} \right\rfloor$, by property 3.1, two vertices are mother-child or father-child relationship. Insert the two labels to the labels set of the shortest paths ($LSSP_m(h)$). Noticing that $LSSP_m(0) = \emptyset$ and m is an integer increasing from one. Go to step 6th.
3. Find out MCSG when $l = \left\lfloor \frac{k}{2^{t_i-t_j}} \right\rfloor$, namely, $t_j.l$ is the $(t_i - t_j)th$ generation of maternal ancestor of $t_i.k$.

   If $k \in \{(l-1) \times 2^{t_i-t_j} + 2^{t_i-t_j-1} + 2\}$, or $k \in \{(l-1) \times 2^{t_i-t_j} + 2^{t_i-t_j-1} + 2, (l-1) \times 2^{t_i-t_j-1} + 4\}$, or $\ldots$, or $k \in \{(l-1) \times 2^{t_i-t_j} + 2^{t_i-t_j-1} + 2^{t_i-t_j-2} + 1, l \times 2^{t_i-t_j}\}$, MCSG is the embedded sub-graph from $F(2)$ to $F(t_i - t_j)$. $t_j.l$ is the initial vertex 0.0 and $t_i.k$ is an outermost layer vertex in MCSG.

   If $k \in \{(l-1) \times 2^{t_i-t_j} + 2^{t_i-t_j-1} - 2\}$, or $k \in \{(l-1) \times 2^{t_i-t_j} + 2^{t_i-t_j-1} - 3, (l-1) \times 2^{t_i-t_j} + 2^{t_i-t_j-1} - 4\}$, or $\ldots$, or $k \in \{(l-1) \times 2^{t_i-t_j} + 1, (l-1) \times 2^{t_i-t_j} + 2^{t_i-t_j-1}\}$, MCSG is also sub-graphs from $F(2)$ to $F(t_i - t_j)$, but $t_j.l$ is the other initial vertex 0.1. Go to step 5th.
4. Find out MCSG when $\left\lfloor l \neq m = \frac{k}{2^{t_i-t_j}} \right\rfloor$.

   For $t_j.m$ is the $(t_i - t_j)th$ generation of maternal ancestor of $t_i.k$, then $F^{mcsg}(t_{min}) = F(t_i - t_j + p + 1)$, in which $2^{p-1} \leq \|m - l\| \leq 2^p$. Go to step 5th.
5. Determine X, Y_1 and Y_2 of MCSG are whether on the shortest paths or not.

   Map the labels of $F^{mcsg}(t_{min})$ into labels of $F(t_{min})$, divide all the vertices in $F(t_{min})$ into six sets as above: $V_\eta^x(t_{min} - 1)$, $V_\eta^{xy}(t_{min} - 1)$ and $V_\eta^y(t_{min} - 1)$. Then, decide X, Y_1 and Y_2 are whether on the shortest paths by property 3.3 or not.

   If X is on the paths, insert the label of X, assuming $t_p.q$, in the middle of $t_i.k$ and $t_j.l$ in $LSSP_m(h)$, and $h = h + 1$. Therefore, go to step 1st with two new pairs of labels:$t_i.k$ and $t_p.q$, $t_p.q$ and $t_j.l$.

   If Y_1 and Y_2 in it, insert the labels of Y_1 and Y_2, $t_{p1}.ql$ and $t_{p2}.q2$, in the middle of $t_i.k$ and $t_j.l$ in $LSSP_m(h)$, and $h = h + 2$. Get two new pairs of labels, $t_i.k$ and $t_{p1}.ql$, $t_{p2}.q2$ and $t_j.l$. Go to step 1st.

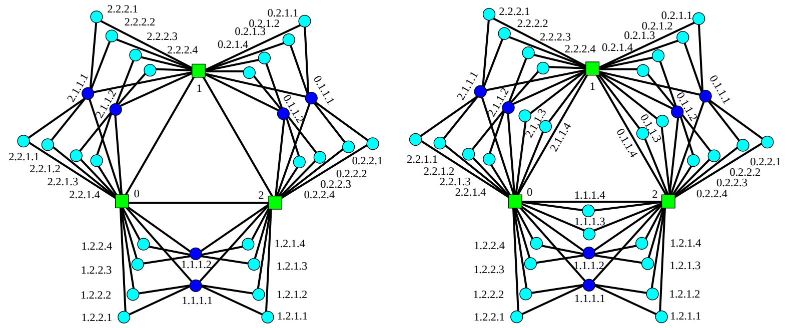
   If X, Y_1 and Y_2 are all on the paths at same time, insert $t_p.q$ into $LSSP_m(h)$ and $h = h + 1$, then insert $t_{p1}.q1$ and $t_{p2}.q2$ into $LSSP_{m+1}(h)$, $h = h + 2$. Go to step 1st with four pairs of labels: $t_i.k$ and $t_p.q$, $t_p.q$ and $t_j.l$, $t_i.k$ and $t_{p1}.q1$, $t_{p1}.q1$, $t_{p2}.q2$ and $t_j.l$.
6. Ascertain the shortest paths routing.

   The shortest paths are traversed every elements in every set of $LSSP_m(h)$ in order, where m is the number of the shortest paths and h is the distance between $t_i.k$ and $t_j.l$.

   The time complexity of the shortest paths routing algorithm is related to the maximum number of the shortest paths between any two vertices in Farey graphs. The number is exactly the product of two Fibonacci numbers (F_n, in which $F_n = F_{n-1} + F_{n-2}$ and $F_0 = F_1 = 1$) in FG. From the construction mechanism, all the vertices on the shortest paths shape rhombuses which are zigzagged adjacent, so that the maximum number of rhombuses from $t_i.k$ to 0. $\left\lfloor \frac{k}{2^{t_i-2}} \right\rfloor$ is $\left\lfloor \frac{t_i}{2} \right\rfloor$ and from $t_j.l$ to 0. $\left\lfloor \frac{l}{2^{t_j-2}} \right\rfloor$ is $\left\lfloor \frac{t_j-3}{2} \right\rfloor$. Therefore, the number of shortest paths from $t_i.k$ to 0. $\left\lfloor \frac{k}{2^{t_i-2}} \right\rfloor$ is $F_{\left\lfloor \frac{t_i}{2} \right\rfloor+1} = \frac{1}{\sqrt{5}} \left[(\frac{1+\sqrt{t}}{2})^{\left\lfloor \frac{t_i}{2+l} \right\rfloor} - (\frac{1-\sqrt{5}}{2})^{\left\lfloor \frac{t_i}{2} \right\rfloor+1} \right]$, while the number from $t_j.l$ to

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 407; doi:10.3390/sym10090407



(a) $GF(2,2)$. (b) $EF(2,2)$.

Figure 6. The labeling of $GF(t,k)$ and $EF(t,k)$ at step t=2 for k=2.

0. $\left\lfloor \frac{l}{2^{t_j-2}} \right\rfloor$ is $F_{\left\lfloor \frac{t_j-3}{2} \right\rfloor+1} = \frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{\left\lfloor \frac{t_j-3}{2} \right\rfloor+1} - (\frac{1-\sqrt{5}}{2})^{\left\lfloor \frac{t_j-3}{2} \right\rfloor+1} \right]$, so that the maximum number is $F_{\left\lfloor \frac{t_i}{2} \right\rfloor+1} \times F_{\left\lfloor \frac{t_j-3}{2} \right\rfloor+1} = \frac{1}{5}[(\frac{1+\sqrt{5}}{2})^{\lfloor \frac{t_i}{2} \rfloor+1} - (\frac{1-\sqrt{5}}{2})^{\lfloor \frac{t_i}{2} \rfloor+1}] \left[(\frac{1+\sqrt{5}}{2})^{\left\lfloor \frac{t_j-3}{2} \right\rfloor+1} - (\frac{1-\sqrt{5}}{2})^{\left\lfloor \frac{t_j-3}{2} \right\rfloor+1} \right]$.
When $t_i = t_j = t$, the number in FG is $F_{\lfloor \frac{t}{2} \rfloor+1} \times F_{\lfloor \frac{t-3}{2} \rfloor+1} = \frac{1}{5}[(\frac{1+\sqrt{5}}{2})^{\lfloor \frac{t}{2} \rfloor+1} - (\frac{1-\sqrt{5}}{2})^{\lfloor \frac{t}{2} \rfloor+1}]$ $[(\frac{1+\sqrt{5}}{2})^{\lfloor \frac{t-3}{2} \rfloor+1} - (\frac{1-\sqrt{5}}{2})^{\lfloor \frac{t-3}{2} \rfloor+1}]$, and it increases almost exponentially. Fortunately, only at most $2t+1$ vertices are needed to be ascertaining in the routing algorithm, and only several operations of additions and multiplications are needed for determining one vertex whether on the shortest paths or not. As a result, all the shortest paths between any pair of vertices can be determined in linear time of $O(n)$.

Property 3.6 The shortest paths routing algorithm between any two vertices of Farey graphs runs in linear time.

4. Labeling of $GF(t,k)$ and $EF(t,k)$

Definition 4.1 (The labeling of GFG) The labeling of any vertex in $GF(t,k)$ shows as the following rules:

- The three initial vertices are labeled with 0, 1 and 2.
- At any step $t \geq 1$, a vertex in $GF(t,k)$ is marked with $a.b.c.d$ according to the $group(a)$, the $subgroup(b.c)$ and the precise $positions(d)$ from down to top in the same subgroup, in which $a \in \{0,1,2\}$, $b = \{1,2,\ldots,t\}$ $c =\in \{1,2,\ldots,2^{b-1}\}$ and $d \in \{1,2,\ldots,k^b\}$.

Definition 4.2 (The labeling of EFG) Vertices in $EF(t,k)$ are labeled as follows:

- Label three initial vertices as 0, 1 and 2.
- At step $t \geq 1$, a vertex is tagged with $a.b.c.d$, in which $a \in \{0,1,2\}$, $b \in \{1,2,\ldots,t\}$, $c \in \{1,2,\ldots,2^{b-1}\}$ and $d \in \{1,2,\ldots,(t-b+1) \times k^b\}$.

Labeling of $GF(2,2)$ and $EF(2,2)$ are illustrated in Fig. 6.

Remark For GFG have more pedals than EFG, so that $d_{max} = (t-b+1) \times k^b$ in EFG, while $d_{max} = k^b$ in GFG.

When $t \geq 2$, any new vertex, adding to GFG/EFG at step t_i, links to two vertices: a mother and a father. The vertices adding to graphs at the same time are multiple births if they have the same parents. Supposing that two arbitrary vertices in GFG/EFG are labeled with $a_1.b_1.c_1.d_1$ and $a_2.b_2.c_2.d_2$, in which $b_1 \geq b_2$, then, we give several properties satisfying GFG and EFG at once.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 407; doi:10.3390/sym10090407

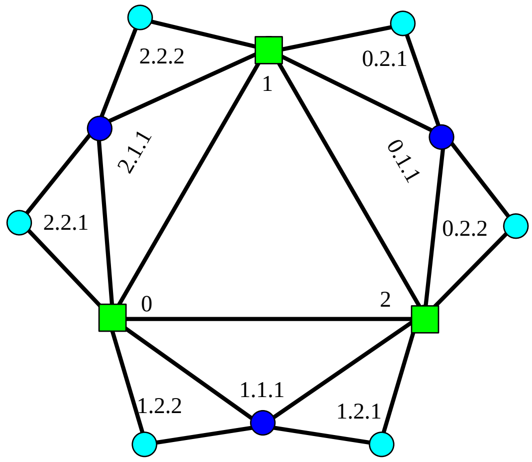


Figure 7. The projected graph of $GF(2,2)$ / $EF(2.2)$.

Property 4.1 (The family of $a.b.c.d$) When $b \geq 1$, $a.b.c.d$ belongs to a set of multiple births $\{a.b.c.(k\left\lfloor \frac{d}{k} \right\rfloor + 1), a.b.c.(k\left\lfloor \frac{d}{k} \right\rfloor + 2), \ldots, a.b.c.(k\left\lfloor \frac{d}{k} \right\rfloor + k)\}$.

When $b \geq 2$, $a.b.c.d$ and its parents shape a triangle, the mother is $a.(b-1).\left\lfloor \frac{c}{2} \right\rfloor.\left\lfloor \frac{d}{k} \right\rfloor$, the father is $a.(b-1).\left\lfloor \frac{c-rem(c,2)}{2^l} \right\rfloor.\left\lfloor \frac{d}{k^l} \right\rfloor$.

If $b \geq b_0$, the $(b-b_0)th$ generation of maternal ancestor of $a.b.c.d$ is $a.b_0.\left\lfloor \frac{c}{2^{b-b_0}} \right\rfloor.\left\lfloor \frac{d}{k^{b-b^0}} \right\rfloor$.

Proof Several results are very obvious besides the father's label, here we only proof it. If $b \geq 2$, let l denotes the difference $b - b_1$, thus, $l \in \{2, \ldots, b-2\}$. When c is even, the time difference l is the sum of one and the number of the continuous zeros from right to left of the binary numbers of c, so that the father's label is $a.(b-l).\left\lfloor \frac{c}{2^l} \right\rfloor.\left\lfloor \frac{d}{k^l} \right\rfloor$; When c is odd but excluding one, l is the sum of one and the number of the continuous zeros from right to left of the binary numbers of $c-1$, the father labels with $a.(b-l).\left\lfloor \frac{c-l}{2^l} \right\rfloor.\left\lfloor \frac{d}{k^l} \right\rfloor$; When $c = 1$ and with any b, the fathers are all the initial vertex. In summary, the father of $a.b.c.d$ is $a.(b-l).\left\lfloor \frac{c-rem(c,2)}{2^l} \right\rfloor.\left\lfloor \frac{d}{k^l} \right\rfloor$ when $b \geq 2$.

Remark The vertex $a.1.1.d$ has two mothers $\{0,1,2\}/\{a\}$ and no father. Three initial vertices have no parents and multiple births. Therefore, the neighbors of $a.b.c.d$ are derived by property 4.2.

Property 4.2 (The neighbors of $a.b.c.d$)

When $b \geq 2$, the set of neighbors of $a.b.c.d$ is $\{a.(b-l).\left\lfloor \frac{c-rem(c,2)}{2^l} \right\rfloor.\left\lfloor \frac{d}{k^l} \right\rfloor,\ a.(b-1)\left\lfloor \frac{c}{2} \right\rfloor.\left\lfloor \frac{d}{k} \right\rfloor,\ a.(b+x).2^{x-1}(ac-1).d,\ a.(b+x).[2^{x-1}(2c-1)+1].d\}$, in which $x \in \{1,2,\ldots,t-b\}$, $d \in \{(b+x-1)\times k^x+1, (b+x-1)\times k^x+2.\ldots,(b+x)\times k^x\}$.

The neighbors of initial vertex a are $\{0,1,2\}/\{a\} \cup \{a.b.1.d\} \cup \{a.b.2^{b-1}.d\}$, where $b \in \{1,2,\ldots,t\}$, $d \in \{1,2,\ldots,k^b\}$.

The neighbors of $a.1.1.d$ are $\{0,1,2\}/\{a\}$ and $\{a.(1+x).2^{x-1}.d,\ a.(1+x).(2^{x-1}+1).d\}$, where $x \in \{1,2,\ldots,t-1\}$, $d \in \{(b+x-1)\times k^x+1,(b+x-1)\times k^x+2,\ldots,(b+x)\times k^x\}$.

Property 4.3 (The projection of GFG/EFG) By merging vertices $a.b.c.d$, which have same $a.b.c$ but different d, into a vertex labeling with $a.b.c$, $GF(t.k)/EF(t,k)$ is projected into a graph which is exactly the combination of three Farey graphs starting from every edge of a triangle.

Proof From the spatial relationship between vertices in different pedals, all the vertices of $a.1.1.d$ are linked to common vertices till two initial vertices $\{0,1,2\}/\{a\}$, so that all of $a.1.1.d$ can merged into a vertex of $a.1.1$.By recursively using the spatial relationship, $GF(t,k)/EF(t,k)$ is projected into a combination of three Farey graphs.

Example 4.1 The projection graph of $GF(2,2)$ $/EF(2,2)$ is shown as Fig. 7.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at Symmetry 2018, 10, 407; doi:10.3390/sym10090407

Property 4.4 (The slice shapes by $a.b.c.d$) For any vertex $a.b.c.d$, the slice is obtained by recursively finding out all triangles which are shaped by a vertex and its parents, till two initial vertices.

Proof Any vertex $a.b.c.d$ has a father $a.(b-l).\left\lfloor \frac{c-rem(c,2)}{2^l} \right\rfloor . \left\lfloor \frac{d}{k^l} \right\rfloor$ and a mother $a.(b-1).\lfloor \frac{c}{2} \rfloor . \left\lfloor \frac{d}{k} \right\rfloor$ y property 4.1, and the three vertices shaped a triangle. Then, the father's mother, or the mother's father, is got by recursively using property 4.1. From the spatial relationship of vertices, $a.b.c.d$ and all these parents shape a slice.

5. Routing of $GF(t,k)$ and $EF(t,k)$

Although the labels of three initial vertices are slightly different from labels of two initial vertices in FG, the shortest paths protocol in GFG/EFG also benefit from the routing algorithm in FG. From the generation mechanism, GFG/EFG is divided into three groups by symmetry, and each group is consisted of k or $t \times k$ pedals, furthermore, vertices in each group can also be divided into three sets by the distance differences, similar to that of FG: $V^x(t)$, $V^{xy}(t)$ and $V^y(t)$.

Property 5.1 (The characteristic of shortest paths when two vertices are located in different groups)

If $a_1 \neq a_2$, the initial vertex $\{0,1,2\}/\{a_1,a_2\}$ is on the shortest roads between $a_1.b_1.c_1.d_1$ and $a_2.b_2.c_2.d_2$ of GFG/EFG, if

a) $a_1.b_1.c_1 \in V_1^x(t) \cup V_1^{xy}(t)$ and $a_2.b_2.c_2 \in V_2^x(t) \cup V_2^{xy}(t)$,

b) or $a_1.b_1.c_1 \in V_2^x(t) \cup V_2^{xy}(t)$ and $a_2.b_2.c_2 \in V_1^x(t) \cup V_1^{xy}(t)$,

c) or $a_1.b_1.c_1 \in V_1^x(t)$ and $a_2.b_2.c_2 \in V_2^y(t)/\{a_2\}$,

d) or $a_1.b_1.c_1 \in V_2^x(t)$ and $a_2.b_2.c_2 \in V_1^y(t)/\{a_1\}$.

The two initial vertices a_1 and a_2 are on the shortest ways, if

a) $a_1.b_1.c_1 \in V_1^y(t)$ and $a_2.b_2.c_2 \in V_2^y(t)$,

b) or $a_1.b_1.c_1 \in V_2^y(t)$ and $a_2.b_2.c_2 \in V_1^y(t)$.

The shortest paths pass $\{0,1,2\}/\{a_1,a_2\}$, a_1 and a_2 simultaneously, if

a) $a_1.b_1.c_1 \in V_1^y(t)$ and $a_2.b_2.c_2 \in V_2^{xy}(t)$,

b) or $a_1.b_1.c_1 \in V_2^y(t)$ and $a_2.b_2.c_2 \in V_1^{xy}(t)$.

Proof From the property 4.3, $a_1.b_1.c_1.d_1$ and $a_2.b_2.c_2.d_2$ are projected as $a_1.b_1.c_1$ and $a_2.b_2.c_2$, which are located in different sub-graphs $F_1(t)$ and $F_2(t)$ of Farey graphs $F(t+1)$. imilar as the proof of property 3.3, the conclusions can easily be deduced.

Property 5.2 (The characteristic of shortest paths when two vertices lie in different slices of same group) If $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ are located in different pedals, the shortest paths between them are positioned in two slices which are connected by two common vertices of them.

Proof From the generating algorithm, if $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ are located in different pedals or sub-pedals, the neighbor sets of $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ are ascertained by property 4.2. When two common neighbors are getting at the same step, then $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ are positioned in different slices, and the two slices can be determined by property 4.4. Supposing that the first two common neighbors are $a.b_3.c_3.d_3$ and $a.(b_3+1).c_4.d_4$, the two slices root from it and belong to $F(b_1-b_3)$ and $F(b_2-b_3)$. Compared with the construction schematic diagram in Fig. 2, the linking of $F(b_1-b_3)$ and $F(b_2-b_3)$ has slightly difference. Named the initial vertices of $F(b_1-b_3)$ and $F(b_2-b_3)$ as X_1, X_2 and Y_1, Y_2, respectively.$F(b_1-b_3)$ and $F(b_2-b_3)$ is linked exactly by merging X_1 and Y_1 into $a.b_3.c_3.d_3$ and linking X_2 and Y_2 into $a.(b_3+1).c_4.d_4$.The vertices of $F(b_1-b_3)$ and $F(b_2-b_3)$ can be divided into six parts similarly: $V_\alpha^x(t)$, $V_\alpha^{xy}(t)$, $V_\alpha^y(t)$ and $V_\beta^x(t)$, $V_\beta^{xy}(t)$, $V_\beta^y(t)$, by the distance between $a.b_1.c_1.d_1$ or $a.b_2.c_2.d_2$ to initial vertices X (i.e. $a.b_3.c_3.d_3$) and Y (i.e. $a.(b_3+1).c_4.d_4$), then the shortest paths between $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ go by $a.b_3.c_3.d_3$, if

a) $a.b_1.c_1 \in V_\alpha^x(t)$ and $a.b_2.c_2 \in V_\beta^x(t)$,

b) $a.b_1.c_1 \in V_\alpha^x(t)$ and $a.b_2.c_2 \in V_\beta^{xy}(t)$,

b) or $a.b_1.c_1 \in V_\alpha^{xy}(t)$ and $a.b_2.c_2 \in V_\beta^x(t)$.

The shortest paths go through $a.b_3.c_3.d_3$ and $a.(b_3+1).c_4.d_4$, if

a) $a.b_1.c_1 \in V_\alpha^x(t)$ and $a.b_2.c_2 \in V^y\beta(t)$,

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at *Symmetry* **2018**, *10*, 407; doi:10.3390/sym10090407

b) or $a.b_1.c_1 \in V_\alpha^{xy}(t)$ and $a.b_2.c_2 \in V_\beta^{xy}(t)$,
c) $a.b_1.c_1 \in V_\alpha^{y}(t)$ and $a.b_2.c_2 \in V_\beta^{x}(t)$.
The shortest paths pass $a.(b_3+1).c_4.d_4$, if
a) $a.b_1.c_1 \in V_\alpha^{xy}(t)$ and $a.b_2.c_2 \in V_\beta^{y}(t)$,
b) $a.b_1.c_1 \in V_\alpha^{y}(t)$ and $a.b_2.c_2 \in V_\beta^{xy}(t)$,
b) or $a.b_1.c_1 \in V_\alpha^{y}(t)$ and $a.b_2.c_2 \in V_\beta^{y}(t)$.

Property 5.3 (The characteristic of shortest paths when two vertices are in same slice) If $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ are in the same slice of a pedal or sub-pedal of same group, the shortest paths between them are determined by property 3.5, for they have been projected into a Farey graph.

Proof If we obtained only one common neighbor vertex, labeling with $a.b_3.c_3.d_3$, of vertices $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ at the same step, then $a.b_1.c_1.d$ and $a.b_2.c_2.d_2$ are located in same slice. Assuming $b_1 \geq b_2$, the projected Farey graph is $F(b_1 - b_3 - 1)$ with hub $a.b_1.c_1.d_1$, then, all the shortest paths are located in $F(b_1 - b_3 - 1)$. So that all the shortest paths can be decided by property 3.5.

Then, the detailed shortest routing algorithm in GFG/EFG is described as follows.

Property 5.4 (The shortest paths routing algorithm in GFG/EFG)

- Given any two vertices are $a_1.b_1.c_1.d_1$ and $a_2.b_2.c_2.d_2$. Insert $a_1.b_1.c_1.d_1$ and $a_2.b_2.c_2.d_2$ to the labels set of the shortest paths ($LSSP_m(h)$), $LSSP_m(0) = \emptyset$.
- If $a_1 \neq a_2$ and, $a_1.b_1.c_1.d_1$ and $a_2.b_2.c_2.d_2$ are not initial vertex. This is exactly the condition of two vertices in different groups. Three initial vertices $\{0.1.2\}/\{a_1, a_2\}$, a_1 and a_2 are ascertained on the shortest paths or not by property 5.1.

  If $\{0,1,2\}/\{a_1, a_2\}$ is on the roads, insert the label $\{0,1,2\}/\{a_1, a_2\}$ in the middle of $a_1.b_1.c_1.d_1$ and $a_2.b_2.c_2.d_2$ in $LSSP_m(h)$, $h = h+1$, and generate two new pairs of labels: $a_1.b_1.c_1.d_1$ and $\{0,1,2\}/\{a1, a2\}$, $\{0,1,2\}/\{a_1, a_2\}$ and $a_2.b_2.c_2.d_2$, respectively.

  If a_1 and a_2 are on it, insert the labels a_1 and a_2 in the middle of two labels $a_1.b_1.c_1.d_1$ and $a_2.b_2.c_2.d_2$, $h = h+2$, and get two new pairs of labels: $a_1.b_1.c_1.d_1$ and a_1, a_2 and $a_2.b_2.c_2.d_2$.

  If $\{0,1,2\}/\{a_1, a_2\}$, a_1 and a_2 are all on it, combine the two conditions above together.

  Go to step 1st.
- If $a_1 \neq a_2$, and $a_1.b_1.c_1.d_1$ or $a_2.b_2.c_2.d_2$ is an initial vertex. It is the case of two vertices locating in a same slice of a pedal or sub-pedal in same group. The common Farey graphs $F(b_1)$ or $F(b_2)$ is obtained by property 5.3, then, the shortest paths are deduced by property 3.5.
- If $a_1 = a_2 = a$ and $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ are not initial vertices, find out neighbors of $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ by property 4.2.

  If two common neighbors are obtained at the same step, then $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ are located in different slices. Whether the two common neighbors are positioned on the shortest paths or not are determined by property 4.6. Assuming two common neighbors are $a.b_3.c_3.d_3$ and $a.b_4.c_4.d_4$, if $a.b_3.c_3.d_3$ (or $a.b_4.c_4.d_4$) is on the shortest paths, insert the label $a.b_3.c_3.d_3$ (or $a.b_4.c_4.d_4$) in the middle of $a_1.b_1.c_1.d_1$ and $a_2.b_2.c_2.d_2$, $h = h+1$, and generate two new pair labels of $a_1.b_1.c_1.d_1$ and $a.b_3.c_3.d_3$ (or $a.b_4.c_4.d_4$), and $a.b_3.c_3.d_3$ (or $a.b_4.c_4.d_4$) or $a_2.b_2.c_2.d_2$. If $a.b_3.c_3.d_3$ and $a.b_4.c_4.d_4$ are on the shortest paths at the same time, insert the label $a.b_3.c_3.d_3$ and $a.b_4.c_4.d_4$ in the set $LSSP_m(h)$, $h = h+1$, $m = m+1$, and make up four new pair labels of $a_1.b_1.c_1.d_1$ and $a.b_3.c_3.d_3$, $a_1.b_1.c_1.d_1$ and $a.b_4.c_4.d_4$, $a.b_3.c_3.d_3$ or $a_2.b_2.c_2.d_2$, $a.b_4.c_4.d_4$ and $a_2.b_2.c_2.d_2$. Go to step 1st.

  If only a common neighbor is obtained at the same time, then $a.b_1.c_1.d_1$ and $a.b_2.c_2.d_2$ are located in same slice. The shortest paths are projected into a Farey graph, and then all the shortest paths are derived by Property 3.5.

Remark The label-based routing protocol for expanded deterministic Apollonian networks is deduced in[31], the graphs are the generalization of the Apollonian networks and are simultaneously scale-free, small-world, and highly clustered. The networks, denoted by $A(d,t)$ after t iterations

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 1 August 2018 doi:10.20944/preprints201808.0007.v1

Peer-reviewed version available at Symmetry 2018, 10, 407; doi:10.3390/sym10090407

with $d \geq 1$ and $t \geq 0$, are constructed as follows. For $t = 0$, $A(d,0)$ is a complete graph K_{d+2} (or $d+1$-clique).For $t \geq 1$, $A(d,t)$ is obtained from $A(d,t-1)$. For each of the existing subgraphs of $A(d,t-1)$ that is isomorphic to $a(d+1)$-clique and created at step $t-1$, a new vertex is created and connected to all the vertices of this subgraph. Apparently, $A(1,t)$ is exactly the same as the special case of the generalization of Farey graphs $GF(t,1)$, however, the algorithm in[31] can only get one of shortest paths in it.

The recursive clique-trees, which have small-world and scale-free properties and allow a fine tuning of the clustering and the power-law exponent of their discrete degree distribution[32]. The recursive clique-tree $K(q,t)(q \geq 1, t \geq 0)$ is the graph constructed as follows: For $t = 0$, $K(q,0)$ is the complete graph K_q (or q-clique).For $K(q,t)$ is obtained from by $K(q,t-1)$ (i) adding for each of its existing subgraphs isomorphic to a q-clique a new vertex and (ii) joining it to all the vertices of this subgraph. From the construction mechanisms, $K(2,t)$ is the same as the extended Farey graphs of $EF(t,1)$.

6. Conclusion

We presented label-based routing algorithm for Farey-type graphs, including Farey graphs, the generalization of Farey graphs and the extended Farey graphs. Our results can be extended easily to several Farey-type deterministic models, such as the model created by edge iterations, evolving graphs with geographical attachment preference, general geometric growth model for pseudofractal scale-free web, the graphs with multidimensional growth and so on. Different from the former research results, they can only get one shortest path from the labels of any pair vertices; we can ascertain all the shortest paths only by their labels in all Farey-type graphs.

For all Farey-type graphs are structure isomorphic, the time complexities of the routing algorithm in the generalization of Farey graphs and the extended Farey graphs are essentially equivalent to which on Farey graphs, thus, the routing algorithms runs in linear time $O(n)$.

Our solutions can also be easily extended to several weighted or delayed models, such as weighted scale-free small-world graphs[33] and delayed pseudofractal graphs[34].

Author Contributions: Wenchao Jiang contribute for conceptualization, designing the experiments, analyzed the data curation and wrote the paper. Yinhua Zhai and Zhigang Zhuang contribute for performed experiments, resources, software, some computations. Paul Martin, Zhiming Zhao and Jia-Bao Liu contribute for supervision, methodology, validation and formalanalysing. All authors read and approved the final version of the paper.

Funding: This work was supported by the National Science Foundation of China under grant No.11601006; the Natural Science Foundation of Guangdong Province [grant no.2016A030313703]; the Guangdong Science and Technology Plan [grant no.2016B030305002, 2017B010124001 and 2017B090901005]; the European Union's Horizon 2020 research and innovation program [grant No.654182 (ENVRIPLUS project)] and [grant No.676247 (VRE4EIC project)]; the China Postdoctoral Science Foundation under grant No. 2017M621579 and Postdoctoral Science Foundation of Jiangsu Province under grant No.1701081B.

Conflicts of Interest: The authors declare no conflict of interest.

1. Comellas, F., Ozon, J., & Peters, J. G. (2000). Deterministic small-world communication networks. Information Processing Letters, 76(1), 83-90.
2. Barabási, A. L., Ravasz, E., & Vicsek, T. (2001). Deterministic scale-free networks. Physica A: Statistical Mechanics and its Applications, 299(3), 559-564.
3. Andrade Jr, J. S., Herrmann, H. J., Andrade, R. F., & Da Silva, L. R. (2005). Apollonian networks: Simultaneously scale-free, small world, Euclidean, space filling, and with matching graphs. Physical Review Letters, 94(1), 018702.
4. Zhang, Z., Gao, S., Chen, L., Zhou, S., Zhang, H., & Guan, J. (2010). Mapping Koch curves into scale-free small-world networks. Journal of Physics A: Mathematical and Theoretical, 43(39), 395101.
5. Zhou, T., Wang, B. H., Hui, P. M., & Chan, K. P. (2006). Topological properties of integer networks. Physica A: Statistical Mechanics and its Applications, 367, 613-618.

6. Zhang, Z., & Comellas, F. (2011). Farey graphs as models for complex networks. Theoretical Computer Science, 412(8), 865-875.
7. Zhang, Z., Wu, B., & Lin, Y. (2012). Counting spanning trees in a small-world Farey graph. Physica A: Statistical Mechanics and its Applications, 391(11), 3342-3349.
8. Zhang, Z., Rong, L., & Guo, C. (2006). A deterministic small-world network created by edge iterations. Physica A: Statistical Mechanics and its Applications, 363(2), 567-572.
9. Zhang, Z. Z., Rong, L. L., & Comellas, F. (2006). Evolving small-world networks with geographical attachment preference. Journal of Physics A: Mathematical and General, 39(13), 3253.
10. Peng, A., & Zhang, L. (2011). Deterministic multidimensional growth model for small-world networks. arXiv preprint arXiv:1108.5450.
11. Zhang, Z., Rong, L., & Zhou, S. (2007). A general geometric growth model for pseudofractal scale-free web. Physica A: Statistical Mechanics and its Applications, 377(1), 329-339.
12. Havlin, S., & ben-Avraham, D. (2007). Fractal and transfractal recursive scale-free nets. New Journal of Physics, 9(6), 175.
13. Xiao, Y., & Zhao, H. (2013, July). Counting the number of spanning trees of generalization Farey graph. In Natural Computation (ICNC), 2013 Ninth International Conference on (pp. 1778-1782). IEEE.
14. Andrade Jr, J. S., Herrmann, H. J., Andrade, R. F., & Da Silva, L. R. (2005). Apollonian networks: Simultaneously scale-free, small world, Euclidean, space filling, and with matching graphs. Physical Review Letters, 94(1), 018702.
15. Auto, D. M., Moreira, A. A., Herrmann, H. J., & Andrade Jr, J. S. (2008). Finite-size effects for percolation on Apollonian networks. Physical Review E, 78(6), 066112.
16. Almeida, G. M., & Souza, A. M. (2013). Quantum transport with coupled cavities on an Apollonian network. Physical Review A, 87(3), 033804.
17. Wong, W. K., Guo, Z. X., & Leung, S. Y. S. (2010). Partially connected feedforward neural networks on Apollonian networks. Physica A: Statistical Mechanics and its Applications, 389(22), 5298-5307.
18. Mendes, G. A., Da Silva, L. R., & Herrmann, H. J. (2012). Traffic gridlock on complex networks. Physica A: Statistical Mechanics and its Applications, 391(1), 362-370.
19. de Oliveira, I. N., de Moura, F. A. B. F., Lyra, M. L., Andrade Jr, J. S., & Albuquerque, E. L. (2010). Bose-Einstein condensation in the Apollonian complex network. Physical Review E, 81(3), 030104.
20. De Oliveira, I. N., De Moura, F. A. B. F., Lyra, M. L., Andrade Jr, J. S., & Albuquerque, E. L. (2009). Free-electron gas in the Apollonian network: Multifractal energy spectrum and its thermodynamic fingerprints. Physical Review E, 79(1), 016104.
21. Knuth, D. E. (1977). A generalization of Dijkstra's algorithm. Information Processing Letters, 6(1), 1-5.
22. Yen, J. Y. (1970). An algorithm for finding shortest routes from all source nodes to a given destination in general networks. Quarterly of Applied Mathematics, 27(4), 526.
23. Delling, D., Sanders, P., Schultes, D., & Wagner, D. (2009). Engineering route planning algorithms. In Algorithmics of large and complex networks (pp. 117-139). Springer Berlin Heidelberg.
24. Zwick, U. (2002). All pairs shortest paths using bridging sets and rectangular matrix multiplication. Journal of the ACM (JACM), 49(3), 289-317.
25. Chan, T. M. (2010). More algorithms for all-pairs shortest paths in weighted graphs. SIAM Journal on Computing, 39(5), 2075-2089.
26. Yen, J. Y. (1971). Finding the k shortest loopless paths in a network. management Science, 17(11), 712-716.
27. Bern, M. W., & Graham, R. L. (1989). The shortest-network problem. Scientific American, 260(1), 84-89.
28. Zwick, U. (2001). Exact and approximate distances in graphs—a survey. In Algorithms—ESA 2001 (pp. 33-48). Springer Berlin Heidelberg.
29. Zhang, Z., Comellas, F., Fertin, G., Raspaud, A., Rong, L., & Zhou, S. (2008). Vertex labeling and routing in expanded Apollonian networks. Journal of Physics A: Mathematical and Theoretical, 41(3), 035004.
30. Comellas, F., & Miralles, A. (2009). Vertex labeling and routing in self-similar outerplanar unclustered graphs modeling complex networks. Journal of Physics A: Mathematical and Theoretical, 42(42), 425001.
31. Comellas, F., & Miralles, A. (2011). Label-based routing for a family of scale-free, modular, planar and unclustered graphs. Journal of Physics A: Mathematical and Theoretical, 44(20), 205102.
32. Comellas, F., Fertin, G., & Raspaud, A. (2003). Vertex Labeling and Routing in Recursive Clique-Trees, a New Family of Small-World Scale-Free Graphs. In SIROCCO (pp. 73-87).

33. Zhang, Y., Zhang, Z., Zhou, S., & Guan, J. (2010). Deterministic weighted scale-free small-world networks. Physica A: Statistical Mechanics and its Applications, 389(16), 3316-3324.
34. Sun, W., Wu, Y., Chen, G., & Wang, Q. (2011). Deterministically delayed pseudofractal networks. Journal of Statistical Mechanics: Theory and Experiment, 2011(10), P10032.