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Article

Payoff Distribution in a Multi-Company Extraction Game with Uncertain Duration

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Abstract: A nonrenewable resource extraction game model is analyzed in a differential game theory framework with random duration. If the c.d.f. of the final time is discontinuous, the related subgames are differentiated based on the position of the initial instant with respect to the jump. We investigate properties of optimal trajectories and of imputation distribution procedures if the game is played cooperatively.

Keywords: differential games; non zero-sum games; cooperative games; resource extraction; random duration; IDP procedure.

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1. Introduction

In a differential game of extraction, the standard scenario involves a dynamic competition among players (or, more precisely, companies) which exert effort aimed at extracting a natural resource. If the resource does not regenerate over time, such as natural gas or earth minerals, it is called exhaustible or nonrenewable.

Economic literature has been dealing with effects and characteristics of exhaustible resource extraction since 1817, when David Ricardo [21] addressed the issue in his essay *The principles of political economy and taxation*. In the 20$^{\text{th}}$ century, the debate was relaunched by Harold Hotelling [8], then subsequently a vast stream of static and dynamic models was conceived and developed over the years.

If we only focus on models described through differential games, the basic framework includes a population of companies extracting the same resource, having the extraction effort levels as their strategic variables, which directly affect their respective payoffs, which increase as the extracted quantity increases. On the other hand, the state variables represent the stocks of resources, which are depleted over time by extraction. In the easiest representation, there is a unique resource and all companies aim to pick it up as much as possible. In order to describe a more realistic economic behaviour, a key element was introduced in economic literature: the random duration of the game.

The seminal paper on this extension of the standard optimal control problem is due to Yaari [27] in 1965. In the same time in Russia in 1966 Petrosyan and Murzov [16] first studied differential zero-sum games with terminal payoff at random time horizon. Subsequently, further studies have been provided: in the work of Boukas *et al.* [1] in 1990, an optimal control problem with random duration was studied in general terms. Cooperative differential games with random time horizon were first studied by Petrosyan and Shevkoplyas [17] in 2000, whereas the concept of time consistency in differential games with prescribed duration was introduced in [15].

Such a concept is particularly relevant because most literature treats stability of the cooperative solutions in static cooperative settings. On the other hand, stable cooperation in the problem is a key requirement when the scenario is dynamic as well. In cooperative differential games, cooperating players wish to establish a dynamically stable (time-consistent) cooperative agreement (e.g., the dynamic versions of the Shapley Value, core etc.).

Time consistency implies that, as cooperation evolves, cooperating partners are guided by the same optimality principle at each instant of time and hence do not have any incentive to deviate from the previously adopted cooperative behaviour.

After Petrosyan's seminal paper in 1977, such topic was actively developed by a number of researchers. In a paper by Jorgensen et al. [9] the problem of time-consistency and agreeability of the solution in linear-state class of differential games had been investigated. In a paper by Petrosjan and Zaccour [18] a similar problem of ecological management was studied as well as in the more recent papers by Zaccour [29] and the book by Petrosyan and Yeung [28]. Recently, the notion of time consistency was extended to the case of discrete games, see, e.g., [20]. An extension of the time consistency problem to the case of differential games with random duration was first undertaken in [17], subsequently further investigation and results were accomplished in [11,12,14,24,25]. In [4] a random time horizon hybrid[1] differential game was considered such that the probability distribution can change with time. The present contribution locates itself in this line of research.

In this paper, we intend to propose a description and an analysis of a scenario which differs from the previous treatments: the random variable which indicates the stopping time of extraction has a c.d.f. which is not continuous over the whole time interval. Specifically, we assume that there is a jump at an internal point, and we carry out an analysis which is differentiated based on the initial time of the game, i.e. before or after the jump. This formulation can represent any situation in which the distribution of the random variable is affected by external factors such as a Parliament bill which makes an extraction technique illegal. An example may be provided by the controversial fracking process for gas extraction.

In this setting, standard models take into account an oligopolistic competition among firms, where each firm aims to maximize its own profit. However, there exist some different approaches in literature which also involve the possibility of cooperation among agents.

Because of the depletion of oil and gas resources on the mainland, the active development of oil-and-gas fields on continental shelves are to begin in the near future. Today there are about seventy developing and potential oil-and-gas fields on continental shelves of Azerbaidzhan, Canada, Kazakhstan, Mexico, Norway, Russia, Saudi Arabia, the USA etc. For example, today the firms which are involved in the development of Sakhalin oil-and-gas fields (Russia) are "Gazprom", "Shell", "Mitsui" and "Mitsubishi".

Moreover, the task of oil and gas exploitation in Arctic is a key issue nowadays, especially relevant for Canada, Denmark, Norway, Russia and the USA. We believe that the source of economic success of the development of pool in Arctic should bring about a cooperative collaboration of participating countries. Collaboration in Arctic is important at least in the sense that an accident at one borehole can lead to serious problems or complete stoppage of resource exploitation for all neighbors. Thus, the involved countries have to collaborate to provide security for oil and gas exploitation in Arctic, otherwise environmental disasters and huge economic losses for all participants might occur. This is the main motivation to consider the cooperative form of the non-renewable resource extraction game.

However, despite all above, the oil and gas extraction on a continental shelf is a high-risk economic activity and reconsideration of existing models of non-renewable resource extraction is required. Stochastic framework may be useful in the sense that it increases the validity of models (see, for example, [3]). As usual, game-theoretical models with infinite or fixed time horizon are used for modelling of renewable or exhausted resource exploitation. Although they provide numerous insights for equilibrium and stability, such an approach is not very realistic. Namely, the contract date is never

[1] see also [5] for a general treatment of hybrid differential games

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equal to the real period of field exploitation, because either exploitation is prematurely finished by accident or unprofitability or is the period of exploitation extended.

Here we specifically consider the occurrence of a cooperative game structure, where companies agree on a collective strategy to maximize the aggregate payoff. The agreement establishes that, after maximization, the total payoff is supposed to be redistributed among the cooperating firms. As in standard theory of cooperative games, the distribution of the total worth is the problem to be addressed (see for example [17]). In a differential game, the total worth simply corresponds to the sum of the integral payoffs of all players, and the distribution of the total worth has to be implemented by using a suitable solution concept. Our main focus is on the cooperative setup, where we will describe the determination of an IDP (imputation distribution procedure, which was first introduced by Petrosyan in [15]), which is a dynamic way to attribute players their respective shares gained in the game. We will also determine the relations to explicitly calculate IDPs in the above different cases, also discussing the issue of time consistency. Finally, we outline a complete example where N companies compete over extraction of a unique exhaustible resource, comparing the results in the non-cooperative and cooperative scenarios.

The paper is organized as follows. Section 2 introduces the notation of the game, whose non-cooperative setup is exposed. The cooperative setup is proposed in Section 3, where the main findings, including a theorem which establishes the existence of a time-consistent imputation, are laid out in detail. In Section 4 we propose a model to employ the above-mentioned procedure. Section 5 concludes and proposes some possible future developments.

2. Notation and non-cooperative setup

2.1. Problem Statement

Consider the following standard notation for the N-players differential game $\Gamma^T(t_0, x_0)$, starting at initial time instant t_0 and at initial state x_0:

- $u_{11} \in U_{11}, u_{12} \in U_{12}, \ldots, u_{1M} \in U_{1M}, \ldots, u_{N1} \in U_{N1}, \ldots, u_{NM} \in U_{NM}$ are the extraction effort levels of the N companies involved in pulling out M exhaustible resources. More precisely, u_{ij} is the effort exerted by firm i to extract resource j. The only requirement for the control sets U_{ij}, for $i = 1, \ldots, N$, $j = 1, \ldots, M$, concerns the non-negativity of effort levels[2], so we can assume $U_{ij} \subseteq \mathbb{R}_+$, for all i, j;
- $x(t) = (x_1(t), \ldots, x_M(t))$ is the state vector indicating the quantities of the exhaustible resources available to be extracted by the companies. We assume $x \in X \subseteq \mathbb{R}^M_+$;
- the M dynamic constraints of the game are given by

$$\begin{cases} \dot{x}(t) = g(x(t), u_{11}(t), \ldots, u_{NM}(t)) \\ x(t_0) = x_0 \in \mathbb{R}^M_+ \end{cases}, \tag{1}$$

  where $x \in \mathbb{R}^M_+$, $u_{ij} \in U_{ij} \subseteq \mathbb{R}_+$, and $g : R^M \times R^N \to R^{NM}$ is a vector-valued function. The state equations (1) are ODEs whose solutions satisfy the standard existence and uniqueness requirements[3];
- the interval over which the game is played is $[t_0, T] \subset \mathbb{R}_+$, where $t_0 \geq 0$ and $T < \infty$;
- the final instant of the game, i.e. the exact time at which all companies stop the extraction, is described by the random variable $\hat{t} \in [t_0, T]$. The cumulative distribution function (c.d.f.) of $\hat{t}$ is

[2] We are not going to impose any other constraint both on the control sets and on the state set, thus admitting any possible level. Because such sets are not compact in principle, maximum points may fail to exist, hence the choice of the payoff functions is crucial in order to have an equilibrium structure.

[3] The standard requirements are simply satisfied when dealing with a linear-quadratic structure such as the one we are going to take into account in Section 4.

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given by $F^p(t)$, which is assumed to have a break (jump) of length $p > 0$. The jump occurs at instant $t_1 \in [t_0,\ T]$, i.e. it can be described as follows:

$$F^p(t) = \begin{cases} F(t), & t \in [t_0, t_1) \\ F(t) + p, & t \in [t_1, T] \end{cases},$$

where $F(t)$ is a sufficiently regular function. By construction, there exists $q > 0$ such that $F(T) = q$, $p + q = 1$.

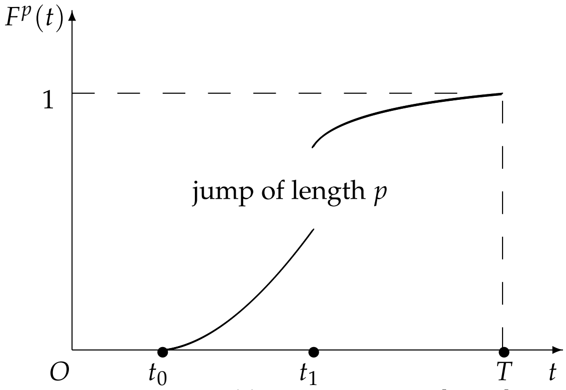

Figure 1. An example of a c.d.f. $F^p(t)$ in the interval $[t_0,\ T]$

- the instantaneous payoff of the i-th player at the moment $\tau \in [t_0, T]$ is defined as $h_i(x(\tau), u_{i1}(\tau), \ldots, u_{iM}(\tau))$. To shorten the notation, we write

$$h_i(x(\tau), u_{i1}(\tau), \ldots, u_{iM}(\tau)) = h_i(\tau).$$

The i-th related integral function is:

$$H_i(t) = \int_{t_0}^{t} h_i(\tau)d\tau; \tag{2}$$

- the i-th objective function is represented by the following integral payoff to be maximized:

$$K_i(t_0, x_0, u_{11}, ..., u_{NM}) = \int_{t_0}^{T} \left(\int_{t_0}^{t} h_i(x(\tau))d\tau \right) dF^p(t). \tag{3}$$

The transformation of integral functional in the form of double integral (3) to the standard for dynamic programming form is important for further study of the game (see also [6]).

Proposition 1. *The integral payoff (3) has the following form:*

$$K_i(t_0, x_0, u_{11}, ..., u_{NM}) = \int_{t_0}^{T} h_i(t)(1 - F(t))dt - p \int_{t_1}^{T} h_i(t)dt. \tag{4}$$

Proof. Keeping in mind that $H_i(t_0) = 0$, $F^p(t_0) = 0$, $F^p(T) = 1$, the payoffs $K_i(\cdot)$ can be rearranged by a simple manipulation:

$$K_i(t_0, x_0, u_{11}, ..., u_{NM}) = \int_{t_0}^{t_1} H_i(t)dF^p(t) + \int_{t_1}^{T} H_i(t)dF^p(t) =$$

$$= [H_i(t)F^p(t)]_{t_0}^{t_1} - \int_{t_0}^{t_1} h_i(t)F^p(t)dt + [H_i(t)F^p(t)]_{t_1}^{T} - \int_{t_1}^{T} h_i(t)F^p(t)dt =$$

$$= \int_{t_0}^{T} h_i(t)(1 - F^p(t))dt = \int_{t_0}^{t_1} h_i(t)(1 - F(t))dt + \int_{t_1}^{T} h_i(t)(1 - F(t) - p)dt =$$

$$= \int_{t_0}^{T} h_i(t)(1 - F(t))dt - p\int_{t_1}^{T} h_i(t)dt.$$

□

It can be helpful to provide a piece of justification for this model. Namely, this problem statement intends to take into account a common situation that there are certain events that happen at fixed time instants and that can be decisive for the game to stop or to proceed.

For instance, political activity or controversy may affect the situation: suppose that the Parliament passes a bill, or the outcome of a referendum establishes that would seriously impede or forbid the extraction activity (for example, prohibition of the fracking process). Obviously, the companies know that the decision will be taken in a certain day and they can also estimate the probability of a negative outcome. Hence it can be readily embedded into the ex-ante estimation of the terminal time probability distribution. Furthermore, the interpretation of such a scenario can also be extended towards other dynamic models involving environmental aspects. For example, even settings where the objective is pollution reduction can be affected by temporary shocks which modify the p.d.f. of some relevant variable: if the state variable is the pollution stock and we have a p.d.f. of its diffusion over the environment *ex ante*, a natural event may cause a jump in the distribution and, consequently, the need for a change of strategy. Other applications in other fields (such as insurance theory) can be hypothesized as well, but that goes far beyond the scope of our paper.

Back to our modelling, the jump in the probability distribution can also occur at the initial time, and this implies that there is a finite probability that the game does not start at all. Such a situation can be very interesting from the theoretical point of view as this corresponds to a non-proper probability function, i.e. a situation that was never addressed before in literature.

Finally, an interesting interpretation can be attached to the c.d.f. $F^p(t)$: basically, $p \in [0, 1)$, suggesting that it can represent the probability that the jump occurs. Namely, if $t_1 = t_0$ the game stops immediately after the start, and since $F(t_1) = 0$, $p = 1$. On the other hand, p decreases as time goes on, because $F(\cdot)$ is increasing: if $t_1 = T$, no jump occurs and $F(T) = 1$, so $p = 0$.

2.2. *Problem Statement for a Subgame*

The important notation in dynamic (differential) games is a notion of subgame [28] which takes non-trivial form for our problem statement for the reason of stochastic elements relating to time of a game duration. Let the game evolves along the trajectory $\tilde{x}(t)$. In order to better identify subgames of $\Gamma^T(t_0, \tilde{x})$, we are going to distinguish two main cases, which are differentiated based on the payoff flows: when the subgame starts before the jump instant t_1 and after t_1.

Subgame starting at $\theta < t_1$: Consider a subgame $\Gamma^T(\theta, \tilde{x})$ such that $\theta \in [t_0; t_1)$. The conditional c.d.f. in the considered subgame takes the following form:

$$F_\theta^p(t) = \frac{F^p(t) - F^p(\theta)}{1 - F^p(\theta)},$$

where

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$$F_\theta^p(t) = \begin{cases} \dfrac{F(t) - F(\theta)}{1 - F(\theta)}, & t \in [\theta, t_1) \\ \dfrac{F(t) + p - F(\theta)}{1 - F(\theta)}, & t \in [t_1, T] \end{cases}.$$

Therefore, recalling that $q = 1 - p$, the expected integral payoff accruing to the player i in this subgame is given by the following formula:

$$K_i(\theta, \tilde{x}, u_{11}, ..., u_{NM}) = \int_\theta^T h_i(t)(1 - F_\theta^p)dt =$$

$$= \int_\theta^{t_1} h_i(t) \left(1 - \frac{F(t) - F(\theta)}{1 - F(\theta)}\right) dt + \int_{t_1}^T h_i(t) \left(1 - \frac{F(t) + p - F(\theta)}{1 - F(\theta)}\right) dt =$$

$$= \frac{1}{1 - F(\theta)} \int_\theta^{t_1} h_i(t)(1 - F(t))dt + \frac{1}{1 - F(\theta)} \int_{t_1}^T h_i(t)(q - F(t))dt =$$

$$= \frac{1}{1 - F(\theta)} \left[\int_\theta^{t_1} h_i(t)(1 - F(t))dt + \int_{t_1}^T h_i(t)(1 - F(t))dt - p \int_{t_1}^T h_i(t)dt\right] =$$

$$= \frac{1}{1 - F(\theta)} \left[\int_\theta^T h_i(t)(1 - F(t))dt - p \int_{t_1}^T h_i(t)dt\right].$$

Subgame starting at $\widehat{\theta} \geq t_1$: Consider a subgame $\Gamma^T(\widehat{\theta}, \tilde{x})$ such that $\widehat{\theta} \in [t_1,\ T]$. The conditional cumulative distribution function in the considered subgame takes the following form:

$$F_{\widehat{\theta}}^p(t) = \frac{F(t) - F(\widehat{\theta})}{1 - p - F(\widehat{\theta})}.$$

Therefore, player i's expected integral payoff is provided by the formula:

$$K_i(\widehat{\theta}, \tilde{x}, u_{11}, ..., u_{NM}) = \frac{1}{1 - p - F(\widehat{\theta})} \int_{\widehat{\theta}}^T h_i(t)(1 - p - F(t))dt.$$

Thus, we prove the following proposition.

Proposition 2. *The expected integral payoff of the player i in the subgame $\Gamma^T(\theta, \tilde{x}), \theta \in [t_0, T]$ has the following form:*

$$K_i(\theta, \tilde{x}, u_{11}, ..., u_{NM}) = \begin{cases} \dfrac{1}{1 - F(\theta)} \left[\int_\theta^{t_1} h_i(t)(1 - F(t))dt + \int_{t_1}^T h_i(t)(1 - F(t))dt - p \int_{t_1}^T h_i(t)dt\right], \\ if\ \theta\ \in [t_0; t_1); \\ \dfrac{1}{1 - p - F\theta)} \int_\theta^T h_i(t)(1 - p - F(t))dt, \\ if\ \theta\ \in [t_1,\ T]. \end{cases} \tag{5}$$

2.3. Open-loop Nash equilibrium

To find the equilibrium in the non-cooperative setup of the game we will use the definition of a time-consistent Nash equilibrium from [28] adopted for the new problem statement as defined in Sec. 2.1. Let us consider the case of $M = 1$ (the definition can be easily extended for the case with several resources).

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Definition 1. *A set of strategies* $\{u_1^*(s), u_2^*(s), \ldots, u_N^*(s)\}$ *is said to constitute a* Nash equilibrium solution *for the n-person differential game (1)–(4), if the following inequalities are satisfied for all* $u_i(s) \in U^i$, $i \in N, s \in [t_0, T]$*:*

$$\begin{aligned} K_1(s, x^*(s), u_1^*, u_2^*, \ldots, u_N^*) &\geq K_1(s, x^*(s), u_1, u_2^*, \ldots, u_N^*), \\ K_2(s, x^*(s), u_1^*, u_2^*, \ldots, u_N^*) &\geq K_2(s, x^*(s), u_1^*, u_2, u_3^*, \ldots, u_N^*), \\ &\vdots \\ K_n(s, x^*(s), u_1^*, u_2^*, \ldots, u_N^*) &\geq K_n\left(s, x^*(s), u_1^*, u_2^*, \ldots, u_{n-1}^*, u_N\right); \end{aligned}$$

where

$$\dot{x}^*(s) = f(s, x^*(s), u_1^*(s), u_2^*(s), \ldots, u_N^*(s)), \quad x^*(t_0) = x_0,$$

The set of strategies $\{u_1^*(s), u_2^*(s), \ldots, u_N^*(s)\}$ *is said to be a* Nash equilibrium *of the game.*

3. Main results in the cooperative setup

Suppose that the game $\Gamma^T(t_0, x_0)$ is played in a cooperative scenario. Generally speaking, cooperation means that a group of companies agree to form a coalition before starting the game. In this case, we assume that such a group is the *grand coalition*, i.e. the totality of the involved players. Clearly, any dynamic model in which players form coalitions that are subgroups of the grand coalition deserves a special attention as well, but it is outside the scope of this paper (for the construction of the value functions in cooperative games, see, for example, [19], and [7] for cooperative differential games).

From now on, in order to simplify the notation and to reconcile the ongoing discussion with a standard case, we assume a unique exhaustible resource, which is extracted by N different companies, hence $M = 1$ and $u_1, \ldots, u_N$ are the effort levels. The cooperating players decide to use optimal strategies $(u_1^*, ..., u_N^*)$, which are defined as the strategies maximizing the sum of all payoffs, i.e.

$$(u_1^*, ..., u_N^*) = \arg \max_{u \in U_1 \times \cdots \times U_N} \sum_{i=1}^{N} K_i(t_0, x_0, u_1, ..., u_N).$$

As is standard in cooperative games, all players in the coalition jointly agree on a distribution method to share the total payoff. It is possible that in some instant, the solution of the current game is not optimal according to the optimality principle which was initially selected, meaning that the optimality principle may lose time-consistency. Because we are investigating a dynamic setting, it is necessary to define and to determine an imputation distribution procedure which is supposed to be compliant with the payoff form (4).

Before proceeding, we briefly recall the notion of imputation: in an N-players cooperative game, an *imputation* is a distribution $\xi = (\xi_1, \ldots, \xi_N)$ among players such that the sum of its coordinates is equal to the value of the grand coalition and each ξ_i assigns to the i-th player a quantity which is not smaller than the one she would achieve by playing as a singleton. In other words, if N is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function of the game, ξ is an imputation if $\xi_1 + \cdots + \xi_N = v(N)$ and $\xi_i \geq v(\{i\})$ for all $i = 1, \ldots, N$. The first property is called *efficiency* and guarantees that the imputation is a method of distribution of the total gain among all players (for an exhaustive overview on cooperative games, see [13]). Different imputations are usually employed in cooperative games, because not all solution concepts fit all models. However, the most useful one

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seems to be the *Shapley value*, first introduced by Nobel laureate L.S. Shapley in [23] in 1953, and which has been utilized in a huge number of economic and financial applications[4].

Definition 2. *Given an imputation* $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N_+$ *in a game* $\Gamma^T(t_0, x^*)$*, such that for all* $i = 1, \ldots, N$ *we have that:*

$$\xi_i = \int_{t_0}^{T} (1 - F(\tau))\beta_i(\tau)d\tau - p \int_{t_1}^{T} \beta_i(\tau)d\tau,$$

then the vector function $\beta(t) = (\beta_1(t), \ldots, \beta_N(t)) \in \mathbb{R}^N_+$ *is called an* ***imputation distribution procedure (IDP)****.*

The next Definition intends to expose the property of *time-consistency* for imputations.

Definition 3. *An imputation* $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N_+$ *in a game* $\Gamma^T(t_0, x^*)$ *is* ***time-consistent*** *if there exists an IDP* $\beta(t) = (\beta_1(t), \ldots, \beta_N(t)) \in \mathbb{R}^N_+$ *such that:*

1. *for all* $\theta \in [t_0, t_1)$ *the vector* $\xi^\theta = (\xi^\theta_1, \ldots, \xi^\theta_N)$*, where*

$$\xi^\theta_i = \frac{1}{1 - F(\theta)} \left[\int_\theta^T \beta_i(t)(1 - F(t))dt - p \int_{t_1}^T \beta_i(t)dt \right].$$

for all $i = 1, \ldots, N$*, belongs to the same optimality principle in the subgame* $\Gamma^T(\theta, x^*)$*, i.e.* ξ^θ *is an imputation in* $\Gamma^T(\theta, x^*)$*;*

2. *for all* $\widehat{\theta} \in [t_1, T]$ *the vector* $\xi^{\widehat{\theta}} = \left(\xi^{\widehat{\theta}}_1, \ldots, \xi^{\widehat{\theta}}_N\right)$*, where*

$$\xi^{\widehat{\theta}}_i = \frac{1}{1 - p - F(\widehat{\theta})} \int_{\widehat{\theta}}^T \beta_i(t)(1 - p - F(t))dt,$$

for all $i = 1, \ldots, N$*, belongs to the same optimality principle in the subgame* $\Gamma^T(\widehat{\theta}, x^*)$*, i.e.* $\xi^{\widehat{\theta}}$ *is an imputation in* $\Gamma^T(\widehat{\theta}, x^*)$*.*

The next step consists in the determination of a relation between ξ and β. Also in this case, we will have to distinguish the cases when the subgame starts either before or after the jump at instant t_1. Firstly, we prove a Lemma which is helpful to reformulate imputation ξ. The subsequent Proposition intend to explicitly outline the forms for the IDPs of the game.

Lemma 1. *If* $t_0 \le \theta \le t_1 \le \widehat{\theta} \le T$*, for all* $i = 1, \ldots, N$*, the coordinates of imputation* ξ *can be written as follows:*

$$\xi_i = \int_{t_0}^{\theta} \beta_i(t)(1 - F(t))dt + (1 - F(\theta))\xi^\theta_i, \tag{6}$$

$$\xi_i = \int_{t_0}^{t_1} \beta_i(t)(1 - F(t))dt + \int_{t_1}^{\widehat{\theta}} \beta_i(t)(q - F(t))dt + (q - F(\theta))\xi^{\widehat{\theta}}_i. \tag{7}$$

[4] An extensive treatment of the Shapley value and of other relevant solution **concepts** can be found in [13].

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Proof. We can write the following:

$$\xi_i = \int_{t_0}^{T} \beta_i(t)(1 - F(t))dt - p \int_{t_1}^{T} \beta_i(t)dt =$$

$$= \int_{t_0}^{\theta} \beta_i(t)(1 - F(t))dt + \int_{\theta}^{T} \beta_i(t)(1 - F(t))dt - p \int_{t_1}^{T} \beta_i(t)dt =$$

$$= \int_{t_0}^{\theta} \beta_i(t)(1 - F(t))dt + (1 - F(\theta))\xi_i^{\theta}.$$

198 and finally the expression (6).

For the second case we can write the following:

$$\xi_i = \int_{t_0}^{t_1} \beta_i(t)(1 - F(t))dt + \int_{t_1}^{T} \beta_i(t)(q - F(t))dt =$$

$$= \int_{t_0}^{t_1} \beta_i(t)(1 - F(t))dt + \int_{t_1}^{\widehat{\theta}} \beta_i(t)(q - F(t))dt + \int_{\widehat{\theta}}^{T} \beta_i(t)(q - F(t))dt =$$

$$= \int_{t_0}^{t_1} \beta_i(t)(1 - F(t))dt + \int_{t_1}^{\widehat{\theta}} \beta_i(t)(q - F(t))dt + (q - F(\widehat{\theta}))\xi_i^{\widehat{\theta}}.$$

199 and finally the expression (7). □

Proposition 3. *If $\theta \in [t_0, t_1)$, then for all $i = 1, \ldots, N$, the i-th coordinate of the IDP is given by:*

$$\beta_i(\theta) = \frac{f(\theta)}{1 - F(\theta)}\xi_i^{\theta} - (\xi_i^{\theta})'. \tag{8}$$

If $\theta \in [t_1, T]$, then for all $i = 1, \ldots, N$, the i-th coordinate of the IDP is given by:

$$\beta_i(\theta) = \frac{f(\theta)}{q - F(\theta)}\xi_i^{\theta} - (\xi_i^{\theta})'. \tag{9}$$

Proof. When $\theta \in [t_0,\ t_1)$, we can differentiate the expression (6) with respect to θ, thus obtaining:

$$0 = \beta_i(\theta)(1 - F(\theta)) - f(\theta)\xi_i^{\theta} + (1 - F(\theta))(\xi_i^{\theta})'.$$

Then, solving for $\beta_i(\theta)$ yields:

$$\beta_i(\theta) = \frac{f(\theta)}{1 - F(\theta)}\xi_i^{\theta} - (\xi_i^{\theta})'.$$

When $\theta \in [t_1,\ T)$, we can differentiate the expression (7) with respect to $\widehat{\theta}$, thus obtaining:

$$0 = \beta_i(\widehat{\theta})(q - F(\widehat{\theta})) - f(\widehat{\theta})\xi_i^{\widehat{\theta}} + (q - F(\widehat{\theta}))(\xi_i^{\widehat{\theta}})'.$$

Then, solving for $\beta_i(\widehat{\theta})$ yields:

$$\beta_i(\widehat{\theta}) = \frac{f(\widehat{\theta})}{q - F(\widehat{\theta})}\xi_i^{\widehat{\theta}} - (\xi_i^{\widehat{\theta}})'.$$

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□

The above results can be collected as follows:

Theorem 1. *Let the imputation $\xi(t, x^*(t), T)$ of the game $\Gamma^T(t_0, x^*)$ be an absolutely continuous function of t, $t \in [t_0, T]$. If the IDP has one of the following forms:*

1. *if $\tau \in [t_0, t_1)$,*

$$\beta_i(\tau) = \frac{f(\tau)}{1 - F(\tau)}\xi_i(\tau, x^*(\tau), T) - \xi_i'(\tau, x^*(\tau), T), \tag{10}$$

2. *if $\tau \in [t_1, T]$*

$$\beta_i(\tau) = \frac{f(\tau)}{1 - p - F(\tau)}\xi_i(\tau, x^*(\tau), T) - \xi_i'(\tau, x^*(\tau), T), \tag{11}$$

then $\xi(t_0, x_0, T)$ is a time-consistent imputation with IDP given by either (10) or (11).

The problem of stable cooperation in differential games with random duration where c.d.f. is continuous (without break (jump)) was studied by in [12,17,24]. Assuming in our model p is equal to zero, the obtained results coincide with the results in the above-mentioned work. Moreover, new results cover the results for a fully deterministic models. Namely, for the problem with prescribed duration for $f(\tau) = 0$ in (10) (11) we obtain results published in [15]. For the problem with constant discounting see work [18] and (10) with $\frac{f(\tau)}{1-F(\tau)} = \lambda$.

4. An example

We are going to consider a simple model of common-property nonrenewable resource extraction published in [2] in 2000, then further investigated in successive papers such as [22] and [11].

Also in this case, $M = 1$, that is we have a unique state variable $x(t)$ indicating the stock of a nonrenewable resource at time t. The companies' strategic variables $u_i(t)$, for $i = 1, \ldots, N$ denote the rates of extraction, or extraction efforts, at time t. The state equation has the form:

$$\dot{x}(t) = -\sum_{i=1}^{N} u_i(t), \tag{12}$$

the initial condition, i.e. the amount of resource at time t_0 is $x(t_0) = x_0$. The differential equation (12) is the most standard and simple dynamics in nonrenewable resource extraction games, where all players concur to extract and deplete the resource with the same intensity. When the involved resource is renewable, it also regenerates at a growth rate δ, hence a positive linear term in the state variable also appears in (12), and the model must be treated differently (see for example [10] or the survey [26]).

Back to the model, we suppose that the game ends at the random time instant t, a random variable having exponential distribution $F(t)$ on the interval $[t_0, t_1]$, i.e. we are investigating the first case, before the jump in the distribution. We also assume that the jump takes place in the end of the interval $[t_0; T]$, i.e. $t_1 = T$. Hence the discontinuity occurs at the terminal time. The c.d.f. of the random variable t is given by:

$$F(t) = e^{-t_0}\left(1 - e^{-(t-t_0)}\right),$$

which turns into $F(t) = 1 - e^{-t}$ for $t_0 = 0$. From now on we will consider this case, i.e. $t_0 = 0$.

Note that we can provide the complete formulation of the discontinuous c.d.f. as in the previous Section:

$$F^p(t) = \begin{cases} 1 - e^{-t}, & t \in [0, t_1) \\ 1 - e^{-t} + e^{-T}, & t \in [t_1, T] \end{cases},$$

meaning that in this case $p = e^{-T}$.

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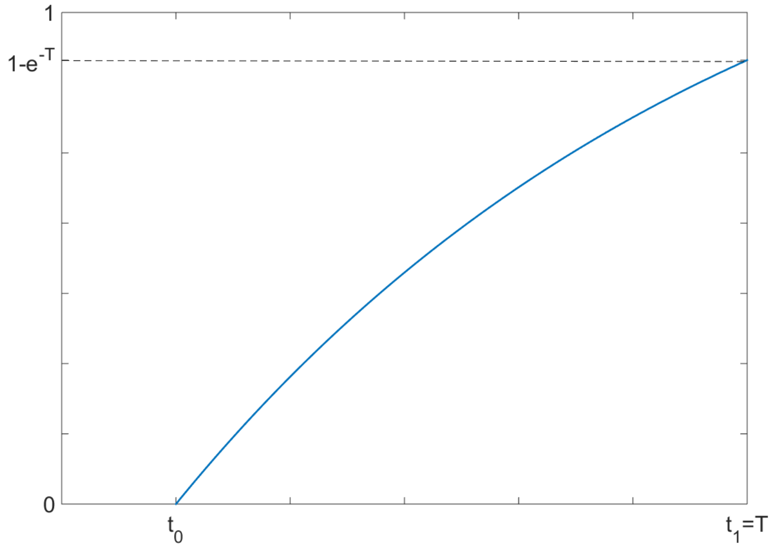

Figure 1. The exponential c.d.f. $F(t) = 1 - e^{-(t-t_0)}$ in the interval $[t_0,\ t_1]$

In this game, each player i has a utility function

$$h_i(x(\tau), u_i(\tau)) = k_i u_i(t) - \frac{1}{2}u_i(t)^2 - \delta_i x(t),$$

where k_i and δ_i are positive constants depending on the specific scenario and on the companies'characteristics.

The expected integral payoff of player i is[5]:

$$K_i = \int_0^{t_1} (k_i u_i(t) - \frac{1}{2}u_i(t)^2 - \delta_i x(t))e^{-t} dt. \tag{13}$$

We are going to find **noncooperative** open-loop optimal trajectories of state and controls in relation to noncooperative form of the game using Pontryagin's maximum principle, which is one of the 2 major procedures for equilibrium structure in differential games (see [2]). In this model, this method is suitable, because the open-loop trajectories are easily visualized in $K_i(\cdot)$. Each company aims to solve the following problem:

$$\max_{u_i} \int_0^{t_1} (k_i u_i(t) - \frac{1}{2}u_i(t)^2 - \delta_i x(t))e^{-t} dt.$$

Each player has a Hamiltonian function of the form:

$$H_i(\cdot) = -\psi_i(t) \sum_{j=1}^{n} u_j(t) + \left(k_i u_i(t) - \frac{1}{2}u_i(t)^2 - \delta_i x(t)\right) e^{-t},$$

where $\psi_i(t)$ is the i-th adjoint variable attached by company i to the resource dynamics or, in line with a standard economic interpretation, the related shadow price.

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[5] To lighten the notation, we will omit redundant arguments whenever possible.

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Differentiating each Hamiltonian with respect to u_i and then equating to 0 yields the first order conditions:

$$\frac{\partial H_i}{\partial u_i} = -\psi_i(t) + (k_i - u_i(t))e^{-t} = 0,$$

then, solving for $u_i(t)$, we obtain:

$$u_i(t) = k_i - \psi_i(t)e^{t}.$$

The second order conditions hold, because for all $i = 1, \ldots, N$:

$$\frac{\partial H_i^2}{\partial^2 u_i} = -e^{-t} < 0.$$

The adjoint equations and the related transversality conditions read as:

$$\begin{cases} \dot{\psi}_i(t) = \delta_i e^{-t} \\ \psi_i(t_1) = 0 \end{cases},$$

hence the optimal costates are $\psi_i^*(t) = \delta_i \left(e^{-t_1} - e^{-t}\right)$, for all $i = 1, \ldots, N$.

Plugging $\psi_i^*(t)$ into the FOCs yields the optimal controls, i.e.

$$u_i^*(t) = k_i - \delta_i(e^{t-t_1} - 1). \tag{14}$$

In order to determine the optimal state $x^*(t)$, it suffices to substitute (14) into the state dynamics (12) and subsequently integrate both sides, employing the initial condition:

$$\begin{cases} \dot{x}(t) = (e^{t-t_1} - 1)\sum_{j=1}^{N} \delta_j - \sum_{j=1}^{N} k_j \\ x(0) = x_0 \end{cases},$$

so the optimal stock of resource amounts to:

$$x^*(t) = x_0 - t\sum_{j=1}^{N}(\delta_j + k_j) + \sum_{j=1}^{N} \delta_j \left(e^{t} - 1\right) e^{-t_1}. \tag{15}$$

Now we are going to take into account a cooperative version of the game, that is a scenario where all companies agree to play strategies such that their aggregate payoff is maximized. The sum of all payoffs is:

$$\sum_{j=1}^{N} K_j = \sum_{j=1}^{N} \int_0^{t_1} \left(k_j u_j(t) - \frac{1}{2}u_j(t)^2 - \delta_j x(t)\right) e^{-t} dt.$$

The approach for the determination of the open-loop equilibrium structure is analogous to the one adopted in the noncooperative case. From now on, we are going to use the notation u_i^C, $x^C(t)$ to avoid confusion with the previous quantities.

$$u_i^C(t) = k_i - \sum_{j=1}^{N} \delta_j(e^{t-t_1} - 1). \tag{16}$$

$$x^C(t) = x_0 - t\sum_{j=1}^{N}(N\delta_j + k_j) + N\sum_{j=1}^{N} \delta_j \left(e^{t} - 1\right) e^{-t_1}. \tag{17}$$

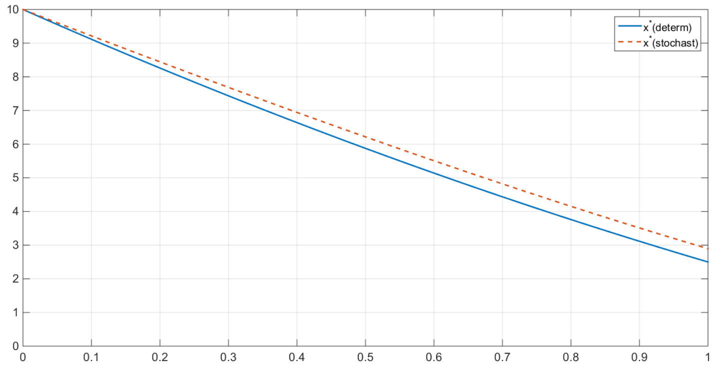

Figure 2. Comparison between deterministic and stochastic settings for state $x^*(t)$

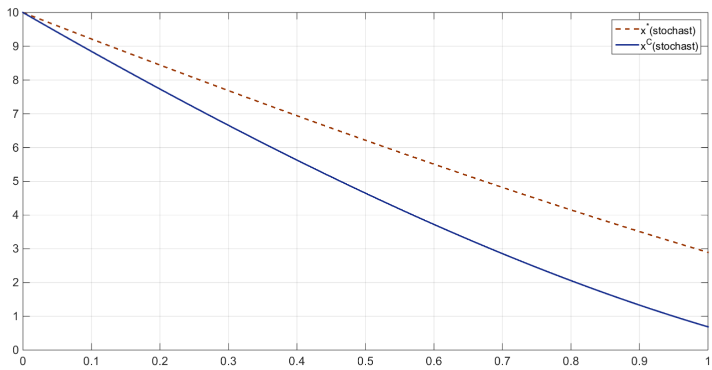

Figure 3. Comparison between Nash equilibrium and cooperative equilibrium for $x(t)$

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The comparison between the resource stocks in the 2 scenarios can be illustrated by a simple inequality, highlighting that the noncooperative resource stock exceeds the cooperative one. Namely, at all $t \in [t_0,\ t_1]$ we have that:

$$x^*(t) \geq x^C(t)$$
$$\Updownarrow$$
$$x_0 - t\sum_{j=1}^{N}(\delta_j + k_j) + \sum_{j=1}^{N}\delta_j\left(e^t - 1\right)e^{-t_1} \geq x_0 - t\sum_{j=1}^{N}\left(N\delta_j + k_j\right) + N\sum_{j=1}^{N}\delta_j\left(e^t - 1\right)e^{-t_1}$$
$$\Updownarrow$$
$$t(N-1)\sum_{j=1}^{N}\delta_j \geq (N-1)\sum_{j=1}^{N}\delta_j\left(e^t - 1\right)e^{-t_1}$$
$$\Updownarrow$$
$$e^{t_1} \geq \frac{e^t - 1}{t}.$$

Such an estimate always holds for $t \geq t_0$, because

$$e^{t_1} > e^t > e^t - 1 \geq \frac{e^t - 1}{t}.$$

An investigation of a suitable IDP requires the definition of an imputation in this model. If we choose an egalitarian distribution, we can define the shares of the imputation as fractions of the total payoff equally divided by the number of players, i.e.

$$\xi_i = \frac{\max_u \sum_{j=1}^{N} K_j(x_0, u_1, ..., u_N)}{N} = \frac{\sum_{j=1}^{N}\int_{t_0}^{t_1}(k_j u_j^C(t) - \frac{1}{2}u_j^C(t)^2 - \delta_i x^C(t))e^{-t}dt}{N}.$$

The case we are taking into account is the first one in the previous Section, i.e. $\theta \in [t_0,\ t_1]$, where constant $D = 0$. Furthermore, the exponential c.d.f. at hand has a relevant property: since $f(t) = e^{-t}$, the ratio $f(t)/\left(1 - F(t)\right) = 1$, hence the expression (9) for IDP takes the form:

$$\beta_i(\theta) = \xi_i^\theta - (\xi_i^\theta)'.$$

Evaluating $h_i^*(\cdot)$ at the optimal controls and states amounts to:

$$\begin{aligned}
h_i^*(t) &= k_i\left(k_i - \sum_{j=1}^{N}\delta_j(e^{t-t_1} - 1)\right) - \frac{1}{2}\left(k_i - \sum_{j=1}^{N}\delta_j(e^{t-t_1} - 1)\right)^2 \\
&\quad - \delta_i\left(x_0 - t\sum_{j=1}^{N}\left(N\delta_j + k_j\right) + N\sum_{j=1}^{N}\delta_j\left(e^t - 1\right)e^{-t_1}\right) \\
&= k_i^2 - k_i\sum_{j=1}^{N}\delta_j(e^{t-t_1} - 1) - \frac{1}{2}\left(k_i^2 - 2k_i\sum_{j=1}^{N}\delta_j(e^{t-t_1} - 1) + \left(\sum_{j=1}^{N}\delta_j\right)^2(e^{t-t_1} - 1)^2\right) \\
&\quad - \delta_i x_0 + t\delta_i\sum_{j=1}^{N}\left(N\delta_j + k_j\right) - N\delta_i\sum_{j=1}^{N}\delta_j\left(e^t - 1\right)e^{-t_1} \\
&= \frac{k_i^2}{2} + \frac{(e^{t-t_1} - 1)^2\left(\sum_{j=1}^{N}\delta_j\right)^2}{2} - \delta_i x_0 + t\delta_i\sum_{j=1}^{N}\left(N\delta_j + k_j\right) - N\delta_i\sum_{j=1}^{N}\delta_j\left(e^t - 1\right)e^{-t_1}.
\end{aligned}$$

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By employing $h_i^*(t)$ in $K_i(\cdot)$, we can determine the expression of the expected integral payoff of company i for a subgame starting at $\theta \in [0,\ t_1]$:

$$K_i^*(\theta) = \frac{1}{e^{-\theta}} \int_\theta^{t_1} \left(\frac{k_i^2}{2} + \frac{(e^{t-t_1}-1)^2 \left(\sum_{j=1}^N \delta_j\right)^2}{2} - \delta_i x_0 + \right.$$

$$\left. + t\delta_i \sum_{j=1}^N (N\delta_j + k_j) - N\delta_i \sum_{j=1}^N \delta_j \left(e^t - 1\right) e^{-t_1} \right) e^{-t} dt = \cdots =$$

$$= \left(\frac{k_i^2}{2} - \delta_i x_0\right) \left[1 - e^{\theta - t_1}\right] + \frac{\left(\sum_{j=1}^N \delta_j\right)^2 \left[2(\theta - t_1)e^{\theta - t_1} + 1 - e^{2\theta - 2t_1}\right]}{2} +$$

$$+ \delta_i \sum_{j=1}^N (N\delta_j + k_j) \left[\theta + 1 - (t_1 + 1)e^{\theta - t_1}\right]$$

$$- N\delta_i \sum_{j=1}^N \delta_j \left[(t_1 - \theta)e^\theta + e^{\theta - t_1} - 1\right].$$

Subsequently, we have to determine $(K_i^*(\theta))'$, by a simple differentiation:

$$(K_i^*(\theta))' = -\left(\frac{k_i^2}{2} - \delta_i x_0\right) e^{\theta - t_1} + \left(\sum_{j=1}^N \delta_j\right)^2 \left[(1 + \theta - t_1)e^{\theta - t_1} - e^{2(\theta - t_1)}\right] +$$

$$+ \delta_i \sum_{j=1}^N (N\delta_j + k_j)(1 - (t_1 + 1)e^{\theta - t_1}) - N\delta_i \sum_{j=1}^N \delta_j \left[(t_1 - \theta - 1)e^\theta + e^{\theta - t_1}\right].$$

Finally, employing the found forms for $K_i^*(\theta)$ and $(K_i^*(\theta))'$ we get:

$$K_j^*(\theta) - (K_j^*(\theta))' = \frac{k_i^2}{2} - \delta_i x_0 + \frac{1}{2} \left(\sum_{j=1}^N \delta_j\right)^2 \left[e^{\theta - t_1} - 1\right]^2 +$$

$$+ \theta\delta_i \sum_{j=1}^N (N\delta_j + k_j) - N\delta_i \sum_{j=1}^N \delta_j (e^\theta - 1).$$

Thus, IDP takes form

$$\beta_i(\theta) = \xi_i^\theta - (\xi_i^\theta)' = \frac{\sum_{j=1}^N K_j^*(\theta)}{N} - \left(\frac{\sum_{j=1}^N K_j^*(\theta)}{N}\right)' = \frac{\sum_{j=1}^N (K_j^*(\theta) - (K_j^*(\theta))')}{N} =$$

$$= \frac{1}{N} \sum_{j=1}^N \left(\frac{k_j^2}{2} - \delta_j x_0 + \frac{1}{2} \left(\sum_{l=1}^N \delta_l\right)^2 \left[e^{\theta - t_1} - 1\right]^2 + \right.$$

$$\left. + \theta\delta_j \sum_{l=1}^N (N\delta_l + k_l) - N\delta_i \sum_{l=1}^N \delta_l (e^\theta - 1) \right).$$

Figure 3, which was created with Matlab R2016a, portrays a sketch of the behavior of the imputation and of the IDP over time. The numerical simulation was performed for the following parameters: $N = 3$, $\sum_{j=1}^3 \delta_j = 0.000069$, $t_1 = 20$, $k_1 = 1$, $k_2 = 2$, $k_3 = 3$.

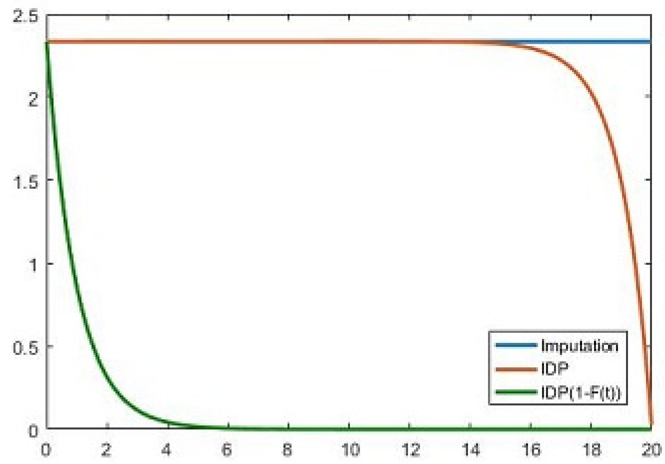

Figure 3. IDP and Imputation for one player

On this picture we can see that the amount of imputation is equal to the integral of IDP multiplied by discount probability factor.

5. Conclusions and further developments

We proposed an analysis of a class of extraction differential games with uncertain duration possibly involving a discontinuous c.d.f. for the random variable indicating the duration of the game. Then we focused our attention on the cooperative aspects of the game to identify the appropriate IDP and applied such a theory to a standard nonrenewable resource extraction model.

There exists a number of possible improvements, both from theoretical and applied viewpoints, regarding the feedback information structure of such a class of games, the solution concepts (i.e. Shapley value, Banzhaf value, core) to be employed, the models which represent scenarios different from the extraction of an exhaustible resource and also models of processes with more complex and realistic c.d.f. All of them are left for future research.

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