ON p-ADIC INTEGRAL REPRESENTATION OF q-BERNOULLI NUMBERS ARISING FROM TWO VARIABLE q-BERNSTEIN POLYNOMIALS

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Abstract. In this paper, we study the p-adic integral representation on $\mathbb{Z}_p$ of q-Bernoulli numbers arising from two variable q-Bernstein polynomials and investigate some properties for the q-Bernoulli numbers. In addition, we give some new identities of q-Bernoulli numbers.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80, 33C45.
Key words and phrases : q-Bernoulli numbers, q-Bernoulli polynomials, Bernstein polynomials, q-Bernstein polynomials, p-adic integral on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{N}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the set of natural numbers, the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Assume that q is an indeterminate in $\mathbb{C}_p$ such that $|1-q|_p < p^{-\frac{1}{p-1}}$.

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† This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

It is known that the q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x, (\text{ see [9, 10]}). \tag{1}$$

As $q \to 1$ in (1), we have the p-adic integral on $\mathbb{Z}_p$ which is give by

$$I_1(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \ (\text{see [7-12]}). \tag{2}$$

From (2), we note that

$$I_1(f_1) - I_1(f) = f'(0), \ (\text{see [9]}), \tag{3}$$

where $f_1(x) = f(x+1)$ and $f'(0) = \frac{df(x)}{dx}|_{x=0}$.

Thus, by (3), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \ (\text{see [6, 9]}), \tag{4}$$

where $B_n(x)$ are ordinary Bernoulli polynomials.

From (4), we note that

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y) = B_n(x), (n \geq 0), \ (\text{see [7-12, 15]}). \tag{5}$$

When $x = 0$, $B_n = B_n(0), (n \geq 0)$, are called the ordinary Bernoulli numbers.

The equation (3) implies the following recurrence relation for Bernoulli numbers:

$$B_0 = 1, \quad (B+1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{6}$$

with the usual convention about replacing B^n by B_n.

In [3, 4, 5], L. Carlitz introduced the q-Bernoulli numbers which are given by the recurrence relation to be

$$\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{7}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$.

He also defined q-Bernoulli polynomials as

$$\beta_{n,q}(x) = (q^x\beta_q + [x]_q)^n = \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}\beta_{l,q},\ \text{(see [3, 4, 5])}. \tag{8}$$

In 1999, Kim proved the following formula.

$$\int_{\mathbb{Z}_p}[x+y]_q^n d\mu_q(y) = \beta_{n,q}(x),\ (n \geq 0)\ ,\ \text{(see [10])}. \tag{9}$$

In the viewpoint of (5), we define the q-Bernoulli polynomials which are different from Carlitz's q-Bernoulli polynomials to be

$$B_{n,q}(x) = \int_{\mathbb{Z}_p}[x+y]_q^n d\mu_1(y),\ (n \geq 0),\ \text{(see [8, 9])}. \tag{10}$$

When $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the q-Bernoulli numbers.

From (3) and (10), we have

$$B_{0,q} = 1, (qB_q + 1)^n - B_{n,q} = \begin{cases} \frac{\log q}{q-1} & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{11}$$

with the usual convention about replacing B_q^n by $B_{n,q}$.

By (10), we easily get

$$B_{n,q}(x) = (q^x B_q + [x]_q)^n = \sum_{l=0}^{n}\binom{n}{l}[x]_q^{n-l}q^{lx}B_{l,q},\ \text{(see [9])}. \tag{12}$$

As is known, the p-adic q-Bernstein operator is given by

$$\mathbb{B}_{n,q}(f|x) = \sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)[x]_q^k[1-x]_{q^{-1}}^{n-k} = \sum_{k=0}^{n}f\left(\frac{k}{n}\right)B_{k,n}(x|q),$$

where $n, k \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{Z}_p$, and f is continuous function on $\mathbb{Z}_p$(see [7]). Here

$$B_{k,n}(x|q) = \binom{n}{k}[x]_q^k[1-x]_{q^{-1}}^{n-k},\ (n, k \geq 0),$$

are called the p-adic q-Bernstein polynomials of degree n(see [7]). Note that $\lim_{q\to 1} B_{k,n}(x|q) = B_{k,n}(x)$, where $B_{k,n}$ is Bernstein polynomials(see [1, 2, 13-16]).

In this paper, we study the p-adic integral representation on $\mathbb{Z}_p$ of q-Bernoulli numbers arising from two variable q-Bernstein polynomials and investigate some properties for the q-Bernoulli numbers. In addition, we give some new identities of q-Bernoulli numbers.

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Peer-reviewed version available at Symmetry 2018, 10, 451; doi:10.3390/sym10100451

2. Some integral representations of q-Bernoulli numbers and polynomials

First, we consider two variable q-Bernstein operator of order n which is given by

$$\mathbb{B}_{n,q}(f|x_1,x_2)=\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)B_{k,n}(x_1,x_2|q),$$

where $n,k\in\mathbb{N}\cup\{0\}$ and $x_1,x_2\in\mathbb{Z}_p$.

Here,

$$B_{k,n}(x_1,x_2|q)=\binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k},\ (\text{see [6, 7]}),\tag{13}$$

are called two variable q-Bernstein polynomials of degree n.

In (13), if $x_1=x_2=x$, then $B_{k,n}(x,x|q)=B_{k,n}(x|q)$ are the q-Bernstein polynomials.

It is not difficult to show that the generating function of $B_{k,n}(x_1,x_2|q)$ is given by

$$F_q^{(k)}(x_1,x_2|t)=\frac{(t[x_1]_q)^k}{k!}e^{(t[1-x_2]_{q^{-1}})}=\sum_{n=k}^{\infty}B_{k,n}(x_1,x_2|q)\frac{t^n}{n!},\tag{14}$$

where $k\in\mathbb{N}\cup\{0\}$(see [6, 7]).

Thus, by (14), we get

$$B_{k,n}(x_1,x_2|q)=\begin{cases}\binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}, & \text{if } n\geq k,\\ 0, & \text{if } n<k,\end{cases}\tag{15}$$

where $n,k\in\mathbb{N}\cup\{0\}$

By (15), we easily get

$$B_{n-k,n}(1-x_2,1-x_1|q^{-1})=B_{k,n}(x_1,x_2|q),\ (n,k\in\mathbb{N}\cup\{0\}).$$

For $1\leq k\leq n$, we have the following properties(see [6, 7]):

$$[1-x_2]_{q^{-1}}B_{k,n-1}(x_1,x_2|q)+[x_1]_qB_{k-1,n-1}(x_1,x_2|q)=B_{k,n}(x_1,x_2|q),\tag{16}$$

$$\frac{\partial}{\partial x_1}B_{k,n}(x_1,x_2|q)=\frac{\log q}{q-1}n\left((q-1)[x_1]_qB_{k-1,n-1}(x_1,x_2|q)+B_{k-1,n-1}(x_1,x_2|q)\right),\tag{17}$$

$$\frac{\partial}{\partial x_2}B_{k,n}(x_1,x_2|q)=\frac{\log q}{1-q}n((q-1)[x_2]_qB_{k,n-1}(x_1,x_2|q)+B_{k,n-1}(x_1,x_2|q)).\tag{18}$$

From (13) and q-Bernstein operator, we note that

$$\mathbb{B}_{n,q}(1|x_1,x_2)=(1+[x_1]_q-[x_2]_q)^n,$$

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$$\mathbb{B}_{n,q}(t|x_1,x_2) = [x_1]_q \sum_{k=0}^{n-1} \binom{n-1}{k} [x_1]_q^k [1-x_2]_{q^{-1}}^{n-k-1},$$

$$\mathbb{B}_{n,q}(t^2|x_1,x_2) = \frac{n-1}{n}[x_1]_q^2(1+[x_1]_q-[x_2]_q)^{n-2} + \frac{[n]_q}{n}(1+[x_1]_q-[x_2]_q)^{n-1},$$

and

$$\mathbb{B}_{n,q}(f|x_1,x_2) = \sum_{l=0}^{n} \binom{n}{l} [x_2]_q^l \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} f\left(\frac{k}{n}\right) \left(\frac{[x_1]_q}{[x_2]_q}\right)^k, \tag{19}$$

where $n \in \mathbb{N} \cup \{0\}$ and f is continuous on $\mathbb{Z}_p$.

It is easy to show that

$$\frac{1}{(1+[x_1]_q-[x_2]_q)^{n-j}} \sum_{k=j}^{n} \frac{\binom{k}{j}}{\binom{n}{j}} B_{k,n}(x_1,x_2|q) = [x_1]_q^j, \tag{20}$$

where $j \in \mathbb{N} \cup \{0\}$ and $x_1, x_2 \in \mathbb{Z}_p$.

From (2), we have

$$\int_{\mathbb{Z}_p} [1-x+y]_{q^{-1}}^n d\mu_1(y) = (-1)^n q^n \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_1(y), \ (n \geq 0). \tag{21}$$

By (10) and (21), we get

$$B_{n,q^{-1}}(1-x) = (-1)^n q^n B_{n,q}(x), \ (n \geq 0). \tag{22}$$

Again, from (11) and (12), we can derive the following equation.

$$B_{n,q}(2) = nq\frac{\log q}{q-1} + (qB_q+1)^n = nq\frac{\log q}{q-1} + B_{n,q}, \ (n > 1). \tag{23}$$

Thus, by (23), we obtain the following lemma.

Lemma 2.1. *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$B_{n,q}(2) = nq\frac{\log q}{q-1} + B_{n,q}.$$

By (2), (10), and (22), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} [1-x]_{q^{-1}}^n d\mu_1(x) &= (-1)^n q^n \int_{\mathbb{Z}_p} [x-1]_q^n d\mu_1(x) \\ &= (-1)^n q^n B_{n,q}(-1) \\ &= B_{n,q^{-1}}(2). \end{aligned} \tag{24}$$

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For $n \in \mathbb{N}$ with $n > 1$, by (21), Lemma 2.1, and (24), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}[1-x]_{q^{-1}}^n d\mu_1(x) &= \int_{\mathbb{Z}_p}[x+2]_{q^{-1}}^n d\mu_1(x)\\ &= (-1)^n q^n \int_{\mathbb{Z}_p}[x-1]_q^n d\mu_1(x)\\ &= n\frac{\log q}{q-1} + \int_{\mathbb{Z}_p}[x]_{q^{-1}}^n d\mu_1(x)\\ &= \frac{n\log q}{q-1} + B_{n,q^{-1}}.\end{aligned} \tag{25}$$

Let us take the double p-adic integral on $\mathbb{Z}_p$ for the two variable q-Bernstein polynomials. Then we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)\\ &= \binom{n}{k}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_1)d\mu_1(x_2)\\ &= \binom{n}{k}B_{k,q}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_2)\\ &= \begin{cases}\binom{n}{k}B_{k,q}(B_{n-k,q^{-1}} + \frac{\log q}{q-1}(n-k)), & \text{if } n > k+1,\\ (k+1)B_{k,q}B_{1,q^{-1}}(2), & \text{if } n = k+1,\\ B_{k,q}, & \text{if } n = k,\\ 1, & \text{if } n = k = 0.\end{cases}\end{aligned} \tag{26}$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For $n, k \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)\\ &= \begin{cases}\binom{n}{k}B_{k,q}(B_{n-k,q^{-1}} + \frac{\log q}{q-1}(n-k)), & \text{if } n > k+1,\\ (k+1)B_{k,q}B_{1,q^{-1}}(2), & \text{if } n = k+1,\\ B_{k,q} & \text{if } n = k,\\ 1, & \text{if } n = k = 0.\end{cases}\end{aligned}$$

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For $n,k \in \mathbb{N}\cup\{0\}$, we have

$$
\begin{aligned}
&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)\\
&=\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\binom{n}{k}[x_1]_q^k[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_1)d\mu_1(x_2)\\
&=\binom{n}{k}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}(1-[1-x_1]_{q^{-1}})^k[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_1)d\mu_1(x_2)\\
&=\binom{n}{k}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{k-l}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_1)d\mu_1(x_2)\\
&=\binom{n}{k}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_2)\\
&\times\left\{1-k\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_1(x_1)+\sum_{l=0}^{k-2}\binom{k}{l}(-1)^{k-l}\left((k-l)\frac{\log q}{q-1}+B_{k-l,q^{-1}}\right)\right\}.
\end{aligned}
\tag{27}
$$

Thus, by (27), we get

$$
\begin{aligned}
&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_2)}\\
&=\binom{n}{k}\left(1-k\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_1(x_1)+\sum_{l=0}^{k-2}\binom{k}{l}(-1)^{k-l}\left((k-l)\frac{\log q}{q-1}+B_{k-l,q^{-1}}\right)\right)\\
&=\binom{n}{k}\left(1-k\frac{\log q-q+1}{(q-1)^2}+k\sum_{l=0}^{k-2}\binom{k-1}{l}(-1)^{k-l}\frac{\log q}{q-1}+\sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}}\right)\\
&=\binom{n}{k}\left(1-k\frac{\log q-q+1}{(q-1)^2}+k\frac{\log q}{q-1}+\sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}}\right)\\
&=\binom{n}{k}\left(1-k\left(\frac{2\log q-q-q\log q+1}{(q-1)^2}\right)+\sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}}\right).
\end{aligned}
\tag{28}
$$

Therefore, by (28), we obtain the following theorem.

Theorem 2.3. *For $n,k \in \mathbb{N}\cup\{0\}$ with $k>1$, we have*

$$
\begin{aligned}
&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\binom{n}{k}\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n-k}d\mu_1(x_2)}\\
&=\binom{n}{k}\left(1-k\left(\frac{2\log q-q-q\log q+1}{(q-1)^2}\right)+\sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}}\right).
\end{aligned}
$$

Therefore, by Theorem 2.2 and Theorem 2.3, we obtain the following corollary.

Corollary 2.4. *For $k \in \mathbb{N}$ with $k > 1$, we have*

$$B_{k,q} = 1 - k\left(\frac{2\log q - q\log q - q + 1}{(q-1)^2}\right) + \sum_{l=0}^{k-2}(-1)^{k-l}\binom{k}{l}B_{k-l,q^{-1}}.$$

For $m, n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)\\ &= \binom{n}{k}\binom{m}{k}\int_{\mathbb{Z}_p}[x_1]_q^{2k}d\mu_1(x_1)\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_2).\end{aligned} \tag{29}$$

Thus, by (29), we get

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_2)}\\ &= \binom{n}{k}\binom{m}{k}B_{2k,q}.\end{aligned}$$

Hence, we have the following proposition.

Proposition 2.5. *For $m, n, k \in \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_2)}\\ &= \binom{n}{k}\binom{m}{k}B_{2k,q}.\end{aligned}$$

Let $m, n, k \in \mathbb{N} \cup \{0\}$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)\\ &= \sum_{l=0}^{2k}\binom{n}{k}\binom{m}{k}\binom{2k}{l}(-1)^{2k-l}\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{2k-l}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_1)d\mu_1(x_2)\end{aligned} \tag{30}$$

Thus, from (30), we have

$$\begin{aligned}
&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_2)}\\
&=\binom{n}{k}\binom{m}{k}\left(1-2k\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_1(x_1)+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{2k-l}d\mu_1(x_1)\right)\\
&=\binom{n}{k}\binom{m}{k}\left(1-2k\frac{\log q-q+1}{(q-1)^2}+2k\frac{\log q}{q-1}+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}B_{2k-l,q^{-1}}\right)\\
&=\binom{n}{k}\binom{m}{k}\left(1-2k\left(\frac{2\log q-q\log q-q+1}{(q-1)^2}\right)+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}B_{2k-l,q^{-1}}\right).
\end{aligned} \tag{31}$$

By (31), we have the following proposition.

Proposition 2.6. *For $m,n,k\in\mathbb{N}\cup\{0\}$, we have*

$$\begin{aligned}
&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p} B_{k,n}(x_1,x_2|q)B_{k,m}(x_1,x_2|q)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n+m-2k}d\mu_1(x_2)}\\
&=\binom{n}{k}\binom{m}{k}\left(1-2k\left(\frac{2\log q-q\log q-q+1}{(q-1)^2}\right)+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}B_{2k-l,q^{-1}}\right).
\end{aligned}$$

Therefore, by Proposition 2.5 and Proposition 2.6, we obtain the following corollary.

Corollary 2.7. *For $k\in\mathbb{N}$, we have*

$$B_{2k,q}=1-2k\left(\frac{\log q-q+1}{(q-1)^2}\right)+\sum_{l=0}^{2k-2}\binom{2k}{l}(-1)^{2k-l}\left((2k-l)\frac{\log q}{q-1}+B_{2k-l,q^{-1}}\right).$$

For $m\in\mathbb{N}$, let $n_1,n_2,\cdots,n_m,k\in\mathbb{N}\cup\{0\}$. Then we note that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m}B_{k,n_i}(x_1,x_2|q)\right)d\mu_1(x_1)d\mu_1(x_2)\\
&=\sum_{l=0}^{mk}\left(\prod_{i=1}^{m}\binom{n_i}{k}\right)\binom{mk}{l}(-1)^{mk-l}\\
&\times\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{mk-l}[1-x_2]_{q^{-1}}^{n_1+n_2+\cdots+n_m-mk}d\mu_1(x_1)d\mu_1(x_2)
\end{aligned} \tag{32}$$

Thus, by (32), we have

$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m} B_{k,n_i}(x_1,x_2|q)\right)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n_1+n_2+\cdots+n_m-mk}d\mu_1(x_2)}$$

$$=\left(\prod_{i=1}^{m}\binom{n_i}{k}\right)\sum_{l=0}^{mk}\binom{mk}{l}(-1)^{mk-l}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{mk-l}d\mu_1(x_1)$$

$$=\left(\prod_{i=1}^{m}\binom{n_i}{k}\right)\left(1-mk\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}d\mu_1(x_1)+\sum_{l=0}^{mk-2}\binom{mk}{l}(-1)^{mk-l}\int_{\mathbb{Z}_p}[1-x_1]_{q^{-1}}^{mk-l}d\mu_1(x_1)\right)$$

$$=\left(\prod_{i=1}^{m}\binom{n_i}{k}\right)\left(1-mk\frac{\log q-q+1}{(q-1)^2}+\sum_{l=0}^{mk-2}\binom{mk}{l}(-1)^{mk-l}\left((mk-l)\frac{\log q}{q-1}+B_{mk-l,q^{-1}}\right)\right).$$

Therefore we obtain the following theorem.

Theorem 2.8. *For $n_1, n_2, \cdots, n_m, k \in \mathbb{N}\cup\{0\}$ and $m\in\mathbb{N}$, we have*

$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m} B_{k,n_i}(x_1,x_2|q)\right)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n_1+n_2+\cdots+n_m-mk}d\mu_1(x_2)}$$

$$=\left(\prod_{i=1}^{m}\binom{n_i}{k}\right)\left(1-mk\left(\frac{\log q-q+1}{(q-1)^2}\right)+\sum_{l=0}^{2k-2}\binom{mk}{l}(-1)^{mk-l}\left((mk-l)\frac{\log q}{q-1}+B_{mk-l,q^{-1}}\right)\right).$$

On the other hand, we easily get

$$\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m} B_{k,n_i}(x_1,x_2|q)\right)d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}[1-x_2]_{q^{-1}}^{n_1+n_2+\cdots+n_m-mk}d\mu_1(x_2)}=\prod_{i=1}^{m}\binom{n_i}{k}B_{mk,q}. \tag{33}$$

Therefore, by Theorem 2.8 and (33), we obtain the following corollary.

Corollary 2.9. *For $m, k\in\mathbb{N}$ with $k>1$, we have*

$$B_{mk,q}=1-mk\left(\frac{\log q-q+1}{(q-1)^2}\right)+\sum_{l=0}^{mk-2}\binom{mk}{l}(-1)^{mk-l}\left((mk-l)\frac{\log q}{q-1}+B_{mk-l,q^{-1}}\right).$$

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