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Quantum Correlations and Permutation Symmetries

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In this paper, the connections between quantum non-locality and permutation symmetries are explored. This includes two types of symmetries: permutation across a superposition and permutation of qubits in a quantum system. An algorithm is proposed for finding the separability class of a quantum state using a method based on factorizing an arbitrary multipartite state into possible partitions, cyclically permuting qubits of the vectors in a superposition to check which separability class it falls into and thereafter using a reduced density-matrix analysis of the system is proposed. For the case of mixed quantum states, conditions for separability are found in terms of the partial transposition of the density matrices of the quantum system. One of these conditions turns out to be the Partial Positive Transpose (PPT) condition. A graphical method for analyzing separability is also proposed. The concept of permutation of qubits is shown to be useful in defining a new entanglement measure in the 'engle'.

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I. INTRODUCTION

Quantum non-locality is an empirically verified aspect of nature [1–5] and entanglement is a key foundational concept relating to the same [6, 7]. Entanglement has been used in a number of applications such as quantum information processing [8–14] and even studied as the phenomenon possibly underlying the emergence of spacetime and gravity [15, 16]. However, the problem of knowing how entangled a certain quantum state is, is of utmost importance before the entanglement within the quantum state can be used. This constitutes the entanglement characterization or *quantum separability problem*, which is of central importance in quantum information theory [10, 17–20]. Different kinds of entanglement can be used for different applications. Maximally entangled states can be used for applications such as teleportation [21] and remote state preparation [22] while partially entangled states are used for applications such as measurement-based quantum computation [23–25] and separable states are usually used as ancilla qubits in various quantum computing protocols [26, 27]. The separability problem for the bipartite case is well understood and Singular Value/ Schmidt decomposition is an efficient computational method to determine separability [28–30]. An N qubit pure state, $|\Psi\rangle = \sum_{n=0}^{2^N-1} a_n |n\rangle$ is said to be separable under a particular bipartition if it can be written in the form,

$$|\Psi\rangle = \left|\Psi^A\right\rangle \otimes \left|\Psi^B\right\rangle. \tag{1}$$

If it cannot be written in this form then it is entangled. The bipartitions of a bipartite separable system can form various separability classes, $[m, N-m]$; $m, N \in \mathbb{Z}$ denoting a general separability class formed of an m qubit subsystem and an $N-m$ qubit subsystem. Generally, there are $\frac{N!}{m!(N-m)!}$ ways to arrange N qubits into a subsystem of size m and one of size $N-m$.

The case of multipartite separability is more complicated and has been shown by Gurvits to be NP-hard [31–33]. There is a considerably richer structure to the separability classes in the multipartite case, due to the exponential growth of the Hilbert Space. Different kinds of measures, be it geometric/graphical or algebraic, for characterizing separability and entanglement like with the Bloch representation of vectors, have been studied over the years. Before moving on to the idea of permutations and separability of subsystems in a quantum state, let me briefly share a new method of characterization of separability using what we call the *thread-and-bead* model.

II. THREAD-AND-BEAD MODEL

In this model an arbitrary quantum system is represented in terms of threads and beads. Each state in the superposition in the system is represented by a single thread. Every qubit in these states has an associated bead: a 'single' bead for $|0\rangle$ and a 'double' bead for $|1\rangle$. The important property in this representation is that if two or more beads have the same nature ('single' or 'double'), they join together only for that bead. This is an effective way to see how separable a state is: the more it is connected in being 'beaded together', the more separable they are.

The interesting part about this approach is that more entangled these threads are, more separable are the corresponding states, and vice versa! Let us look at an illustration, in Figure 3.1. The primary problem with this approach that we found relates to the probability factors associated with the beads. Entanglement depends significantly on these factors. A general two-qubit state $\alpha|0\rangle + \beta|1\rangle$ does not have the same entanglement as state $\alpha'|0\rangle + \beta'|1\rangle$ for $\alpha \neq \alpha', \beta \neq \beta'$. To tackle this problem, we proposed a method of changing the girth of the thread associated with a superposition term depending on the probability factor. This helps in

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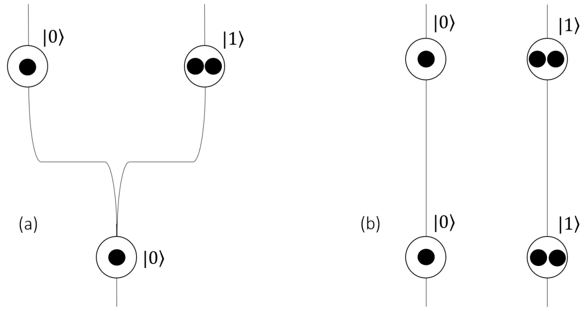

FIG. 1. Illustration of the Thread-and-Bead Model for two qubits: (a) separable state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$ and (b) maximally entangled Bell-state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

understanding not only separability but also nuances of how entangled a quantum state is.

III. SYMMETRY UNDER PERMUTATION ACROSS SUPERPOSITIONS

A separable quantum state can be represented as $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle \otimes ... \otimes |\psi_n\rangle$ for an n-partite separable state. Let us say that we have for each separable sub-system, a general expansion $|\psi_i\rangle = \sum_j \alpha_j^{(i)} |\psi_{i,j}\rangle$. We now introduce the idea of a local permutation across superposition of vector-states.

Although permutation symmetry is usually mentioned with respect to qubit permutations, in this paper local permutation of vector states across a superposition is discussed. Now, if we define local permutation as the permutation of k_i superposition terms by one place within a subsystem, keeping the rest of the system unchanged. We find that a local permutation operator on the i^{th} qubit keeps the state invariant for a separable state. Let us see what happens if we take an entangled state, with the example of the maximally entangled two-qubit Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and operate U_2: $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \xrightarrow{U_2} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. This clearly changes the state, taking it to another Bell state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Thus local permutations of a state leave it invariant if it is separable in the qubit over which this operation is being performed, but changes the state if the state is an entangled state. This formalism can be extended to more than one qubit and can include subsystems comprising multiple qubits. For example, $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \xrightarrow{U_{23}} \frac{1}{\sqrt{3}}(|000\rangle + |001\rangle + |110\rangle)$. Since this state is the well known entangled W-state, the local permutation over the subsystem comprising qubits 2 and 3 changes the state.

A. Extension to Partial Transposition and PPT Criterion

It is found that formalism can be extended to the case of mixed states using density matrices and the concept of partial positive transpose. In this case, unlike in the permutation of a state, the partial transpose of the density matrix of a separable state is not equal to the matrix itself, but has certain shared characteristic properties. For an N-qubit quantum state $|\psi\rangle = \sum_j |q_1^{(j)}\rangle|q_2^{(j)}\rangle...|q_N^{(j)}\rangle$, the density matrix is given by $\rho = |\psi\rangle\langle\psi|$. We can now take a partial positive transpose with respect to the i^{th} qubit: $(\sum_j |q_1^{(j)}\rangle...|q_i^{(j)}\rangle...|q_N^{(j)}\rangle)(\sum_{j'} \langle r_1^{(j')}|...\langle r_i^{(j')}|...\langle r_N^{(j)}|) \xrightarrow{\rho^{T_i}} (\sum_j |q_1^{(j)}\rangle...|r_i^{(j)}\rangle...|q_N^{(j)}\rangle)(\sum_{j'} \langle r_1^{(j')}|...\langle q_i^{(j')}|...\langle r_N^{(j)}|)$, where a general $\langle r| = |q\rangle^\dagger$ is used for ease of writing the terms. Now, let us look at two properties of matrices under partial transposition: determinant and eigenspectrum.

Statement 3.1. A state is separable if and only if the partial transpose of its density matrix has a non-negative determinant.

Proof. We know that a characteristic equation of a matrix A is given by $det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n)$ where I is the identity matrix of the same dimensions as A, λ is a variable and $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ are the eigenvalues of the matrix A. Setting $\lambda = 0$, $det(A) = \lambda_1\lambda_2...\lambda_n$. Now, if we take the specific case of a density matrix $\rho = |\psi\rangle\langle\psi|$ and find its eigenvalues, corresponding to the eigenvector $|\phi\rangle$, we see that the eigenvalues given by $\langle\phi|\rho|\phi\rangle = \langle\phi|\psi\rangle\langle\psi|\phi\rangle = |\langle\phi|\psi\rangle|^2 \geq 0$ lead to the condition $det(\rho) \geq 0$. For a general Kronecker Product $A \otimes B$ for n-dimensional A and m-dimensional B,we have the expansion: $A \otimes B = (I_m \otimes B)(A \otimes I_q)$, where I_q is the identity matrix of dimension q. Considering the determinant of this Kronecker Product, $det(A \otimes B) = det(I_m \otimes B)det(A \otimes I_p) = det(B)^p det(B)^m$. We use this for the case of a separable density matrix of the form $\rho = \rho_1 \otimes \rho_2$ for an two-partite system. Taking the partial transpose over any one subsystem, say, the second subsystem, and finding the determinant of the same using equation, $det(\rho_1 \otimes (\rho_2)^T) = det(\rho_1)^{d_2} det((\rho_2)^T)^{d_1} = det(\rho_1)^{d_2} det(\rho_2)^{d_1} = det(\rho_1 \otimes \rho_2) = det(\rho) \geq 0$, where d_i represents the dimensions of the density matrix of the i^{th} subsystem.□

Statement 3.2. A state is separable if and only if the partial transpose of its density matrix has only non-negative eigenvalues.

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Proof. Considering any general matrix A and using the characteristic polynomial of the transpose of the matrix, $det(A^T - \lambda I) = det((A - \lambda I)^T) = det(A - \lambda I)$, where the symmetric property of the identity matrix has been used, besides the property that $det(M^T) = det(M)$ for a general matrix M. Thus, the characteristic polynomials of both A and A^T are the same. As a result, the eigenvalues of A^T and A are the same. Using this result for a density matrix $\rho = \rho_1 \otimes \rho_2$, the partial transpose with respect to the second subsystem: $\rho_1 \otimes (\rho_2)^T$ preserves the eigenspectrum of ρ. Since the eigenvalues of ρ are non-negative, the eigenvalues of the partial transpose with respect to the i^{th} subsystem ρ^{T_i} are also non-negative. □

This is actually the statement of the famous Partial Positive Transpose (PPT) condition [3], which is a necessary and sufficient condition for separability. Thus, we have, starting with the idea of permuting subsystem-states and partial transposition, independently reached a set of conditions that corroborate with a necessary and sufficient condition in the PPT criterion.

One of the key problems in contemporary quantum entanglement and separability characterization is in the realization of conceptual tools and maps, such as the partial transpose, in physical systems. Horodecki and Ekert [34] proposed the method known as structural physical approximation (SPA) which provides a way in which non-physical operations that can detect entanglement such as the partial transpose can be approximated systematically by physical operations.*Lim et al* [35] presented a practical scheme to realize a physical approximation to the partial transpose.They did this using local measurements on individual quantum systems and classical communication.

Given this way of realizing the partial transpose, an algorithm for determining whether a state is entangled and thereafter which entanglement class it belongs to was developed on some of these ideas. The physical realization of the non-unitary operator that we call the *Combination-Key Operator* would rely on local measurements and state preparation as well. This is something that we will not discuss beyond the theoretical formulation of the concept, in this letter. Before going on to the *Quantum Combination-Key* algorithm, we will define the *Combination-Key operator* that will play a pivotal role in the algorithm.

B. Quantum Combination Key Operator

We introduce the 'combination-key' operator in this section. A multiqubit state $|\chi\rangle$, with M superposition states, has a bipartite separability in the partitions $\epsilon^{(k)} = \{\phi, \psi\}$ for the form $|\chi\rangle = |\phi\rangle \otimes |\psi\rangle$ if after the operation of the Quantum Combination-Key Operation $U_{rs_{\epsilon^{(k)}}}$ (where $\epsilon^{(k)}$ represents the partition that has the separable qubits), where

$$U_{rs_{\epsilon^{(k)}}} = \mathbb{I} \otimes \sum_i \frac{\langle \epsilon_{i+1}^{(k)}|\chi\rangle}{\langle \epsilon_i^{(k)}|\chi\rangle} |\epsilon_{i+1}^{(k)}\rangle\langle\epsilon_i^{(k)}| \tag{2}$$

the state maps back onto itself. $|\epsilon_i^{(k)}\rangle$ represents the i^{th} superposition state in the $\epsilon^{(k)}$ partition.

Proof. Let us take the composite quantum system: $|\chi\rangle = |\phi\rangle \otimes |\psi\rangle$ where $|\phi\rangle = \sum_{i=1}^m \alpha_i|\phi_i\rangle$ and $|\psi\rangle = \sum_{j=1}^n \beta_j|\psi_j\rangle$. Considering the *Combination-Key Operator* U_{rs} for the subsystem $|\psi\rangle$:

$$U_{rs_{|\psi\rangle}} = \mathbb{I} \otimes \sum_i \frac{\langle \psi_{i+1}|\chi\rangle}{\langle \psi_i|\chi\rangle} |\psi_{i+1}\rangle\langle\psi_i| \tag{3}$$

An important point here is that there is a certain cyclicity in the indices. So, this operator takes the n^{th} state to the first superposition vector-state.

Thus, the action of this operator is given by

$$\begin{aligned} |\chi\rangle &= \sum_{i=1}^m \sum_{j=1}^n \alpha_i\beta_j|\phi_i\rangle \otimes |\psi_j\rangle \\ &\xrightarrow{U_{rs_{|\psi\rangle}}} \sum_{i=1}^m \sum_{j=1}^{n-1} \alpha_i\beta_{j+1}|\phi_i\rangle\otimes|\psi_{j+1}\rangle + \sum_{i=1}^m \alpha_i\beta_1|\phi_i\rangle\otimes|\psi_1\rangle \end{aligned} \tag{4}$$

In effect, this gives back the same state $|\chi\rangle$. If the coefficients (physically, the relative phases) for each of the superposition states are removed (in a modified state measurement and preparation scheme), then the operator for the biseparable state $|\phi\rangle \otimes |\psi\rangle$ becomes $U = \mathbb{I} \otimes P_\psi$ where P_ψ is the permutation operator over partition ψ.

In the definition of the *Combination-Key Operator* in equation (2), we assumed the absence of all superposition states being distinct. But what if $|\epsilon_i^{(k)}\rangle = |\epsilon_j^{(k)}\rangle$? Then the Combination-Key-mechanism which should have given the states (without the relative phases considered) of the form $|\chi\rangle = ...+|\phi_i\rangle\otimes|\psi_i\rangle+...+|\phi_j\rangle\otimes|\psi_j\rangle+... \xrightarrow{U_{rs_\psi}} ...+|\phi_i\rangle\otimes|\psi_{i+1}\rangle+...+|\phi_j\rangle\otimes|\psi_{j+1}\rangle+...$ but instead we get $|\chi\rangle = ...+|\phi_i\rangle\otimes|\psi_i\rangle+...+|\phi_j\rangle\otimes|\psi_j\rangle+... \xrightarrow{U_{rs_\psi}} ...+|\phi_i\rangle\otimes|\psi_{i+1}\rangle+|\phi_i\rangle\otimes|\psi_{j+1}\rangle+...+|\phi_j\rangle\otimes|\psi_{j+1}\rangle+|\phi_j\rangle\otimes|\psi_{i+1}\rangle+....$ The additional terms are generated due to the inherent similarity of certain superposition states. However, it turns out that these additional terms are useful for speeding up the permutation operator! If we define a new V_{rs_ψ} operator such that,

$$V_{rs_\psi} = \mathbb{I} \otimes \sum_i \sum_j |\epsilon_j^{(k)}\rangle\langle\epsilon_i^{(k)}| \tag{5}$$

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Here we are not considering the relative phases but the relevant corrections can be added if the phases are considered. Then the operation of V_{rs_ψ} on $|\chi\rangle$ generates every cyclic permutation at once. More correctly, V_{rs_ψ} generates multiple copies of every cyclic permutation since there will still be cases where $|\psi_i\rangle \neq |\psi_j\rangle$.

If a state $|\psi\rangle$ is considered as an m qubit state with its superposition states the various $|\psi_i\rangle$ states, then the operator V_{rs_ψ} may be considered to be an outer product of the form

$$V_{rs_\psi} = tr_\phi |\chi\rangle\langle\chi| \tag{6}$$

Determination of closure after these permutation operations could be achieved in a variety of ways:

1. In the *classical* algorithm, one can carry out $M-1$ sequential permutation operations, after which one has the newly generated set of states, denoted $|\chi'\rangle$ would have to be compared to the original after each permutation operation. A possible test is binary subtraction, namely if $|\chi'\rangle - |\chi\rangle \neq 0$ after all $M-1$ permutations then the state is not closed under that partition.

2. A possible speed-up may be possible by using certain quantum operations. Considering the most general symmetric N-qubit state with 2^N superposition states, $|N\rangle = \frac{1}{\sqrt{2^N}}((|0\rangle + |1\rangle)^{\otimes N})$. We can then use the given state $|\chi\rangle$, without consideration of relative phases, to form the state $|\chi_-\rangle = |N\rangle - |\chi\rangle$, which contains every N qubit superposition state not present in $|\chi\rangle$. It is then reasonable to state to reason that if a new state is generated after a permutation operation then $|\langle\chi|\chi_-\rangle| \neq 0$ and the state is not closed. If a state is closed under a particular partition of the system then $|\langle\chi_-|V_{rs_\psi}|\chi\rangle| = 0$.

3. If two copies of the same state $|\chi\rangle$ are taken and the *Quantum Combination-Key Operator* V_{rs_ψ} is used on one copy, we have the states $|\chi'\rangle = V_{rs_\psi}|\chi\rangle$ and $|\chi\rangle$. A point to be noted here is that the relative phases are not considered in this case. These states can now be compared to see whether the new state is the same as the reference state. Quantum state comparison relies on measurements based on projections and Positive-Operator Valued Measures (POVMs). No quantum measurement can unambiguously confirm when two quantum systems have been prepared in the same state when each system is prepared in some unknown pure state. It is only possible to detect when the states of the two systems are different, with a certain probability. Thus, given the states $|\chi'\rangle = V_{rs_\psi}|\chi\rangle$ and $|\chi\rangle$, we can find out using POVMs whether they are different. If so, then the state is not separable in the partition under consideration.

These methods can be generalized to the multipartite case. Let us consider a particular partition of an n-partite quantum state into an $[\alpha_1, \alpha_2, \alpha_3, ..]$ separability class. Then $|\chi\rangle$ can be written as $|\chi\rangle = \sum_{i=1}^{M} |\phi_i\rangle \otimes |\psi_i\rangle \otimes |\delta_i\rangle ....$

We can then define the set of $n-1$ operators

$$U_\phi = P_\phi \otimes \mathbb{I}_\psi \otimes \mathbb{I}_\delta ... \tag{7}$$

$$U_\psi = \mathbb{I}_\phi \otimes P_\psi \otimes \mathbb{I}_\delta ... \tag{8}$$

$$U_\delta = \mathbb{I}_\phi \otimes \mathbb{I}_\psi \otimes P_\delta \otimes ... \tag{9}$$

and so on, where P_α is the permutation operator on the partition α. If we apply this sequentially in a total of $(M-1)^{n-1}$ steps, we generate every permutation of the quantum system required.

Generalising the quantum permutation operator to the case applicable to an n-partite system, we have

$$U_n = tr_\psi |\chi\rangle\langle\chi| = |\phi\rangle|\delta\rangle ... \otimes \langle\phi|\langle\delta| ... \tag{10}$$

we find that one operation is required to carry out all $(M-1)^{n-1}$ permutation operations at once. We can then test for the closure of a particular n-partition for a given separability class using the equation: $\langle\chi'|U_n|\chi\rangle = 0$.

C. A Hybrid Permutation-Based Algorithm for Separability Analysis

Even with the speed-up possible with the elements and concepts presented in the last section, the problem of the presence of a large number of possible partitions possible for each separability class remains. This can be further optimized and made quicker using a combination of quantum and classical steps for a comprehensive hybrid algorithm for determining the particular separability class *and* partition for a quantum state.

1. *Determining the possible separability classes given a superposition state*

We determine which separability classes are possible for an N qubit state. This step involves factorising J, the number of superposition states in the system, into its prime factors. Efficient factorisation is possible with an algorithm like Shors algorithm [36]. We know that $J = J_{\alpha_1} J_{\alpha_2} J_{\alpha_3} ...$, where J_{α_i} is the number of superposition states for the partition α_i. If J is a prime number, this factorisation would be either be the case for the completely entangled state $[N]$ or the state is $[1, N-1]$ separable. There will be instances where this step of prime factorisation of J does not significantly reduce the number of separability classes within this method.

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2. Next, we look into ways of reducing possibilities of partitions that the state can have, in terms of separability patterns. For a single qubit to be separable, the number of $|1\rangle$ for that qubit in the superposition is found to be 0, J or $\frac{J}{2}$. Generally, if a qubit is part of a larger n_α qubit subsystem for the partition α, then the number of $|1\rangle$'s for that qubit is $\frac{x}{J_\alpha}J, 0 \geq x \leq J_\alpha$, where J_α are the number of superposition states in the partition α. The qubit is trivially separable if $x = 0$ or $x = J_\alpha$. Since J_α can only take values in the range $1 \geq J_\alpha \leq 2^{n_\alpha}$, this limits the set of qubits which be part of a separable n_α qubit subsystem. If we denote the number of $|1\rangle$ for the i^{th} qubit as $N_i^{(1)}$ then the condition that this column can form part of a separable n_α qubit subsystem is $\frac{N_i^{(1)}}{J} = \frac{x}{J_\alpha}J, 0 \geq x \leq J_\alpha$.

This step can limit the number of possible partitions of the system given a particular separability class.

3. We shall now look at which of the separability classes and partitions a state lies in. For this, we will define what we call the *Trace-Space Coordinates*:

$$Q_i^{(\alpha)} = \begin{cases} |1\rangle, \ U_{rs_{\alpha_{(i)}}}|\chi\rangle \text{ is closed} \\ |0\rangle, \ U_{rs_{\alpha_{(i)}}}|\chi\rangle \text{ is not closed} \end{cases} \tag{11}$$

where α refers to a particular separability class and (i) refers to the particular permutation or instance of this. So, if the placeholder values for each qubit is given from right-to-left, as per convention, we will assign the index i accordingly, depending on the larger of the bipartitions. For example, for the three qubit case and [21] separability, we take the '$2'$ qubits and assign $i = 1$ for $[23, 1]$, $i = 2$ to $[12, 3]$ and $i = 3$ to $[13, 2]$.

We then find all the coordinates this way and then assign relative phases to the coordinates for the same separability class and add the states to form a sum L^α. We then use positive-operator valued measure (POVMs) with basis-states based on these relative phases. So, for instance, for the three qubit states, we have the separability classes [3], [21] and [111]. However, the trace-method works only for bipartitions and thus we will only have cases for [21] with $i = 1$ for $[23, 1]$, $i = 2$ to $[12, 3]$ and $i = 3$ to $[13, 2]$. Let us say we have the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |100\rangle)$, we have $Q_1^{([21])} = |0\rangle, Q_2^{([21])} = |1\rangle$ and $Q_3^{([21])} = |0\rangle$. Thus, the relative-phase based construction would be of the form: $L^{[21]} = Q_1^{([21])} + e^{i\phi_1}Q_2^{([21])} + e^{2i\phi_1}Q_3^{([21])} = |0\rangle + e^{i\phi_1}|1\rangle + e^{2i\phi_1}|0\rangle$. This can be determined using POVMs with basis vectors: $|\nu_{k\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{ik\phi_\alpha}|1\rangle$ where ϕ_α denotes the separability class specific phase factor.

This completely describes the separability class and partition that a particular state falls into.

This method can be generalized to mixed states and higher-dimensional multiqubit states. In the case of the former, we can consider the separability of the pure states constituting the mixed states and analysing their separability. For the latter, we have to take bipartitions over various cases (a point considered within our algorithm itself) to find which multipartite separability class a multiqubit state falls into.

IV. SYMMETRY UNDER PERMUTATION OF QUBITS

Permutation of qubits and correlations in a quantum state are found to have a connection that has been explored extensively in the past [37–39]. In this paper, we look at two specific nuances of this subject: a new entanglement measure called the 'engle' and a general investigation of permutation symmetry and entanglement.

A. Engle

Two of the prominent entanglement measures that have been used widely have been the tangle and concurrence, which have a connection with the idea of the hyperdeterminant of the quantum state. The 'residual entanglement' or 'tangle' of an N-qubit state is defined as

$$\tau(\psi) = 2|\sum a_{\alpha_1\alpha_2\ldots\alpha_n} a_{\beta_1\beta_2\ldots\beta_n} a_{\gamma 1\gamma_2\ldots\gamma_n} a_{\delta_1\delta_2\ldots\delta_n} \times \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\ldots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\ldots\epsilon_{\gamma_{n-1}\delta_{n-1}}\epsilon_{\alpha_n\gamma_n}\epsilon_{\beta_n\delta_n}| \tag{12}$$

where the a terms are the coefficients in the standard basis $|\psi\rangle = \sum_{i_1\ldots i_N} a_{i_1 i_2\ldots i_N}|i_1 i_2 \ldots i_N\rangle$ and $\epsilon_{10} = -\epsilon_{01} = 1$ and $\epsilon_{00} = \epsilon_{11} = 0$.

This measure is defined so that
1. $0 \leq \tau(\psi) \leq 1$
2. $\tau(\psi) = 0$ for separable states.
3. $\tau(\psi) = 1$ for maximally entangled states.
4. $\tau(\psi)$ is invariant under qubit permutations and local unitary operations.

The tangle for three-qubit states is defined as:

$$\begin{aligned}\tau(\psi) = 4|a_{011}^2 a_{100}^2 + a_{010}^2 a_{101}^2 + a_{001}^2 a_{110}^2 + a_{000}^2 a_{111}^2 \\ -2a_{001}a_{011}a_{110}a_{100} + 4a_{001}a_{010}a_{111}a_{100} - 2a_{000}a_{011}a_{111}a_{100} \\ - 2a_{010}a_{011}a_{101}a_{100} - 2a_{001}a_{010}a_{101}a_{110} \\ +4a_{000}a_{011}a_{101}a_{110} - 2a_{000}a_{010}a_{101}a_{111} - 2a_{000}a_{001}a_{110}a_{111}|\end{aligned} \tag{13}$$

We find the tangle for our three-qubit vector states. For maximally entangled, separable as well as partially entangled states, the tangle value vanishes. This shows that it is not as good a witness, based on the conditions associated with it for different kinds of entanglement. This can be refined and corrected by studying the manner in which the various kinds of states contribute to the tangle. We create a functionally useful witness that we call the '*engle*'. Before moving to the definition of the witness, we would like to prove a result:

Given the same Hamming weight for all vector-states in superposition in a quantum system, the idea that maximum permutation symmetry leads to maximal entanglement, as seen in the case of the Dicke States, helps us in defining the new entanglement witness. Using this concept, we know that the product of coefficients of quantum states would be maximum when they are equal, which correspond to the maximally entangled case. We define define a new entanglement witness called 'engle' as

$$\zeta(\tau) = (1 - \frac{1}{{}^N P_n})^{-{}^N P_n}(\prod_i (1 - |a_i|^2)) \\ f(\prod_S (\sum_j |T_{|S\rangle_1,|S\rangle_j} - T_{|S_r\rangle_1,|S_r\rangle_j}|)) \quad (14)$$

where n represents the number of $|1\rangle$ qubits and

$$T_{|S\rangle_i,|S\rangle_j} \;=\; 1\text{if}|S\rangle_i - |S\rangle_j \;=\; |0\rangle\text{or } 0 \text{ otherwise}. \quad (15)$$

The function $f : f(x) = 1 \forall x \neq 0; f(x) = 0, x = 0$. $|S\rangle_i$ is a subsystem of the i^{th} superposition term in a state, with their being N superposition terms, while $|S_r\rangle_i$ denotes the remainder subsystem of the quantum state.

In this quantity, there are two parts of the witness: $\prod_i (1 - \frac{|a_i|}{N})$ and $f(\prod_S (\sum_j |T_{|S\rangle_1,|S\rangle_j} - T_{|S_r\rangle_1,|S_r\rangle_j}|))$. The former tells us about the amount of entanglement, using the symmetry considerations, while the latter removes cases that are separable.

B. Permutation-based Entanglement Witness

Another entanglement witness can be created based on the qubit-swap operation. Let us define a general separable mixed quantum state as

$$\rho = \sum_k p_k |\phi_1^{(k)}\rangle\langle\phi_1^{(k)}| \otimes |\phi_2^{(k)}\rangle\langle\phi_2^{(k)}| \otimes ... \otimes |\phi_N^{(k)}\rangle\langle\phi_N^{(k)}| \quad (16)$$

A general permutation P is found to be a potent tool for probing entanglement of quantum states. We know that a general permutation can expressed as a product of transpositions or 2-cycles $T_{i,j}$. For instance, a three element permutation (123) = (12)(13). Let us look at the expectation value of such an operator on a mixed quantum state,

$$\begin{aligned}\langle T_{i,j}\rangle &= Tr(T_{i,j}\rho) \\ &= Tr(\sum_k p_k |\phi_1^{(k)}\rangle\langle\phi_1^{(k)}| \otimes ...|\phi_j^{(k)}\rangle\langle\phi_i^{(k)}| \otimes \\ &\quad |\phi_i^{(k)}\rangle\langle\phi_j^{(k)}| ... \otimes |\phi_N^{(k)}\rangle\langle\phi_N^{(k)}|) \\ &= p_k |\langle\phi_1^{(k)}\rangle|^2 ... |\langle\phi_i^{(k)}|\phi_j\rangle|^2 ... |\langle\phi_N^{(k)}\rangle|^2 \geq 0 \quad (17)\end{aligned}$$

Thus for a separable state, the expectation value of $\langle T_{i,j}\rangle$ is positive or zero, and if $\langle T_{i,j}\rangle < 0$, then the state is entangled. This idea can be extended to higher dimensions using the idea that any permutation can be written as a transposition of elements (qubits). It then naturally follows that

$$\frac{\langle T_{i_1,j_1}\rangle}{|\langle T_{i_1,j_1}\rangle|^2} + \frac{\langle T_{i_2,j_2}\rangle}{|\langle T_{i_2,j_2}\rangle|^2} + ... \frac{\langle T_{i_S,j_S}\rangle}{|\langle T_{i_S,j_S}\rangle|^2} = -A \quad (18)$$

where A relates to the A cycle being investigated, T_{i_l,j_l} denotes the transposition operator for the l^{th} transposition between qubits i_l and j_l, and the system is entangled over the qubits covered by this permutation A-cycle. This gives another powerful method to check the entanglement of a state.

V. CONCLUSION

In this paper a comprehensive technique, based on the *Combination-Key Algorithm*, to determine the separability class of an arbitrary multipartite quantum state is proposed. For the case of mixed quantum states, conditions for separability are found in terms of the partial transposition of the density matrices of the quantum system. One of these conditions turns out to be the Partial Positive Transpose (PPT) condition. I also propose a graphical method and a couple of permutation symmetry-based entanglement measures. These methods can be crucial in determining the form of separability a state has and subsequently the kinds of applications it can be used for.

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