ON $p$-ADIC INTEGRAL REPRESENTATION OF $q$-BERNOULLI NUMBERS ARISING FROM TWO VARIABLE $q$-BERNSTEIN POLYNOMIALS

C.S. RYOO$^1$, T. KIM$^2$, D.S. KIM$^3$ AND Y. YAO$^4$

$^1$Department of Mathematics, Hannam University, Daejeon 306-791, Korea
$^2$Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China
$^3$Department of Mathematics, Kwangwoon University, Seoul 139-791, Korea
$^4$Department of Mathematics, Sogang University, Seoul 121-742, Korea

Abstract. In this paper, we study the $p$-adic integral representation on $\mathbb{Z}_p$ of $q$-Bernoulli numbers arising from two variable $q$-Bernstein polynomials and investigate some properties for the $q$-Bernoulli numbers. In addition, we give some new identities of $q$-Bernoulli numbers.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80, 33C45.

Key words and phrases : $q$-Bernoulli numbers, $q$-Bernoulli polynomials, Bernstein polynomials, $q$-Bernstein polynomials, $p$-adic integral on $\mathbb{Z}_p$.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{N}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the set of natural numbers, the ring of $p$-adic integers, the field of $p$-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. The $p$-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Assume that $q$ is an indeterminate in $\mathbb{C}_p$ such that $|1 - q|_p < p^{-\frac{1}{p^\infty}}$.

$^*$Corresponding author.

$^\dagger$This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).
It is known that the q-number is defined by
\[ [x]_q = \frac{1 - q^x}{1 - q}. \]
Note that \( \lim_{q \to 1} [x]_q = x \). Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable function on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by Kim to be
\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)q^x, \quad \text{(see [9, 10]).} \tag{1}
\]
As \( q \to 1 \) in (1), we have the \( p \)-adic integral on \( \mathbb{Z}_p \) which is given by
\[
I_1(f) = \lim_{q \to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [7-12]).} \tag{2}
\]
From (2), we note that
\[
I_1(f_1) - I_1(f) = f'(0), \quad \text{(see [9]),} \tag{3}
\]
where \( f_1(x) = f(x+1) \) and \( f'(0) = \frac{df(x)}{dx}|_{x=0} \).

Thus, by (3), we get
\[
\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^\infty B_n(x) \frac{t^n}{n!}, \quad \text{(see [6, 9]),} \tag{4}
\]
where \( B_n(x) \) are ordinary Bernoulli polynomials.

From (4), we note that
\[
\int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y) = B_n(x), \quad (n \geq 0), \quad \text{(see [7-12, 15]).} \tag{5}
\]
When \( x = 0 \), \( B_n = B_n(0) \), \( (n \geq 0) \), are called the ordinary Bernoulli numbers.

The equation (3) implies the following recurrence relation for Bernoulli numbers:
\[
B_0 = 1, \quad (B + 1)^n - B_n = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{6}
\]
with the usual convention about replacing \( B^n \) by \( B_n \).

In [3, 4, 5], L. Carlitz introduced the \( q \)-Bernoulli numbers which are given by the recurrence relation to be
\[
\beta_{0,q} = 1, \quad q(q\beta_q + 1)^n - \beta_{n,q} = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \tag{7}
\]
with the usual convention about replacing \( \beta_q^n \) by \( \beta_{n,q} \).
On $p$-adic integral representation of $q$-Bernoulli numbers

He also defined $q$-Bernoulli polynomials as

$$\beta_{n,q}(x) = (q^x \beta_q + [x]_q)_n = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^l x \beta_{l,q}, \text{ (see [3, 4, 5]).} \quad (8)$$

In 1999, Kim proved the following formula.

$$\int_{\mathbb{Z}_p} [x + y]_q^n d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0), \text{ (see [10]).} \quad (9)$$

In the viewpoint of (5), we define the $q$-Bernoulli polynomials which are different from Carlitz’s $q$-Bernoulli polynomials to be

$$B_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_1(y), \quad (n \geq 0), \text{ (see [8, 9]).} \quad (10)$$

When $x = 0$, $B_{n,q} = B_{n,q}(0)$ are called the $q$-Bernoulli numbers.

From (3) and (10), we have

$$B_{0,q} = 1, (qB_q + 1)^n - B_{n,q} = \begin{cases} \log q & \text{if } n = 1, \\ 0 & \text{if } n > 1, \end{cases} \quad (11)$$

with the usual convention about replacing $B_q^n$ by $B_{n,q}$.

By (10), we easily get

$$B_{n,q}(x) = (q^x B_q + [x]_q)_n = \sum_{l=0}^{n} \binom{n}{l} [x]_q^{n-l} q^l x B_{l,q}, \text{ (see [9]).} \quad (12)$$

As is known, the $p$-adic $q$-Bernstein operator is given by

$$B_{n,q}(f|x) = \sum_{k=0}^{n} \binom{n}{k} f \left( \frac{k}{n} \right) [x]_q^k [1 - x]_q^{n-k} = \sum_{k=0}^{n} \frac{k}{n} B_{k,n}(x|q),$$

where $n, k \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{Z}_p$, and $f$ is continuous function on $\mathbb{Z}_p$ (see [7]). Here

$$B_{k,n}(x|q) = \binom{n}{k} [x]_q^k [1 - x]_q^{n-k}, \quad (n, k \geq 0),$$

are called the $p$-adic $q$-Bernstein polynomials of degree $n$ (see [7]). Note that

$$\lim_{q \to 1} B_{k,n}(x|q) = B_{k,n}(x),$$

where $B_{k,n}$ is Bernstein polynomials (see [1, 2, 13-16]).

In this paper, we study the $p$-adic integral representation on $\mathbb{Z}_p$ of $q$-Bernoulli numbers arising from two variable $q$-Bernstein polynomials and investigate some properties for the $q$-Bernoulli numbers. In addition, we give some new identities of $q$-Bernoulli numbers.
2. Some integral representations of $q$-Bernoulli numbers and polynomials

First, we consider two variable $q$-Bernstein operator of order $n$ which is given by
\[
\mathcal{B}_{n,q}(f[x_1,x_2]) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) [x_1]^k_q [1-x_2]^{n-k}_{q^{-1}} = \sum_{k=0}^{n} \binom{k}{n} B_{k,n}(x_1,x_2|q),
\]
where $n, k \in \mathbb{N} \cup \{0\}$ and $x_1, x_2 \in \mathbb{Z}_p$.

Here,
\[
B_{k,n}(x_1,x_2) = \binom{n}{k} [x_1]^k_q [1-x_2]^{n-k}_{q^{-1}}, \quad \text{(see [6, 7])},
\]
are called two variable $q$-Bernstein polynomials of degree $n$.

In (13), if $x_1 = x_2 = x$, then $B_{k,n}(x,x|q) = B_{k,n}(x|q)$ are the $q$-Bernstein polynomials.

It is not difficult to show that the generating function of $B_{k,n}(x_1,x_2|q)$ is given by
\[
F_q^{(k)}(x_1,x_2|t) = \frac{(t[1]_q^k)^k}{k!} e^{(t[1]_q^2 - 1)} = \sum_{n=0}^{\infty} B_{n,k,n}(x_1,x_2|q) \frac{t^n}{n!},
\]
where $k \in \mathbb{N} \cup \{0\}$ (see [6, 7]).

Thus, by (14), we get
\[
B_{n,k,n}(x_1,x_2|q) = \begin{cases} \binom{n}{k} [x_1]^k_q [1-x_2]^{n-k}_{q^{-1}}, & \text{if } n \geq k, \\ 0, & \text{if } n < k, \end{cases}
\]
where $n, k \in \mathbb{N} \cup \{0\}$.

By (15), we easily get
\[
B_{n-k,n}(1-x_2,1-x_1|q^{-1}) = B_{k,n}(x_1,x_2|q), \quad (n, k \in \mathbb{N} \cup \{0\}).
\]

For $1 \leq k \leq n$, we have the following properties (see [6, 7]):
\[
[1-x_2]^{n-k}_{q^{-1}} B_{n-k-1,n-1}(x_1,x_2|q) + [x_1]^q B_{k-1,n-1}(x_1,x_2|q) = B_k(n,x_1,x_2|q),
\]
where $n, k \in \mathbb{N} \cup \{0\}$.

\[
\frac{\partial}{\partial x_1} B_{k,n}(x_1,x_2|q) = \frac{\log q}{q-1} \left((q-1)[x_1]_q B_{k-1,n-1}(x_1,x_2|q) + B_{k-1,n-1}(x_1,x_2|q)\right),
\]
where $n, k \in \mathbb{N} \cup \{0\}$.

\[
\frac{\partial}{\partial x_2} B_{k,n}(x_1,x_2|q) = \frac{\log q}{1-q} \left((q-1)[x_2]_q B_{k,n-1}(x_1,x_2|q) + B_{k,n-1}(x_1,x_2|q)\right).
\]
From (13) and $q$-Bernstein operator, we note that
\[
\mathcal{B}_{n,q}(1|x_1,x_2) = (1 + [x_1]_q - [x_2]_q)^n,
\]
On $p$-adic integral representation of $q$-Bernoulli numbers

\[ B_{n,q}(t|x_1, x_2) = [x_1]_q \sum_{k=0}^{n-1} \binom{n-1}{k} [x_1]_q^k [1 - x_2]_q^{n-k-1}, \]

\[ B_{n,q}(t^2|x_1, x_2) = \frac{n-1}{n} [x_1]_q^2 (1 + [x_1]_q - [x_2]_q)^{n-2} + \frac{[n]_q}{n} (1 + [x_1]_q - [x_2]_q)^{n-1}, \]

and

\[ B_{n,q}(f|x_1, x_2) = \sum_{l=0}^{n} \binom{n}{l} [x_2]_q^l \sum_{k=0}^{l} \binom{l}{k} (-1)^{l-k} f \left( \frac{k}{n} \right) \left( \frac{x_1}{x_2} \right)^{\frac{k}{n}}, \tag{19} \]

where $n \in \mathbb{N} \cup \{0\}$ and $f$ is continuous on $\mathbb{Z}_p$.

It is easy to show that

\[ \frac{1}{(1 + [x_1]_q - [x_2]_q)^{n-j}} \sum_{k=0}^{n} \binom{n}{k} B_{k,n}(x_1, x_2|q) = [x_1]_q^j, \tag{20} \]

where $j \in \mathbb{N} \cup \{0\}$ and $x_1, x_2 \in \mathbb{Z}_p$.

From (2), we have

\[ \int_{\mathbb{Z}_p} [1 + x + y]_q^{n-1} d\mu_1(y) = (-1)^n q^n \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_1(y), \quad (n \geq 0). \tag{21} \]

By (10) and (21), we get

\[ B_{n,q}^{-1}(1 - x) = (-1)^n q^n B_{n,q}(x), \quad (n \geq 0). \tag{22} \]

Again, from (11) and (12), we can derive the following equation.

\[ B_{n,q}(2) = n q \log \frac{q}{q - 1} + (qB_q + 1)^n = n q \log \frac{q}{q - 1} + B_{n,q}, \quad (n > 1). \tag{23} \]

Thus, by (23), we obtain the following lemma.

**Lemma 2.1.** For $n \in \mathbb{N}$ with $n > 1$, we have

\[ B_{n,q}(2) = n q \log \frac{q}{q - 1} + B_{n,q}. \]

By (2), (10), and (22), we get

\[ \int_{\mathbb{Z}_p} [1 - x]_q^{n-1} d\mu_1(x) = (-1)^n q^n \int_{\mathbb{Z}_p} [x - 1]_q^n d\mu_1(x) \]

\[ = (-1)^n q^n B_{n,q}(-1) \]

\[ = B_{n,q}^{-1}(2). \tag{24} \]
For $n \in \mathbb{N}$ with $n > 1$, by (21), Lemma 2.1, and (24), we have

$$
\int_{\mathbb{Z}_p} [1 - x]_{q^{-1}}^{n-1} d\mu_1(x) = \int_{\mathbb{Z}_p} [x + 2]_{q^{-1}}^{n} d\mu_1(x)
= (-1)^n q^n \int_{\mathbb{Z}_p} [x - 1]_{q^{-1}}^{n} d\mu_1(x)
= \frac{n \log q}{q - 1} + \int_{\mathbb{Z}_p} [x]_{q^{-1}}^{n} d\mu_1(x)
= \frac{n \log q}{q - 1} + B_{n,q^{-1}}.
$$

Let us take the double $p$-adic integral on $\mathbb{Z}_p$ for the two variable $q$-Bernstein polynomials. Then we have

$$
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2) d\mu_1(x_1) d\mu_1(x_2)
= \binom{n}{k} \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} [x_1]_{q}^{k} [1 - x_2]_{q^{-1}}^{n-k} d\mu_1(x_1) d\mu_1(x_2)
= \binom{n}{k} B_{k,q} \int_{\mathbb{Z}_p} [1 - x_2]_{q^{-1}}^{n-k} d\mu_1(x_2)
= \begin{cases} 
\binom{n}{k} B_{k,q} (B_{n-k,q^{-1}} + \frac{\log q}{q - 1} (n - k)), & \text{if } n > k + 1, \\
(k + 1) B_{k,q} B_{1,q^{-1}}(2), & \text{if } n = k + 1, \\
B_{k,q}, & \text{if } n = k, \\
1, & \text{if } n = k = 0.
\end{cases}
$$

Therefore, we obtain the following theorem.

**Theorem 2.2.** For $n, k \in \mathbb{N} \cup \{0\}$, we have

$$
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2) d\mu_1(x_1) d\mu_1(x_2)
= \binom{n}{k} B_{k,q} (B_{n-k,q^{-1}} + \frac{\log q}{q - 1} (n - k)),
= \begin{cases} 
\binom{n}{k} B_{k,q} (B_{n-k,q^{-1}} + \frac{\log q}{q - 1} (n - k)), & \text{if } n > k + 1, \\
(k + 1) B_{k,q} B_{1,q^{-1}}(2), & \text{if } n = k + 1, \\
B_{k,q}, & \text{if } n = k, \\
1, & \text{if } n = k = 0.
\end{cases}
$$
Therefore, by (28), we obtain the following theorem.

**Theorem 2.3.** For $n, k \in \mathbb{N} \cup \{0\}$ with $k > 1$, we have

\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) d\mu_1(x_1) d\mu_1(x_2)
= \left( \frac{n}{k} \right) \int_{\mathbb{Z}_p} [1 - x_1]_{q^{-1}}^{n-k} d\mu_1(x_2)
\leq \left( \frac{n}{k} \right) \int_{\mathbb{Z}_p} \left( 1 - k \int_{\mathbb{Z}_p} [1 - x_1]_{q^{-1}} d\mu_1(x_1) + \sum_{l=0}^{k-2} \left( \frac{k}{l} \right) (-1)^{k-l} \left( \left( k - l \right) \frac{\log q}{q-1} + B_{k-l,q^{-1}} \right) \right).
\]

Thus, by (27), we get

\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) d\mu_1(x_1) d\mu_1(x_2)
= \left( \frac{n}{k} \right) \int_{\mathbb{Z}_p} \left( 1 - k \int_{\mathbb{Z}_p} [1 - x_1]_{q^{-1}} d\mu_1(x_1) + \sum_{l=0}^{k-2} \left( \frac{k}{l} \right) (-1)^{k-l} \left( \left( k - l \right) \frac{\log q}{q-1} + B_{k-l,q^{-1}} \right) \right).
\]

Therefore, by (28), we obtain the following corollary.
Corollary 2.4. For \( k \in \mathbb{N} \) with \( k > 1 \), we have

\[
B_{k,q} = 1 - k \left( \frac{2 \log q - q \log q - 1}{(q-1)^2} \right) + \sum_{l=0}^{k-2} (-1)^{k-l} \binom{k}{l} B_{k-l,q-1}.
\]

For \( m, n \in \mathbb{N} \cup \{0\} \), we have

\[
\int_{Z_p} \int_{Z_p} B_{k,n}(x_1, x_2|q) B_{k,m}(x_1, x_2|q) d\mu_1(x_1) d\mu_1(x_2) = \binom{n}{k} \binom{m}{k} \int_{Z_p} [1 - x_2]^{n+m-2k} d\mu_1(x_2).
\]

Thus, by (29), we get

\[
\int_{Z_p} \int_{Z_p} B_{k,n}(x_1, x_2|q) B_{k,m}(x_1, x_2|q) d\mu_1(x_1) d\mu_1(x_2) = \binom{n}{k} \binom{m}{k} B_{2k,q}.
\]

Hence, we have the following proposition.

Proposition 2.5. For \( m, n, k \in \mathbb{N} \cup \{0\} \), we have

\[
\int_{Z_p} \int_{Z_p} B_{k,n}(x_1, x_2|q) B_{k,m}(x_1, x_2|q) d\mu_1(x_1) d\mu_1(x_2) = \binom{n}{k} \binom{m}{k} B_{2k,q}.
\]

Let \( m, n, k \in \mathbb{N} \cup \{0\} \). Then we get

\[
\int_{Z_p} \int_{Z_p} B_{k,n}(x_1, x_2|q) B_{k,m}(x_1, x_2|q) d\mu_1(x_1) d\mu_1(x_2) = \sum_{l=0}^{2k} \binom{n}{k} \binom{m}{k} \binom{2k}{l} (-1)^{2k-l} \int_{Z_p} \int_{Z_p} [1 - x_1]^{2k-l} [1 - x_2]^{n+m-2k} d\mu_1(x_1) d\mu_1(x_2)
\]

(30)
Thus, from (30), we have
\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) B_{k,m}(x_1, x_2|q) d\mu_1(x_1) d\mu_1(x_2) 
= \left( \frac{n}{k} \right) \left( \frac{m}{k} \right) \left( 1 - 2k \int_{\mathbb{Z}_p} [1 - x_1|q^{-1}] d\mu_1(x_1) + \sum_{l=0}^{2k-2} \binom{2k}{l} (-1)^{2k-l} \int_{\mathbb{Z}_p} [1 - x_1^{2k-l}|q^{-1}] d\mu_1(x_1) \right) 
= \left( \frac{n}{k} \right) \left( \frac{m}{k} \right) \left( 1 - 2k \left( \log q - q + 1 \right) \left( 1 - \frac{q}{q-1} \right) \right) + \sum_{l=0}^{2k-2} \binom{2k}{l} (-1)^{2k-l} B_{2k-l, q^{-1}}. 
\]
(31)

By (31), we have the following proposition.

**Proposition 2.6.** For \( m, n, k \in \mathbb{N} \cup \{0\} \), we have
\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} B_{k,n}(x_1, x_2|q) B_{k,m}(x_1, x_2|q) d\mu_1(x_1) d\mu_1(x_2) 
= \left( \frac{n}{k} \right) \left( \frac{m}{k} \right) \left( 1 - 2k \left( \log q - q + 1 \right) \left( 1 - \frac{q}{q-1} \right) \right) + \sum_{l=0}^{2k-2} \binom{2k}{l} (-1)^{2k-l} B_{2k-l, q^{-1}}. 
\]

Therefore, by Proposition 2.5 and Proposition 2.6, we obtain the following corollary.

**Corollary 2.7.** For \( k \in \mathbb{N} \), we have
\[
B_{2k,q} = 1 - 2k \left( \frac{\log q - q + 1}{(q-1)^2} \right) + \sum_{l=0}^{2k-2} \binom{2k}{l} (-1)^{2k-l} \left( \frac{2k}{q} \right) \left( 1 - \frac{q}{q-1} \right) + B_{2k-l, q^{-1}}. 
\]

For \( m \in \mathbb{N} \), let \( n_1, n_2, \ldots, n_m, k \in \mathbb{N} \cup \{0\} \). Then we note that
\[
\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \left( \prod_{i=1}^{m} B_{k,n_i}(x_1, x_2|q) \right) d\mu_1(x_1) d\mu_1(x_2) 
= \sum_{l=0}^{mk} \binom{m}{k} \left( \binom{n_i}{k} \left( \frac{mk-l}{l} \right) (-1)^{mk-l} \right) 
\times \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} [1 - x_1|x_{1}^{-1}] [1 - x_2|x_{2}^{-1}]^{n_1+n_2+\cdots+n_m} d\mu_1(x_1) d\mu_1(x_2) 
\]
(32)
Thus, by (32), we have
\[
\frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (\prod_{i=1}^{m} B_{k,n_i}(x_1,x_2[q])) \, d\mu_1(x_1) \, d\mu_1(x_2)}{\int_{\mathbb{Z}_p} [1-x_2]^{n_1+n_2+\cdots+n_m-mk} \, d\mu_1(x_2)}
= \left( \frac{m}{k} \right) \sum_{l=0}^{mk-1} (-1)^{mk-l} \int_{\mathbb{Z}_p} [1-x_1]^{mk-l} \, d\mu_1(x_1)
= \left( \frac{m}{k} \right) \left( 1 - mk \int_{\mathbb{Z}_p} [1-x_1] \, d\mu_1(x_1) + \sum_{l=0}^{mk-2} \left( \frac{mk}{l} \right) (-1)^{mk-l} \int_{\mathbb{Z}_p} [1-x_1]^{mk-l} \, d\mu_1(x_1) \right)
= \left( \frac{m}{k} \right) \left[ 1 - mk \log q - q + 1 \right] + \sum_{l=0}^{mk-2} \left( \frac{mk}{l} \right) (-1)^{mk-l} \left( (mk-l) \log q \frac{q}{q-1} + B_{mk-l,q^{-1}} \right).
\]

Therefore we obtain the following theorem.

**Theorem 2.8.** For \( n_1, n_2, \ldots, n_m, k \in \mathbb{N} \cup \{0\} \) and \( m \in \mathbb{N} \), we have
\[
\frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (\prod_{i=1}^{m} B_{k,n_i}(x_1,x_2[q])) \, d\mu_1(x_1) \, d\mu_1(x_2)}{\int_{\mathbb{Z}_p} [1-x_2]^{n_1+n_2+\cdots+n_m-mk} \, d\mu_1(x_2)} = \left( \frac{m}{k} \right) \left( 1 - mk \left( \log q - q + 1 \right) \right) + \sum_{l=0}^{mk-2} \left( \frac{mk}{l} \right) (-1)^{mk-l} \left( (mk-l) \log q \frac{q}{q-1} + B_{mk-l,q^{-1}} \right).
\]

On the other hand, we easily get
\[
\frac{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (\prod_{i=1}^{m} B_{k,n_i}(x_1,x_2[q])) \, d\mu_1(x_1) \, d\mu_1(x_2)}{\int_{\mathbb{Z}_p} [1-x_2]^{n_1+n_2+\cdots+n_m-mk} \, d\mu_1(x_2)} = \prod_{i=1}^{m} \left( \frac{n_i}{k} \right) B_{mk,q}.
\]

Therefore, by Theorem 2.8 and (33), we obtain the following corollary.

**Corollary 2.9.** For \( m, k \in \mathbb{N} \) with \( k > 1 \), we have
\[
B_{mk,q} = 1 - mk \left( \log q - q + 1 \right) \frac{q}{q-1} + \sum_{l=0}^{mk-2} \left( \frac{mk}{l} \right) (-1)^{mk-l} \left( (mk-l) \log q \frac{q}{q-1} + B_{mk-l,q^{-1}} \right).
\]

**References**

On $p$-adic integral representation of $q$-Bernoulli numbers


