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A new sequence and its some congruence properties*

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Abstract

The aim of this paper is to use the elementary method and the properties of the second kind Stirling numbers to study the congruence properties of a new sequence, which is closely related to Fubini polynomials and Euler numbers, and give some interesting congruences for it. This solves a problem proposed by the first author at an unpublished paper.

Keywords: Fubini polynomials, Euler numbers, congruence, elementary method.

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1. Introduction

Let $n \geq 0$ be an integer, the famous Fubini polynomials $F_n(y)$ is defined by the coefficients of the generating function

$$\frac{1}{1-y\left(e^t-1\right)} = \sum_{n=0}^{\infty} \frac{F_n(y)}{n!} \cdot t^n, \tag{1}$$

where $F_0(y) = 1$, $F_1(y) = y$, and so on.

These polynomials are closely related to the Stirling numbers and Euler numbers. For example, if $y = -\frac{1}{2}$, then (1) becomes

$$\frac{2}{1+e^t} = \sum_{n=0}^{\infty} \frac{E_n}{n!} \cdot t^n, \tag{2}$$

where E_n denotes the Euler numbers.

On the other hand, the two variable Fubini polynomials are also defined by means of the following (see [5, 9])

$$\frac{e^{xt}}{1-y\left(e^t-1\right)} = \sum_{n=0}^{\infty} \frac{F_n(x,y)}{n!} \cdot t^n,$$

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and $F_n(y) = F_n(0, y)$ for all integers $n \geq 0$. Many people have studied the properties of $F_n(x, y)$, and have obtained a lot of important works. For example, T. Kim et al. proved a series of identities related to $F_n(x, y)$ (see [9, 10]), one of which is

$$F_n(x, y) = \sum_{l=0}^{n} \binom{n}{l} x^l \cdot F_{n-l}(y), \ n \geq 0.$$

Zhao Jianhong and Chen Zhuoyu [16] studied the computational problem of the sums

$$\sum_{a_1+a_2+\cdots+a_k=n} \frac{F_{a_1}(y)}{(a_1)!} \cdot \frac{F_{a_2}(y)}{(a_2)!} \cdots \frac{F_{a_k}(y)}{(a_k)!},$$

where the summation is over all k-tuples with non-negative integer coordinates $(a_1, a_2, \cdots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$. They proved the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} \frac{F_{a_1}(y)}{(a_1)!} \cdot \frac{F_{a_2}(y)}{(a_2)!} \cdots \frac{F_{a_k}(y)}{(a_k)!}$$
$$= \frac{1}{(k-1)!(y+1)^{k-1}} \cdot \frac{1}{n!} \sum_{i=0}^{k-1} C(k-1, i) F_{n+k-1-i}(y), \tag{3}$$

where the sequence $\{C(k, i)\}$ is defined as follows: For any positive integer k and integers $0 \leq i \leq k$, we define $C(k, 0) = 1$, $C(k, k) = k!$ and

$$C(k+1, i+1) = C(k, i+1) + (k+1)C(k, i), \ \text{ for all } 0 \leq i < k,$$

providing $C(k, i) = 0$, if $i > k$.

For clarity, for $1 \leq k \leq 9$, we list values of $C(k, i)$ in the following table.

$C(k,i)$	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$k=1$	1	1								
$k=2$	1	3	2							
$k=3$	1	6	11	6						
$k=4$	1	10	35	50	24					
$k=5$	1	15	85	225	274	120				
$k=6$	1	21	175	735	1624	1764	720			
$k=7$	1	28	322	1960	6769	13132	13068	5040		
$k=8$	1	36	546	4536	22449	67284	118124	109584	40320	
$k=9$	1	45	870	9450	63273	269325	723680	1172700	1026576	362880

Meanwhile, Zhao Jianhong and Chen Zhuoyu [16] proposed some conjectures related to the sequence. We think this sequence is meaningful because it satisfies some very interesting congruence properties, such as

$$C(p-2, i) \equiv 1 \pmod{p} \tag{4}$$

for all odd primes p and integers $0 \leq i \leq p-2$. The equivalent conclusion is

$$C(p-1,i) \equiv 0 \,(\text{mod } p) \tag{5}$$

for all odd primes p and positive integers $1 \leq i \leq p-2$. Since some related contents can be found in references [1,2,4,7,8,11–15], we would not go through all of them here.

The aim of this paper is to use the elementary method and the properties of the second kind Stirling numbers to prove congruence (4). That is, we will solve the conjectures in [16], which are listed as follows.

Theorem. Let p be an odd prime, for any integer $1 \leq i \leq p-2$, we have the congruence

$$C(p-1,i) \equiv 0 \,(\text{mod } p)\,.$$

From this theorem and (3) we can also deduce the following three corollaries:

Corollary 1. For any positive integer n and odd prime p, we have

$$F_{n+p-1}(y) + F_n(y) \equiv 0 \,(\text{mod } p)\,.$$

Corollary 2. For any positive integer n and odd prime p, we have

$$E_{n+p-1} + E_n \equiv 0 \,(\text{mod } p).$$

Corollary 3. For any odd prime p, we have the congruences

$$E_{p+1} \equiv 0 \,(\text{mod } p), \quad 2E_p \equiv 1 \,(\text{mod } p) \quad \text{and} \quad E_{p-1} \equiv -1 \,(\text{mod } p).$$

2. Several simple lemmas

Lemma 1. For any positive integer k, we have the identity

$$k!y(y+1)^{k-1} = \sum_{i=0}^{k-1} C(k-1,i)F_{k-i}(y).$$

Proof. Taking $n=1$ in (3), and note that $F_0(y)=1$, $F_1(y)=y$, and the equation $a_1+a_2+\cdots+a_k=1$ holds if and only if one of a_i is 1, others are 0. The number of the solutions of this equation is $\binom{k}{1}=k$. So from (3) we have

$$\sum_{a_1+a_2+\cdots+a_k=1} \frac{F_{a_1}(y)}{(a_1)!}\cdot\frac{F_{a_2}(y)}{(a_2)!}\cdots\frac{F_{a_k}(y)}{(a_k)!} = \binom{k}{1}y = ky$$

$$= \frac{1}{(k-1)!(y+1)^{k-1}}\cdot\sum_{i=0}^{k-1} C(k-1,i)F_{k-i}(y)$$

or identity

$$k!y(y+1)^{k-1} = \sum_{i=0}^{k-1} C(k-1,i)F_{k-i}(y).$$

This proves Lemma 1.

Lemma 2. For any positive integer n, we have the identity

$$F_n(y) = \sum_{k=0}^{n} S(n,k)\ k!\ y^k,\ \ (n \geq 0),$$

where $S(n,k)$ are the second kind Stirling numbers.

Proof. See reference [9].

Lemma 3. For any positive integers n and k, we have

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n (-1)^{k-j}.$$

Proof. See Theorem 4.3.12 of [3].

Lemma 4. For any odd prime p and positive integer $2 \leq k \leq p-1$, we have the congruence

$$k!S(p,k) \equiv 0 \pmod{p}.$$

Proof. From the definition and properties of $S(n,k)$, we know that $S(n,k)=0$, if $k>n$. For any integers $0 \leq j \leq p-1$, from the famous Fermat's Little Theorem we have the congruence $j^p \equiv j \pmod{p}$. From this congruence and Lemma 3 we have

$$k!S(p,k) = \sum_{j=0}^{k}\binom{k}{j} j^p(-1)^{k-j} \equiv \sum_{j=0}^{k}\binom{k}{j} j(-1)^{k-j} \equiv k!S(1,k) \equiv 0 \pmod{p},$$

if $k \geq 2$. This proves Lemma 4.

3. Proof of the theorem

In this section, we will prove our theorem by mathematical induction. Taking $k=p$ in Lemma 1 and note that $C(p-1,0)=1$ and $C(p-1,p-1)=(p-1)!$ we have

$$p!y(y+1)^{p-1} = \sum_{i=0}^{p-1} C(p-1,i)F_{p-i}(y)$$
$$= F_p(y) + y(p-1)! + \sum_{i=1}^{p-2} C(p-1,i)F_{p-i}(y). \quad (6)$$

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Note that $(p-1)!+1 \equiv 0 \,(\mathrm{mod}\ p)$, from (6) we have the congruence

$$y - F_p(y) \equiv \sum_{i=1}^{p-2} C(p-1,i) F_{p-i}(y) \,(\mathrm{mod}\ p)\,. \tag{7}$$

From Lemma 2 we have

$$F_p(y) = \sum_{k=0}^{p} S(p,k)\ k!\ y^k \tag{8}$$

and

$$F_p^{(p-1)}(0) = S(p,p-1)\ (p-1)! \cdot (p-1)!, \tag{9}$$

where $F_n^{(k)}(y)$ denotes the k-order derivative of $F_n(y)$ for variable y.

$$F_{p-1}^{(p-1)}(0) = S(p-1,p-1)\ (p-1)! \cdot (p-1)! = (p-1)! \cdot (p-1)!. \tag{10}$$

Then applying Lemma 3 and Lemma 4 and note that $S(1,p-1)=0$ we have

$$\begin{aligned} (p-1)!S(p,p-1) &\equiv \sum_{j=0}^{p-1} \binom{p-1}{j} j^p (-1)^{p-1-j} \equiv \sum_{j=0}^{p-1} \binom{p-1}{j} j (-1)^{p-1-j} \\ &\equiv (p-1)!S(1,p-1) \equiv 0 \,(\mathrm{mod}\ p)\,. \end{aligned} \tag{11}$$

Combining (7), (9), (10) and (11) we have

$$0 \equiv -S(p,p-1)(p-1)!(p-1)! \equiv C(p-1,1)(p-1)! \cdot (p-1)! \,(\mathrm{mod}\ p) \tag{12}$$

or

$$C(p-1,1) \equiv 0 \,(\mathrm{mod}\ p)\,. \tag{13}$$

That is, the theorem is true for $i=1$.

Assume that the theorem is true for all $1 \le i \le s$. That is,

$$C(p-1,i) \equiv 0 \,(\mathrm{mod}\ p)$$

for $1 \le i \le s < p-1$. It is clear that if $s = p-2$, then our theorem is true.

If $1 < s < p-2$, then from (7) we have the congruence

$$y - F_p(y) \equiv \sum_{i=s+1}^{p-2} C(p-1,i) F_{p-i}(y) \,(\mathrm{mod}\ p)\,. \tag{14}$$

In congruence (14), taking the $(p-s-1)$-order derivative with respect to t, then let $y=0$, applying Lemma 2 we have

$$\begin{aligned} & -S(p,p-s-1)(p-s-1)! \cdot (p-s-1)! \\ \equiv\ & C(p-1,s+1)(p-s-1)!(p-s-1)! \pmod{p}. \end{aligned} \tag{15}$$

Note that $((p-s-1)!, p) = 1$, from Lemma 4 and (15) we have the congruence

$$C(p-1,s+1)(p-s-1)! \equiv -(p-s-1)!S(p,p-s-1) \equiv 0 \pmod{p},$$

which implies

$$C(p-1,s+1) \equiv 0 \pmod{p}.$$

That is, our theorem is true for $i=s+1$. This completes the proof of our theorem by mathematical induction.

Now, we prove the corollaries. For any integer $0 \le i \le p-3$, from our theorem and the definition of $C(n,k)$ we have

$$\begin{aligned} & 0 \equiv C(p-1,i+1) = C(p-2,i+1) + (p-1)C(p-2,i) \\ \equiv\ & C(p-2,i+1) - C(p-2,i) \pmod{p}. \end{aligned}$$

Repeated application of this congruence we can deduce that

$$C(p-2,i+1) \equiv C(p-2,i) \equiv \cdots \equiv C(p-2,0) \equiv 1 \pmod{p}$$

for all $0 \le i \le p-3$. Note that $C(p-2,0)=1$ and

$$C(p-2,p-2) = (p-2)! \equiv -(p-1)! \equiv 1 \pmod{p}.$$

Now the proofs of corollaries completes.

Authors' contributions

Writing - original draft, Wenpeng Zhang; Writing - review & editing, Xin Lin. All authors read and approved the final manuscript.

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