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Article

Heronian means as aggregation operators for multi-attribute decision making applications

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Abstract: The Pythagorean fuzzy set (PFS), which is characterized by a membership and a non-membership degree and the square sum of them is less or equal to one, can act as an effective tool to express decision makers' fuzziness and uncertainty. Considering that the Heronian mean (HM) is a powerful aggregation operator which can take the interrelationship between any two arguments, we study the HM in Pythagorean fuzzy environment and propose new operators for aggregating interval-valued Pythagorean fuzzy information. First, we investigate the HM and geometric HM (GHM) under interval-valued intuitionistic fuzzy environment and develop a series of aggregation operators for interval-valued intuitionistic fuzzy numbers (IVIFNs) including interval-valued intuitionistic fuzzy Heronian mean (IVIFHM), interval-valued intuitionistic fuzzy geometric Heronian mean (IVIFGHM), interval-valued intuitionistic fuzzy weighted Heronian mean (IVIFWHM) and interval-valued intuitionistic fuzzy weighted geometric Heronian mean (IVIFWGHM). Second, some desirable and important properties of these aggregation operators are discussed. Third, based on these aggregation operators, a novel approach to multi-attribute decision making (MADM) is proposed. Finally, to demonstrate the validity of the approach, a numerical example is provided and discussed. Moreover, we discuss several real-world applications of these operators within policy-making contexts.

Keywords: interval-valued intuitionistic fuzzy set; aggregation operator; Heronian mean; geometric Heronian mean; multi-attribute decision making

1. Introduction & Context

Decision making (DM) is a very common and significant activity across all walks of economy and society. Individuals and organisations face with a large number of DM events. With the complexity of DM increasing and expansion of choice sets and possibilities of real situations, fuzziness and vagueness have become a common problem in DM. Fuzziness in DM can be an important determinant of uncertainty and perception of uncertain environment. In other words, decision maker's own perception of the possibilities and uncertainties around those possibilities can be fuzzy or vague or ambiguous and may as well be specified by multiple probability functions. Such fuzziness or ambiguity can shape and determine the actions and behaviors of the economic agents. There are two related problems [54] – first, behavior is influenced by ambiguity; second, the economic agents are ambiguity averse. This can lead to conservative choices depending on the extent of ambiguity. Any estimation of the belief represented by convex non-additive probability functions can be tricky through traditional operators. Such cases arise frequently in policy-making across a wide spectrum of sectors, circumstances and stakeholders. Schmeidler [55] presented a model of decision making (Choquet expected utility (CEU) model) with ambiguity aversion. In this paper, we first develop operators that can effectively express DM problem and highlight several areas of application drawing on the relevant literature.

To deal with fuzziness and uncertainty, Zadeh [1] introduced the concept of Fuzzy Set (FS) theory, which allows decision makers to assign a membership degree to an element, representing the degree of the element belonging to a given fixed set. Since the introduction of FS, DM under FS has been widely investigated. Nevertheless, the FS only has a membership degree which makes it difficult to describe fuzziness, vagueness and hesitancy effectively. There are several extensions of FS such as interval-valued fuzzy set (IVFS) [2], type 2 fuzzy set [3] and type n fuzzy set [4]. Take the IVFS as an example, an interval is used to represent the membership degree instead of an accurate value which makes the IVFS contain more information and cause less distortion than FS. In 1986, another important extension of FS, called the intuitionistic fuzzy set (IFS) [5], was introduced by Atanassov. The IFS, which is characterized by a membership degree and a non-membership, is more suitable and useful to deal with fuzziness and hesitancy, which are common in DM for issues related to economy and society, than FS. Since the introduction of IFS, is has been widely investigated and applied to quite a few fields, such as medical diagnosis [6-8], pattern recognition [9-11], data mining [12-14] and MADM [15-18]. Motivated by the IVFS, Atanassov introduced the concept of IVIFS [19], which can be viewed as a combination of IVFS and IFS. The IVIFS has two intervals, denoting the membership degree and non-membership degree respectively. Because of this feature, the IVIFS has been widely investigated and applied to a lot of fields especially MADM [20-23].

In MADM, one of the most crucial aspects is aggregation operators. In the past several decades, several advancements have been reported. Xu [24] extended the traditional ordered weighted averaging (OWA) operator and the ordered weighted geometric averaging (OWG) operator to IFS and introduced a family of aggregation operators for intuitionistic fuzzy numbers (IFNs). Furthermore, Xu et al. [25] and Xu [26] generalized the OWA and OWG operators in interval-valued intuitionistic fuzzy environment and introduced a series of aggregation operators for IVIFNs. Motivated by the induced aggregation for IFNs [27], Yang et al. [28], Cai et al. [29] and Meng et al. [30] introduced some induced aggregation operators for IVIFNs. As relationship between arguments plays a crucial role in the aggregated result, scholars increasingly paid attention to aggregation operators which can incorporate the relationship into account. Xu [31] investigated the power aggregation operators under intuitionistic fuzzy environment and interval-valued intuitionistic fuzzy environment and developed power aggregation operators for IFNs and IVIFNs. Furthermore, Zhou et al. [32] proposed generalized forms of power aggregation operators for IFNs. Then He et al. [33] proposed the generalized power averaging operator for IVIFNs and applied that to MADM. Motivated by the Choquet integral, Xu [34], Tan et al. [35] and Wei et al. [36] introduced the Choquet integral operators for IFNs and applied these aggregation operators to MADM. Then, Meng et al. [37], Tan et al. [38] and Wu et al. [39] developed Choquet integral-based aggregation operators for IVIFNs. Xu et al. [40] investigated the Bonferroni mean (BM) for IFNs and introduced the intuitionistic fuzzy Bonferroni mean (IFBM). Xia et al. [41] and Zhou et al. [42] extended the BM and introduced the geometric Bonferroni mean (GBM). Furthermore, they studied the GBM in intuitionistic fuzzy environment and introduced the intuitionistic fuzzy geometric Bonferroni mean (IFGBM). Then, Xu et al. [43] and Shi et al. [44] generalized the BM and GBM in interval-valued intuitionistic fuzzy environment and introduced some BM and GBM for IVIFNs.

With a better understanding and awareness that the correlation between arguments can be a very significant determinant for the aggregated outcome, these aggregation operators which can build in the interrelationships between arguments have drawn a significant attention. The Heronian mean (HM) [45] operator can belong to these class of aggregation operators. Liu et al. [46] pointed out that the HM can overcome several drawbacks of BM. The HM was introduced for crisp numbers. In the past decade, some crucial advancement about HM in different fuzzy environments have been published. Liu et al. [47] introduced the generalized uncertain linguistic Heronian mean (GULHM), uncertain linguistic geometric Heronian mean (ULGHM), generalized uncertain linguistic weighted Heronian mean (GULWHM) and uncertain linguistic weighted geometric Heronian mean (ULWGHM) for uncertain linguistic variables. Liu et al. [48] investigated the HM in intuitionistic uncertain linguistic environment and developed some HM for intuitionistic uncertain linguistic. Yu et al. [49] generalized the HM in linguistic hesitant fuzzy environment and developed a series of HM aggregation operators for linguistic hesitant fuzzy information. Later, Chu et al. [50] studied the HM in two-dimensional uncertain linguistic fuzzy environment. Liu et al. [46] studied the HM in 2-tuple linguistic environment. Li et al. [51] extended the HM to some single valued neutrosophic number.

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Yu et al. [52] studied HM under dual hesitant fuzzy environment and developed some dual hesitant fuzzy HM aggregation operators. Nevertheless, to the best of our knowledge, there has been no rigorous study about HM under interval-valued intuitionistic fuzzy environment which is the motivation of this paper.

The remainder of the paper is organized as follows. In Section 2, we give a brief description of IVIFS, IVIFN with their operational laws. The HM of IVIFNs with their desirable prosperities are developed and discussed in Section 3. The weighted form of HM for IVIFNs are proposed in Section 4. A novel approach to MADM is presented in Section 5. To illustrate the approach, a numerical example is provided in Section 6. Section 7 highlights several areas of applications with reference to literature and the last section provides concluding remarks.

2. Preliminaries

In this section, we review the basic concepts about IVIFS and HM.

2.1 Interval-valued intuitionistic fuzzy set

Atanassov and Gargov [20] extended the IFS to IVIFS, using two intervals instead of two accurate values to represent the membership degree and non-membership degree respectively.

Definition 1. Let X be a given fixed set, an interval-valued intuitionistic fuzzy set (IVIFS) $\tilde{A}$ over X is defined as

$$\tilde{A}=\left\{\left\langle x,\tilde{\mu}_A(x),\tilde{v}_A(x)\right\rangle \middle| x\in X\right\} \tag{1}$$

where $\tilde{\mu}_A(x)\subset[0,1]$ represents the membership and $\tilde{v}_A(x)\subset[0,1]$ represents the non-membership degree, satisfying $0\le\sup\left(\tilde{\mu}_A(x)\right)+\sup\left(\tilde{v}_A(x)\right)\le 1$, $\forall x\in X$. For convenience, let $\tilde{\mu}_A(x)=[a,b]$ and $\tilde{v}_A(x)=[c,d]$, then $\tilde{A}$ can be denoted as $\tilde{A}=([a,b],[c,d])$, which can be called an IVIFN.

Definition 2. Let $\tilde{\alpha}_1=([a_1,b_1],[c_1,d_1])$, $\tilde{\alpha}_2=([a_2,b_2],[c_2,d_2])$, $\tilde{\alpha}=([a,b],[c,d])$ be any three IVIFNs. The operational law for IVIFNs can be defined as

(1) $\tilde{\alpha}_1\oplus\tilde{\alpha}_2=([a_1+a_2-a_1a_2,b_1+b_2-b_1b_2],[c_1c_2,d_1d_2])$,

(2) $\tilde{\alpha}_1\otimes\tilde{\alpha}_2=([a_1a_2,b_1b_2],[c_1+c_2-c_1c_2,d_1+d_2-d_1d_2])$,

(3) $\lambda\tilde{\alpha}=\left(\left[1-(1-a)^{\lambda},1-(1-b)^{\lambda}\right],\left[c^{\lambda},d^{\lambda}\right]\right)$,

(4) $\tilde{\alpha}^{\lambda}=\left(\left[a^{\lambda},b^{\lambda}\right],\left[1-(1-c)^{\lambda},1-(1-d)^{\lambda}\right]\right)$.

To compare two IVIFNs, Atanassov and Gargov [20] gave the definition of score function and accuracy function of an IVIFN. Then based on score function and accuracy function, a comparison law of two IVIFNs is introduced.

Definition 3. Let $\tilde{\alpha}=([a,b],[c,d])$ be an IVIFN, then a score function S and an accuracy function H can be defined as follows

$$S\left(\tilde{\alpha}\right)=(a-c+b-d)/2 \tag{2}$$

$$H\left(\tilde{\alpha}\right)=(a+b+c+d)/2 \tag{3}$$

Definition 4. Let $\tilde{\alpha}_1=([a_1,b_1],[c_1,d_1])$ and $\tilde{\alpha}_2=$

$([a_2,b_2],[c_2,d_2])$ be two IVIFNs, $S(\tilde{\alpha}_1)$ and $S(\tilde{\alpha}_2)$ be the scores of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ respectively; $H(\tilde{\alpha}_1)$ and $H(\tilde{\alpha}_2)$ be the accuracy of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ respectively. Then the comparison law of two IVIFNs can be defined as

(1) If $S(\tilde{\alpha}_1) > S(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 > \tilde{\alpha}_2$;

(2) If $S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2)$, then

if $H(\tilde{\alpha}_1) > H(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$;

if $H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 = \tilde{\alpha}_2$.

2.2 Heronian mean and geometric Heronian mean

Sykora [45] introduced the generalization form of HM, which is shown in Definition 4. It is worth pointing out that the HM was introduced for crisp numbers.

Definition 5. Let $a_i\,(i=1,2,\dots,n)$ be a collection of crisp numbers, then the Heronian mean (HM) operator is defined as

$$HM^{p,q}(a_1,a_2,\dots,a_n)=\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=i}^{n}a_i^p a_j^q\right)^{\frac{1}{p+q}} \quad (4)$$

where $p,q\geq 0$ and $p+q>0$.

Then, Liu et al. [48] introduced the concept of GHM.

Definition 6. Let $a_i\,(i=1,2,\dots,n)$ be a collection of nonnegative crisp numbers with $p,q\geq 0$ and $p+q>0$, then the geometric Heronian mean (GHM) operator is defined as

$$GHM^{p,q}(a_1,a_2,\dots,a_n)=\frac{1}{p+q}\left(\prod_{i=1,j=i}^{n}(pa_i+qa_j)^{\frac{2}{n(n+1)}}\right) \quad (5)$$

3. Heronian mean operators for interval-valued intuitionistic fuzzy information

As the HM and GHM were introduced for crisp numbers. In this section, we investigate the HM and GHM under interval-valued intuitionistic fuzzy environment and develop some new aggregation operators for IVIFNs.

3.1. The interval-valued intuitionistic fuzzy Heronian mean operator

Definition 7. Let $\tilde{\alpha}_i=([a_i,b_i],[c_i,d_i])\,(i=1,2,\dots,n)$ be a collection of IVIFNs, then the interval-valued intuitionistic fuzzy Heronian mean (IVIFHM) is defined as

$$IVIFHM^{p,q}(\tilde{\alpha}_1,\tilde{\alpha}_2,\dots,\tilde{\alpha}_n)=$$

$$\left(\frac{2}{n(n+1)}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(\tilde{\alpha}_i^p\otimes\tilde{\alpha}_j^q\right)\right)^{\frac{1}{p+q}} \quad (6)$$

where p and q are two crisp numbers, satisfying $p,q\geq 0$ and $p+q>0$.

By Definition 2, we can obtain Theorem 1, which is shown below.

Theorem 1. Let $\tilde{\alpha}_i=([a_i,b_i],[c_i,d_i])\,(i=1,2,\dots,n)$ be a collection of IVIFNs with $p,q\geq 0$ and $p+q>0$, then

The proof is shown in the Appendix.

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The $IVIFHM^{p,q}$ operator also have the following properties.

$$IVIFHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)=$$

$$\left(\begin{array}{c}\left[\left(1-\prod_{i=1,j=i}^{n}\left(1-a_i^p a_j^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1,j=i}^{n}\left(1-b_i^p b_j^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right],\\ \left[1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-c_i\right)^p\left(1-c_j\right)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-d_i\right)^p\left(1-d_j\right)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right]\end{array}\right) \quad (7)$$

Theorem 2. (Idempotency) Let $\tilde{\alpha}_i=\left(\left[a_i,b_i\right],\left[c_i,d_i\right]\right)\left(i=1,2,\ldots,n\right)$ be a collection of IVIFNs with $\tilde{\alpha}_i=\tilde{\alpha}=\left(\left[a,b\right],\left[c,d\right]\right)$ holds for all i, then

$$IVIFHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)=\tilde{\alpha} \quad (8)$$

The proof is shown in the Appendix

Theorem 3. (Monotonicity) Let $\tilde{\alpha}_i=\left(\left[a_i,b_i\right],\left[c_i,d_i\right]\right)\left(i=1,2,\ldots,n\right)$ and $\tilde{\beta}_i=\left(\left[e_i,f_i\right],\left[g_i,h_i\right]\right)\left(i=1,2,\ldots,n\right)$ be two collections of IVIFNS， if $a_i\le e_i$, $b_i\le f_i$, $c_i\ge g_i$ and $d_i\ge h_i$ then

$$IVIFHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)\le IVIFHM^{p,q}\left(\tilde{\beta}_1,\tilde{\beta}_2,\ldots,\tilde{\beta}_n\right) \quad (9)$$

The proof is shown in the Appendix

Theorem 4. (Boundedness)

Let $\tilde{\alpha}_i=\left(\left[a_i,b_i\right],\left[c_i,d_i\right]\right)\left(i=1,2,\ldots,n\right)$ be a collection of IVIFNs, and

$\tilde{\alpha}^-=\min\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)$, $\tilde{\alpha}^+=\max\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)$

Then

$$\tilde{\alpha}^-\le IVIFHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)\le\tilde{\alpha}^+ \quad (10)$$

The proof is shown in the Appendix

The interval-valued intuitionistic fuzzy geometric Heronian mean operator

Definition 8. Let $\tilde{\alpha}_i=\left(\left[a_i,b_i\right],\left[c_i,d_i\right]\right)\left(i=1,2,\ldots,n\right)$ be a collection of IVIFNs, then the interval-valued intuitionistic fuzzy geometric Heronian mean (IVIFGHM) is defined as

$$IVIFGHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)=\frac{1}{p+q}\left(\bigotimes_{i=1}^{n}\bigotimes_{j=i}^{n}\left(p\tilde{\alpha}_i,q\tilde{\alpha}_j\right)^{2/n(n+1)}\right) \quad (11)$$

By Definition 2, we can obtain Theorem 5, which is shown below.

Theorem 5. Let $\tilde{\alpha}_i=\left(\left[a_i,b_i\right],\left[c_i,d_i\right]\right)\left(i=1,2,\ldots,n\right)$ be a collection of IVIFNs with $p,q\ge0$ and $p+q>0$, then

The proof of Theorem 5 is similar to the proof of Theorem 1.

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There are some other desirable properties of IVIFGHM operator.

$$IVIFGHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)=$$
$$\left(\begin{array}{c}\left[1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-a_i)^p\left(1-a_j\right)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-b_i)^p\left(1-b_j\right)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right],\\ \left[\left(1-\prod_{i=1,j=i}^{n}\left(1-c_i^p c_j^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1,j=i}^{n}\left(1-d_i^p d_j^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right]\end{array}\right) \quad (12)$$

Theorem 6. (Idempotency) Let $\tilde{\alpha}_i=([a_i,b_i],[c_i,d_i])\,(i=1,2,\ldots,n)$ be a collection of IVIFNs with $\tilde{\alpha}_i=\tilde{\alpha}=([a,b],[c,d])$ holds for all *i*, then

$$IVIFGHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)=\tilde{\alpha} \quad (13)$$

The proof of Theorem 6 is similar to the proof of Theorem 2.

Theorem 7. (Monotonicity)

Let $\tilde{\alpha}_i=([a_i,b_i],[c_i,d_i])\,(i=1,2,\ldots,n)$ and $\tilde{\beta}_i=([e_i,f_i],[g_i,h_i])\,(i=1,2,\ldots,n)$ be two collections of IVIFNS， if $a_i\le e_i$, $b_i\le f_i$, $c_i\ge g_i$ and $d_i\ge h_i$ then

$$IVIFGHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)\ge IVIFGHM^{p,q}\left(\tilde{\beta}_1,\tilde{\beta}_2,\ldots,\tilde{\beta}_n\right) \quad (14)$$

The proof of Theorem 7 is similar to the proof of Theorem 3.

Theorem 8. (Boundedness)

Let $\tilde{\alpha}_i=([a_i,b_i],[c_i,d_i])\,(i=1,2,\ldots,n)$ be a collection of IVIFNs, and

$$\tilde{\alpha}^{-}=\min\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right),\quad \tilde{\alpha}^{+}=\max\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)$$

Then

$$\tilde{\alpha}^{-}\ge IVIFGHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)\ge\tilde{\alpha}^{+} \quad (15)$$

The proof of Theorem 8 is similar to the proof of Theorem 4.

4. Interval-valued intuitionistic fuzzy weighted Heronian mean

We have introduced the IVIFHM and the IVIFGHM operator for IVIFNs in Section 3. However, the weights of IVIFNs are not taken into account in those two aggregation operators. In this section, we develop some new HM aggregation operators for IVIFNs which can take the weight vector into consideration.

The interval-valued intuitionistic fuzzy weighted Heronian mean

Definition 9. Let $\tilde{\alpha}_i=([a_i,b_i],[c_i,d_i])\,(i=1,2,\ldots,n)$ be a collection of IVIFNs, then the interval-valued intuitionistic fuzzy weighted Heronian mean (IVIFGHM) is defined as

$$IVIFGHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)=\left(\frac{2}{n(n+1)}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(\left(w_i\tilde{\alpha}_i\right)^p\otimes\left(w_j\tilde{\alpha}_j\right)^q\right)\right)^{\frac{1}{p+q}} \quad (16)$$

where, $p,q\ge 0$ and $p+q>0$, $w=(w_1,w_2,\ldots,w_n)^T$ is the weight vector of $\tilde{\alpha}_i\,(i=1,2,\ldots,n)$, satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{n}w_i=1$.

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Theorem 9. Let $\tilde{\alpha}_i = \left([a_i, b_i],[c_i, d_i]\right)\left(i = 1,2,\ldots,n\right)$ be a collection of IVIFNs, $w = \left(w_1, w_2, \ldots, w_n\right)^T$ be the weight vector of $\tilde{\alpha}_i \left(i = 1,2,\ldots,n\right)$ with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then (See Eq. (17)).

The proof is similar to the proof of Theorem1.

$$IVIFWHM^{p,q}\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\right) = \left(\begin{bmatrix} \left(1 - \prod_{i=1, j=i}^{n}\left(1-\left(1-\left(1-a_i\right)^{w_i}\right)^p\left(1-\left(1-a_j\right)^{w_j}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}, \\ \left(1 - \prod_{i=1, j=i}^{n}\left(1-\left(1-\left(1-b_i\right)^{w_i}\right)^p\left(1-\left(1-b_i\right)^{w_j}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}} \end{bmatrix}, \begin{bmatrix} 1 - \left(1 - \prod_{i=1, j=i}^{n}\left(1-\left(1-c_i^{w_i}\right)^p\left(1-c_j^{w_j}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}, \\ 1 - \left(1 - \prod_{i=1, j=i}^{n}\left(1-\left(1-d_i^{w_i}\right)^p\left(1-d_j^{w_j}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}} \end{bmatrix}\right) \quad (17)$$

$$IVIFWGHM^{p,q}\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\right) = \left(\begin{bmatrix} 1 - \left(1 - \prod_{i=1, j=i}^{n}\left(1-\left(1-a_i^{w_i}\right)^p\left(1-a_j^{w_j}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}, \\ 1 - \left(1 - \prod_{i=1, j=i}^{n}\left(1-\left(1-b_i^{w_i}\right)^p\left(1-b_j^{w_j}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}} \end{bmatrix}, \begin{bmatrix} \left(1 - \prod_{i=1, j=i}^{n}\left(1-\left(1-\left(1-c_i\right)^{w_i}\right)^p\left(1-\left(1-c_i\right)^{w_j}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}, \\ \left(1 - \prod_{i=1, j=i}^{n}\left(1-\left(1-\left(1-d_i\right)^{w_i}\right)^p\left(1-\left(1-d_i\right)^{w_j}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}} \end{bmatrix}\right) \quad (19)$$

4.1 The interval-valued intuitionistic fuzzy weighted geometric Heronian mean

Definition 10. Let $\tilde{\alpha}_i = \left([a_i, b_i],[c_i, d_i]\right)\left(i = 1,2,\ldots,n\right)$ be a collection of IVIFNs, then the interval-valued intuitionistic fuzzy weighted geometric Heronian mean (IVIFWGHM) is defined as

$$IVIFWGHM^{p,q}\left(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\right) = \frac{1}{p+q}\left(\bigotimes_{i=1}^{n}\bigotimes_{j=i}^{n}\left(p\left(w_i\tilde{\alpha}_i\right) \oplus q\left(w_j\tilde{\alpha}_j\right)\right)\right)^{2/n(n+1)} \quad (18)$$

Theorem 10. Let $\tilde{\alpha}_i = \left([a_i, b_i],[c_i, d_i]\right)\left(i = 1,2,\ldots,n\right)$ be a collection of IVIFNs, $w = \left(w_1, w_2, \ldots, w_n\right)^T$ be the weight vector of $\tilde{\alpha}_i \left(i = 1,2,\ldots,n\right)$ with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then we can obtain the result shown as Eq. (19).

The proof is similar to the proof of Theorem1.

Authors should discuss the results and how they can be interpreted in perspective of previous studies and of the working hypotheses. The findings and their implications should be discussed in the broadest context possible. Future research directions may also be highlighted.

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5. A novel approach to multi-attribute decision making with interval-valued intuitionistic fuzzy information

In the present section, a novel approach to MADM under interval-valued intuitionistic fuzzy environment is proposed. Let $X=\{x_1,x_2,\ldots x_n\}$ be a set of alternatives, $G=\{G_1,G_2,\ldots G_m\}$ be a set of attributes with the weight vector $w=(w_1,w_2,\ldots,w_m)^T$, satisfying $w_i\in[0,1]$ and $\sum_{i=1}^{m}w_i=1$. The decision maker is required to express their preference information by IVIFNs. All the IVIFNs construct the interval-valued intuitionistic fuzzy decision matrix denoted by $\tilde{R}=(\tilde{r}_{ij})_{n\times m}=([a_{ij},b_{ij}],[c_{ij},d_{ij}])_{n\times m}$ where $[a_{ij},b_{ij}]$ and $[c_{ij},d_{ij}]$ repressively represent the degrees that the alternative x_i satisfies and does not satisfy the attribute G_j, satisfying $[a_{ij},b_{ij}]\subset[0,1]$, $[c_{ij},d_{ij}]\subset[0,1]$, $b_{ij}+d_{ij}\le 1$, $i=1,2,\ldots,n$ and $j=1,2,\ldots,m$.

Next, we utilize the aggregation operators introduced in Section 4 to solve this problem. The steps are shown below.

Step 1. Standardize the decision matrix. Generally, the attributes can be classified into two varieties, the benefit attribute and the cost attribute. The decision matrix should be normalized by

$$\tilde{r}_{ij}=\begin{cases}([a_{ij},b_{ij}],[c_{ij},d_{ij}]), G_i\in I_1\\([c_{ij},d_{ij}],[a_{ij},b_{ij}]), G_i\in I_2\end{cases} \tag{20}$$

where $i=1,2,\ldots n$, $j=1,2,\ldots,m$, I_1 and I_2 represent the benefit attribute and the cost attribute respectively. We denote the standardized matrix by $\tilde{R}=(\tilde{r}_{ij})_{n\times m}=([u_{ij},v_{ij}],[t_{ij},f_{ij}])_{n\times m}$.

Step 2. Utilize the IVIFWHM operator or the IVIFWGH operator to aggregate $\tilde{r}_{ij}$ $(j=1,2,\ldots,m)$ of the ith line and the overall value $\tilde{r}_i$ of alternatives x_i $(i=1,2,\ldots,n)$ can be obtained. (See Eq. (21) and Eq. (22) in the next page)

Step 3. Calculate the scores of $\tilde{r}_i$ $(i=1,2,\ldots,n)$ by definition 3.

Step 4. Rank $\tilde{r}_i$ $(i=1,2,\ldots,n)$ according to their scores by definition 4.

Step 5. Rank the alternatives x_i $(i=1,2,\ldots,n)$ according to the rank of $\tilde{r}_i$ $(i=1,2,\ldots,n)$ and choose the best alternative.

6. Numerical example

In this section, we utilize a numerical example introduced by Sun et al. [53] to illustrate the validity of the approach to MDAM in Section 5.

Decision makers are required to evaluate innovation capability and efficiency of high technology enterprises. There are five enterprises x_i $(i=1,2,3,4,5)$ and the four attributes are innovation resources input ability (G_1), research and development ability (G_2), manufacturing capacity and marketing ability (G_3) and innovation output capacity (G_4). Decision makers use IVIFNs to estimate the enterprise. The decision matrix is given as $\tilde{R}=(\tilde{r}_{ij})_{5\times 4}=([a_{ij},b_{ij}],[c_{ij},d_{ij}])_{5\times 4}$. The weight vector of attributes is $w=(0.15,0.35,0.2,0.3)^T$.

$$\tilde{R}=\begin{bmatrix}([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) & ([0.1,0.3],[0.5,0.6]) & ([0.3,0.4],[0.3,0.5])\\([0.6,0.7],[0.2,0.3]) & ([0.6,0.7],[0.2,0.3]) & ([0.4,0.7],[0.1,0.2]) & ([0.5,0.6],[0.1,0.3])\\([0.3,0.6],[0.3,0.4]) & ([0.5,0.6],[0.3,0.4]) & ([0.5,0.6],[0.1,0.3]) & ([0.4,0.5],[0.2,0.4])\\([0.7,0.8],[0.1,0.2]) & ([0.6,0.7],[0.1,0.3]) & ([0.3,0.4],[0.1,0.2]) & ([0.3,0.7],[0.1,0.2])\\([0.3,0.4],[0.2,0.3]) & ([0.3,0.5],[0.1,0.3]) & ([0.2,0.5],[0.4,0.5]) & ([0.3,0.4],[0.5,0.6])\end{bmatrix}$$

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Next, we first utilize the novel approach to solve this problem. Then, we discuss more results by different values to the parameters p and q.

6.1 Calculation process

Step 1. Because all the attributes are benefit attributes, we do not need to standardize the decision matrix.

$$\tilde{r}_i = IVIFWHM^{p,q}\left(\tilde{r}_{i1},\tilde{r}_{i2},\ldots,\tilde{r}_{im}\right) =$$

$$\left(\begin{array}{l}\left[\left(1-\prod_{k=1,l=k}^{m}\left(1-\left(1-(1-u_{ik})^{w_k}\right)^p\left(1-(1-u_{il})^{w_l}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},\right.\\ \left.\left(1-\prod_{k=1,l=k}^{m}\left(1-\left(1-(1-v_{ik})^{w_k}\right)^p\left(1-(1-v_{il})^{w_l}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right],\\ \left[1-\left(1-\prod_{k=1,l=k}^{m}\left(1-\left(1-t_{ik}^{w_k}\right)^p\left(1-t_{il}^{w_l}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},\right.\\ \left.1-\left(1-\prod_{k=1,l=k}^{m}\left(1-\left(1-f_{ik}^{w_k}\right)^p\left(1-f_{il}^{w_l}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right]\end{array}\right) \quad (21)$$

$$\tilde{r}_i = IVIFWGHM^{p,q}\left(\tilde{r}_{i1},\tilde{r}_{i2},\ldots,\tilde{r}_{im}\right) =$$

$$\left(\begin{array}{l}\left[1-\left(1-\prod_{k=1,l=k}^{m}\left(1-\left(1-u_{ik}^{w_k}\right)^p\left(1-u_{il}^{w_l}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},\right.\\ \left.1-\left(1-\prod_{k=1,l=k}^{m}\left(1-\left(1-v_{ik}^{w_k}\right)^p\left(1-v_{il}^{w_l}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right],\\ \left[\left(1-\prod_{k=1,l=k}^{m}\left(1-\left(1-(1-t_{ik})^{w_k}\right)^p\left(1-(1-t_{il})^{w_l}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},\right.\\ \left.\left(1-\prod_{k=1,l=k}^{m}\left(1-\left(1-(1-f_{ik})^{w_k}\right)^p\left(1-(1-f_{il})^{w_l}\right)^p\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right]\end{array}\right) \quad (22)$$

Step 2. Utilize Eq. (22) to aggregate the preference information of alternative $x_i\ (i=1,2,3,4,5)$. Here, without loss of generality, let p=q=1. We can obtain a series of overall values for the five enterprises.

$\tilde{r}_1$ = ([0.0930, 0.1509]. [0.7346, 0.8249])

$\tilde{r}_2$ = ([0.1544, 0.2201]. [0.6938, 0.7644])

$\tilde{r}_3$ = ([0.1375, 0.1914]. [0.6808, 0.7840])

$\tilde{r}_4$ = ([0.1572, 0.2447]. [0.5657, 0.6937])

$\tilde{r}_5$ = ([0.0799, 0.1422]. [0.7036, 0.7998])

Step 3. Calculate the scores of the overall values of $\tilde{r}_i\ (i=1,2,3,4,5)$.

$S(\tilde{r}_1)=-0.6578 \quad S(\tilde{r}_2)=-0.5419$

$S\left(\tilde{r}_3\right) = -0.5679 \quad S\left(\tilde{r}_4\right) = -0.4287$

$S\left(\tilde{r}_5\right) = -0.6406$

Step 4. Rank the overall values according to their scores, we can obtain $\tilde{r}_4 > \tilde{r}_2 > \tilde{r}_3 > \tilde{r}_5 > \tilde{r}_1$.

Step 5. According to the rank of $\tilde{r}_i \left(i = 1,2,3,4,5\right)$, we have $x_4 \succ x_2 \succ x_3 \succ x_5 \succ x_1$, which means x_2 is the highest technological enterprise.

6.2 Further discussion

It is noted that the parameter p and q play a crucial role in the final result. Decision makers can change the parameters, leading to different outcomes. In this subsection, we discuss the impact of parameters p and q on the final result.

If p and q changes between 0 and 4, we can get different overall values of $\tilde{r}_i \left(i = 1,2,3,4,5\right)$. Thus, the scores of the overall values will be different. More details about the scores of overall values $\tilde{r}_i \left(i = 1,2,3,4,5\right)$ are shown in Figure 1-5. These results are obtained by the IVIFWHM operator.

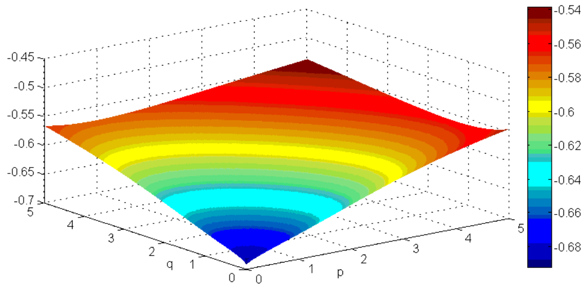

Fig. 1. Score of x_1

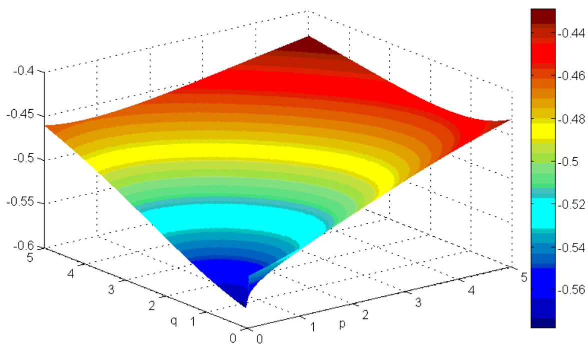

Fig. 2. Score of x_2

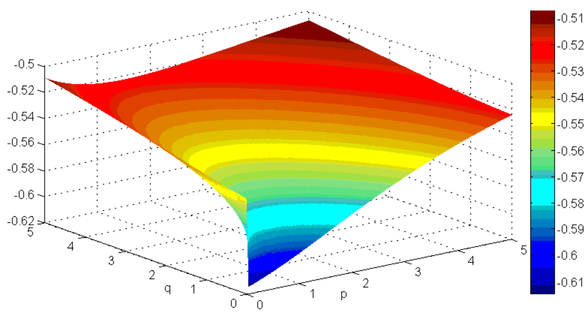

Fig. 3. Score of x_3

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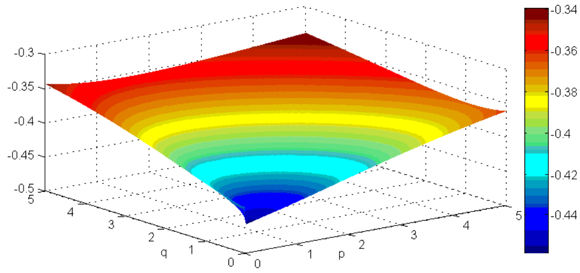

Fig. 4. Score of x_4

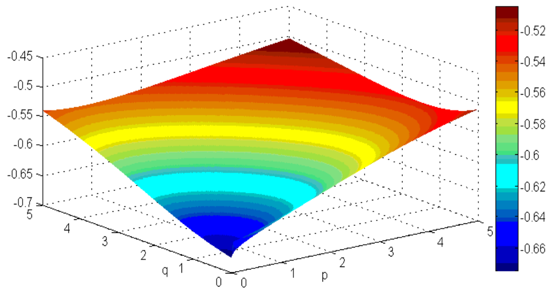

Fig. 5. Score of x_5

Then let p be a fixed value, when p changes, we may get different scores of $x_i\ (i=1,2,3,4,5)$ and different ranking result. Here, let p=1. Figure 5 illustrates details of scores and ranking order.

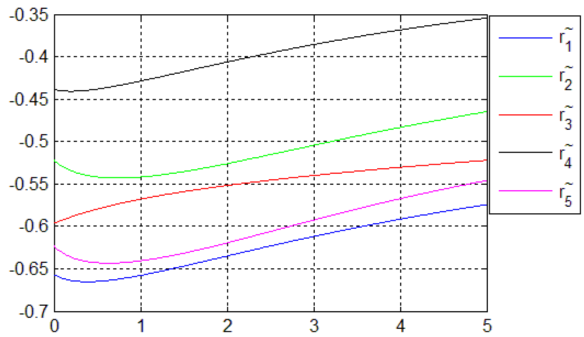

Fig. 6. The scores and ranking order (p=1, $q \in [0,5]$).

From Figure 6, we observe that when p=1 and q changes from 0 to 5, the ranking order is always $x_4 \succ x_2 \succ x_3 \succ x_5 \succ x_1$. However, if let q be a fixed value, we can get a different ranking order. Let q=1 and p changes from 0 to 5. The scores and ranking orders and shown in Figure 7.

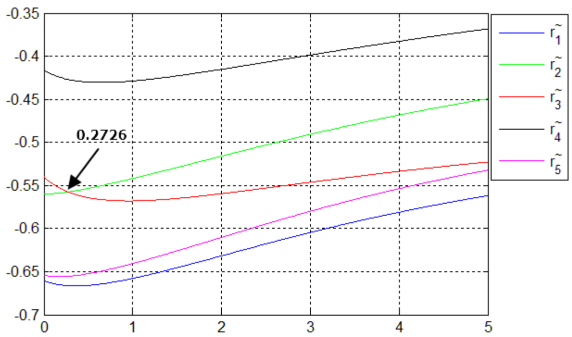

Fig. 7. The scores and ranking order (q=1, $p \in [0,5]$).

From Figure 7, we observe that

(1) when $p \in [0, 0.2726)$, the ranking order is $x_4 \succ x_3 \succ x_2 \succ x_5 \succ x_1$, and the best alternative is x_4;

(2) when $p \in (0.2726, 5]$, the ranking order is $x_4 \succ x_2 \succ x_3 \succ x_5 \succ x_1$ and the best alternative is x_4.

From Figures 6 and 7, it is evident that by assigning different values, scores of aggregated overall values will be different. As the parameters p and q increase in value, the scores of the aggregated values will also increase. This phenomenon demonstrates that the parameters p and q play a very important role in determining the outcome. In real MADM, this has significant implication. Decision makers' perception (i.e. pessimistic vs. optimistic) of the environment attributes will lead to higher or lower values for p and q. It is also possible that parameters can take varying patterns, which would determine the outcomes. This will depend on specific real-world contexts, which we try to indicate in the next section with a few examples.

7. Application areas

Complexity of real-world issues can significantly determine the trajectory of changes and patterns in the parameter values. In cases where there are both cross-sectional (or spatial) and temporal variations, the parameters will require appropriate functional approximation or non-parametric approaches. In following discussion, we highlight a few examples of areas of application for the HM operators developed above and showcase the MADM problems.

- *House price evaluation*: House purchase involves several MADM events. Right from forming a choice set to steps towards preferred search option and subsequent offer-making and mortgage financing processes involve several numerical as well as non-numerical evaluation with fuzziness in linguistic descriptions and uncertainty in outcomes. Such fuzziness can influence search patterns, decision inertia and timings. Asymmetry in information flow and perceptions among economic decision-makers contribute to the uncertainty. Moreover, perception varies significantly among the economic agents i.e. p and q in the above operators. The HM operators developed above can offer improvement in the forecasting precision [56].
- *Investment project appraisal*: Assessment of investment project risk is problematic due to inherent probabilities and possibilities of outcomes. The parameter uncertainty is generally described by probability and possibility distributions. The investment risk assessment is undertaken on the assumption that uncertainty distributions of the effectiveness calculation parameters take the form of fuzzy numbers [57]. Simple aggregation can lead to biased appraisal values.
- *Project scheduling problem*: Project scheduling problem typically involves determination and allocation of resources which lead to balancing of the total cost and the completion time. There are several issues of relevance in the problem - mixed uncertainty with randomness and fuzziness. Moreover, activity duration times also need to be assumed as random fuzzy variables [58]. Such problems require multiple aggregation procedures with often linguistic description of the choice sets.
- *Decision policies in system dynamics models*: System Dynamics (SD) can be defined as the branch of control theory that deals with socioeconomic systems and the branch of science that deals with management issues' controllability [59]. The use of fuzzy systems can enhance the application of system dynamics to understand real-world problems such as complex urban planning issues, designing urban operating system, citizen engagement problems etc. As MADM environments are frequently present in SD applications, a fuzzy-SD integrated methodology can allow a natural language modeling of decision policies [60].

8. Conclusion

In this paper, we investigate the aggregation operators for IVIFNs. Considering that the IVIFNs provided by decision makers are not independent, we focus on the aggregation operators which can capture the interrelationship between IVIFNs. The HM is a class of aggregation operators which can effectively incorporate the interrelationship between arguments. However, the HM was introduced for crisp numbers in previous studies. Thus, we study HM in interval-valued environment and develop a family of new aggregation operators for IVIFNs. Then based on these new aggregation operators, we propose a new approach to MADM in which attributes values take the form of IVIFNs. To illustrate the new method, a numerical example about enterprise evaluation problem is presented. Furthermore, as the parameters play a significant role in the final result, we investigate the impact of the parameters on the ranking order. Several application areas have been highlighted.

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Appendix A

Proof of Theorem 1.

Proof. By Definition 2, we have

$$\tilde{\alpha}_i^p = \left(\left[a_i^p, b_i^p\right], \left[1-\left(1-c_i\right)^p, 1-\left(1-d_i\right)^p\right]\right),$$

$$\tilde{\alpha}_j^q = \left(\left[a_j^q, b_j^q\right], \left[1-\left(1-c_j\right)^q, 1-\left(1-d_j\right)^q\right]\right).$$

Thus,

$$\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q = \begin{pmatrix} \left[a_i^p a_j^q, b_i^p b_j^q\right], \\ \left[1-\left(1-c_i\right)^p\left(1-c_j\right)^q, 1-\left(1-d_i\right)^q\left(1-d_j\right)^q\right] \end{pmatrix}.$$

In the followings, we fist prove that

$$\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q\right) =$$

$$\left(\left[1-\prod_{i=1,j=i}^{n}\left(1-a_i^p a_j^q\right), 1-\prod_{i=1,j=i}^{n}\left(1-b_i^p b_j^q\right)\right], \left[\prod_{i=1,j=i}^{n}\left(1-\left(1-c_i\right)^p\left(1-c_j\right)^q\right), \prod_{i=1,j=i}^{n}\left(1-\left(1-d_i\right)^p\left(1-d_j\right)^q\right)\right]\right) \quad (1)$$

(1) For n=2, we have

$$\bigoplus_{i=1}^{2}\bigoplus_{j=i}^{2}\left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q\right) = \left(\tilde{\alpha}_1^p \otimes \tilde{\alpha}_1^q\right) \oplus \left(\tilde{\alpha}_1^p \otimes \tilde{\alpha}_2^q\right) \oplus \left(\tilde{\alpha}_2^p \otimes \tilde{\alpha}_2^q\right)$$

$$\begin{pmatrix} \left[a_1^p a_1^q, b_1^p b_1^q\right], \\ \left[1-\left(1-c_1\right)^p\left(1-c_1\right)^q, 1-\left(1-d_1\right)^q\left(1-d_1\right)^q\right] \end{pmatrix} \oplus \begin{pmatrix} \left[a_1^p a_2^q, b_1^p b_2^q\right], \\ \left[1-\left(1-c_1\right)^p\left(1-c_2\right)^q, 1-\left(1-d_1\right)^q\left(1-d_2\right)^q\right] \end{pmatrix} \oplus$$

$$\begin{pmatrix} \left[a_2^p a_2^q, b_2^p b_2^q\right], \\ \left[1-\left(1-c_2\right)^p\left(1-c_2\right)^q, 1-\left(1-d_2\right)^q\left(1-d_2\right)^q\right] \end{pmatrix} =$$

$$\left(\left[1-\prod_{i=1,j=i}^{2}\left(1-a_i^p a_j^q\right), 1-\prod_{i=1,j=i}^{2}\left(1-b_i^p b_j^q\right)\right], \left[\prod_{i=1,j=i}^{2}\left(1-\left(1-c_i\right)^p\left(1-c_j\right)^q\right), \prod_{i=1,j=i}^{2}\left(1-\left(1-d_i\right)^p\left(1-d_j\right)^q\right)\right]\right) \quad (2)$$

i.e. Eq. (1) holds for n=2.

(2) If Eq. (1) holds for n=k, i.e.

$$\bigoplus_{i=1}^{k}\bigoplus_{j=i}^{k}\left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q\right) =$$

$$\left(\left[1-\prod_{i=1,j=i}^{k}\left(1-a_i^p a_j^q\right), 1-\prod_{i=1,j=i}^{k}\left(1-b_i^p b_j^q\right)\right], \left[\prod_{i=1,j=i}^{k}\left(1-\left(1-c_i\right)^p\left(1-c_j\right)^q\right), \prod_{i=1,j=i}^{k}\left(1-\left(1-d_i\right)^p\left(1-d_j\right)^q\right)\right]\right).$$

Then when n=k+1, we have

$$\bigoplus_{i=1}^{k+1}\bigoplus_{j=i}^{k+1}\left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q\right) = \bigoplus_{i=1}^{k}\bigoplus_{j=i}^{k}\left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q\right) \oplus \left(\bigoplus_{i=1}^{k+1}\left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_{k+1}^q\right)\right)$$

$$\left(\left[1-\prod_{i=1,j=i}^{k+1}\left(1-a_i^p a_j^q\right), 1-\prod_{i=1,j=i}^{k+1}\left(1-b_i^p b_j^q\right)\right], \left[\prod_{i=1,j=i}^{k+1}\left(1-\left(1-c_i\right)^p\left(1-c_j\right)^q\right), \prod_{i=1,j=i}^{k+1}\left(1-\left(1-d_i\right)^p\left(1-d_j\right)^q\right)\right]\right) \quad (3)$$

i.e. Eq. (1) holds for n=k+1. By Eq. (2) and Eq. (3), we get that Eq. (1) holds for all n.

Furthermore, by Definition 2, we can obtain

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$$\frac{2}{n(n+1)}\bigoplus_{i=1}^{n}\bigoplus_{j=i}^{n}\left(\tilde{\alpha}_i^p\otimes\tilde{\alpha}_j^q\right)=\left(\begin{array}{c}\left[1-\prod_{i=1,j=i}^{n}\left(1-a_i^pa_j^q\right)^{2/n(n+1)},1-\prod_{i=1,j=i}^{n}\left(1-b_i^pb_j^q\right)^{2/n(n+1)}\right],\\ \left[\prod_{i=1,j=i}^{n}\left(1-(1-c_i)^p(1-c_j)^q\right)^{2/n(n+1)},\prod_{i=1,j=i}^{n}\left(1-(1-d_i)^p(1-d_j)^q\right)^{2/n(n+1)}\right]\end{array}\right)$$

Thus,

$$IVIFHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)=$$

$$\left(\begin{array}{c}\left[\left(1-\prod_{i=1,j=i}^{n}\left(1-a_i^pa_j^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1,j=i}^{n}\left(1-b_i^pb_j^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right],\\ \left[1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-c_i)^p(1-c_j)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-d_i)^p(1-d_j)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right]\end{array}\right).$$

Proof of Theorem 2.

Proof. Since $\tilde{\alpha}_i=\tilde{\alpha}=([a,b],[c,d])$ holds for all i, then we have

$$IVIFHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)=IVIFHM^{p,q}\left(\tilde{\alpha},\tilde{\alpha},\ldots,\tilde{\alpha}\right)=$$

$$\left(\begin{array}{c}\left[\left(1-\prod_{i=1,j=i}^{n}\left(1-a^pa^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},\left(1-\prod_{i=1,j=i}^{n}\left(1-b^pb^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right],\\ \left[1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-c)^p(1-c)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},1-\left(1-\prod_{i=1,j=i}^{n}\left(1-(1-d)^p(1-d)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}\right]\end{array}\right)=$$

$$\left(\begin{array}{c}\left[\left(1-\left(1-a^{p+q}\right)^{\frac{n(n+1)}{2}\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}},\left(1-\left(1-b^{p+q}\right)^{\frac{n(n+1)}{2}\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right],\\ \left[1-\left(1-\left(1-(1-c)^{p+q}\right)^{\frac{n(n+1)}{2}\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}},1-\left(1-\left(1-(1-d)^{p+q}\right)^{\frac{n(n+1)}{2}\frac{2}{n(n+1)}}\right)^{\frac{1}{p+q}}\right]\end{array}\right)=([a,b],[c,d])=\tilde{\alpha}$$

Proof of Theorem 3.

Proof. Since $a_i\le e_i$, $a_j\le e_j$, $b_i\le f_i$ and $b_j\le f_j$ for i=1,2,…, n and j=i, i+1, …, n, we have $a_i^pa_j^q\le e_i^pe_j^q$ and $b_i^pb_j^q\le f_i^pf_j^q$.

Then

$$\left(1-\prod_{i=1,j=i}^{n}\left(1-a_i^pa_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}\le\left(1-\prod_{i=1,j=i}^{n}\left(1-e_i^pe_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}$$

and

$$\left(1-\prod_{i=1,j=i}^{n}\left(1-b_i^pb_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}\le\left(1-\prod_{i=1,j=i}^{n}\left(1-f_i^pf_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}$$

Since $c_i\le g_i$, $c_j\le g_j$, $d_i\le h_i$ and $d_j\le h_j$ or i=1,2,…, n and j=i, i+1, …, n, we have $(1-c_i)^p(1-c_j)^q\ge(1-g_i)^p(1-g_j)^q$ and $(1-d_i)^p(1-d_j)^q\ge(1-h_i)^p(1-h_j)^q$.

Then

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$$1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-c_i\right)^p\left(1-c_j\right)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}} \geq 1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-g_i\right)^p\left(1-g_j\right)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}},$$

and

$$1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-d_i\right)^p\left(1-d_j\right)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}} \geq 1-\left(1-\prod_{i=1,j=i}^{n}\left(1-\left(1-h_i\right)^p\left(1-h_j\right)^q\right)^{2/n(n+1)}\right)^{\frac{1}{p+q}}.$$

Let $IVIFHM^{p,q}\left(\tilde{\alpha}_1,\tilde{\alpha}_2,\ldots,\tilde{\alpha}_n\right)=\tilde{\alpha}$, $\tilde{\beta}=IVIFHM^{p,q}\left(\tilde{\beta}_1,\tilde{\beta}_2,\ldots,\tilde{\beta}_n\right)$ then

$$S\left(\tilde{\alpha}\right)=\left(\begin{array}{l}\left(1-\prod_{i=1,j=i}^{n}\left(1-a_i^p a_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}-\left(1-\prod_{i=1,j=i}^{n}\left(1-b_i^p b_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}+\\ \left(1-\prod_{i=1,j=i}^{n}\left(1-e_i^p e_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}-\left(1-\prod_{i=1,j=i}^{n}\left(1-f_i^p f_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}\end{array}\right)\Bigg/2$$

and

$$S\left(\tilde{\beta}\right)=\left(\begin{array}{l}\left(1-\prod_{i=1,j=i}^{n}\left(1-e_i^p e_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}-\left(1-\prod_{i=1,j=i}^{n}\left(1-f_i^p f_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}+\\ \left(1-\prod_{i=1,j=i}^{n}\left(1-g_i^p g_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}-\left(1-\prod_{i=1,j=i}^{n}\left(1-h_i^p h_j^q\right)^{1/n(n+1)}\right)^{\frac{1}{p+q}}\end{array}\right)\Bigg/2$$

Since $s\left(\tilde{\alpha}\right)-s\left(\tilde{\beta}\right)\leq 0$, then $\tilde{\alpha}\leq\tilde{\beta}$, which completes the proof.

Proof of Theorem 4.

Proof. By Theorem 2, we can obtain $\tilde{\alpha}^- = IVIFHM^{p,q}\left(\tilde{\alpha}^-,\tilde{\alpha}^-,\ldots,\tilde{\alpha}^-\right)$ and $\tilde{\alpha}^+ = IVIFHM^{p,q}\left(\tilde{\alpha}^+,\tilde{\alpha}^+,\ldots,\tilde{\alpha}^+\right)$

By Theorem 3, we have $IVIFHM^{p,q}\left(\tilde{\alpha}^-,\tilde{\alpha}^-,\ldots,\tilde{\alpha}^-\right)\leq IFIVHM^{p,q}\left(\tilde{\alpha}\right)\leq IVIFHM^{p,q}\left(\tilde{\alpha}^+,\tilde{\alpha}^+,\ldots,\tilde{\alpha}^+\right)$, which completes the proof.

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