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Monte Carlo Comparison for Nonparametric Threshold Estimators

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Abstract

This paper compares the finite sample performance of three non-parametric threshold estimators via Monte Carlo method. Our results show that the finite sample performance of the three estimators is not robust to the relative position of the threshold level along the distribution of threshold variable, especially when a structural change occurs at the tail part of the distribution.

JEL Classification: C14, C21

Keywords: Difference Kernel Estimator, Integrated Difference Kernel Estimator, M-Estimation, Monte Carlo, Nonparametric Threshold Regression

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1 Introduction

Threshold models are widely used to characterize the potential structural changes in economic relationships. There are many applications of threshold models in both time series and cross-sectional scenario (e.g., Potter(1995), Hansen(2011)). A number of threshold estimators for threshold models have been proposed in the literature, which can be categorized into two groups based on different assumptions. The first group is based on the "fixed threshold effect" assumption. The second group imposes a "diminishing threshold effect" assumption introduced by Hansen (2000). It is well known that, for the least square estimator, the threshold estimate is super consistent with the convergence rate n under "fixed threshold effect" assumption and $n^{1-2\alpha}$ under "diminishing threshold effect" assumption, respectively, where α measures the diminishing rate of the threshold effect.

The asymptotic theory and statistical inference for both groups have been well developed for the least square estimator with exogenous assumptions in both regressors and threshold variable (e.g., Chan (1993), Hansen (2000), Seo & Linton (2007)). Recently, there is a growing interest in studying threshold models with endogeneity. Extending Hansen's (2000) framework, Caner and Hansen (2004) apply the 2SLS method to estimate threshold models with endogenous slope regressors. In the spirit of the sample selection technique of Heckman (1979), imposes joint normality assumption, Kourtellos et al. (2016) explore the case that both threshold variable and slope regressors are endogenous. Seo and Shin (2016) propose a two-step GMM estimator for a dynamic panel threshold model with fixed effects, which allows endogeneity in both slope regressors and threshold variable. It is worth noticing that the GMM method allows both fixed and diminishing threshold effect and the convergence rate for the GMM threshold estimator is not super consistent. By relaxing the joint normality assumption of Kourtellos et al. (2016), Kourtellos et al. (2017) propose a two-step least square estimator based on a nonparametric control function approach to correct the threshold endogeneity. The semiparametric threshold model separates the threshold effect into two parts, namely, exogenous threshold effect and endogenous threshold effect. Therefore, with "small threshold" effect, the convergence rate for the threshold variable depends on both diminishing rates of the two effects.

However, few studies work on the estimation and statistical inference of threshold estimators based on nonparametric estimation methods, which do not rely on the least square method. Delgado and Hidalgo (2000) suggest a difference kernel estimator (DKE), which depends on a chosen point. The convergence rate of Delgado and Hidalgo's (2000) DKE is nh^{d-1}, which depends on both the bandwidth and the dimensionality of regressors, $d \geq 1$. Built upon Delgado and Hidalgo's (2000)

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method, Yu and Phillips (2018) introduce an integrated difference kernel estimator (IDKE). Yu and Phillips (2018) argue that the IDKE can be applied to the case with endogenous threshold variable. The convergence rate of the IDKE is unrelated to either bandwidth or dimensionality of regressors and is super consistent with the rate n. Using recently developed discrete smoothing methods, Henderson et al. (2017) introduce a semiparametric M-estimator of a nonparametric threshold regression model. The threshold estimator of Henderson et al. (2017) can be estimated at the rate $\sqrt{n/h}$ (h is the bandwidth), which is faster than the usual $\sqrt{n}$ rate. One may notice that the aforementioned convergence rate is the same as the smoothed least squares estimator of Seo and Linton (2016). However, they are entirely different. Henderson et al. (2017) focus on the nonparametric threshold model and their proposed estimator bases on a non-smooth objective function. On the contrary, Seo and Linton (2016) work on a linear threshold model and the proposed estimator is based on a smooth objective function with the indicator function replaced by a CDF-type smooth function.

With many applications and simulations available for comparing the parametric threshold estimators, little guidance is available to applied researchers as to the choice of nonparametric threshold estimators. Moreover, to avoid the boundary effect of the threshold estimator, most simulations are designed deliberately with the true threshold level chosen at the middle of the threshold variable distribution, which can be highly doubted in reality. Therefore, the purpose of this paper is to carefully compare the nonparametric threshold estimator of the aforementioned methods using Monte Carlo method. More importantly, we consider the case that the true threshold level is not only at the middle but also at the two tails of the threshold variable distribution.

The rest of the paper is organized as follows. In section 2, we briefly review the estimation strategies of three nonparametric threshold estimators, DKE, IDKE, and M-estimator. In Section 3, we illustrate the possible theoretical reason for the conjecture of the poor finite sample performance of the difference kernel type estimator. Section 4 presents the design of the Monte Carlo simulations. Section 5 reports the finite sample performance. Section 6 concludes.

2 Three Nonparametric Threshold Estimators

To compare the finite sample performance, in this paper, we consider three nonparametric threshold estimators: Henderson et al.'s (2017) semiparametric M-estimator, Delgado and Hidalgo's (2000) difference kernel estimator (DKE) and Yu and Phillips's (2018) integrated difference kernel estima-

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tor (IDKE).

Following Henderson et al. (2017), we consider a generalized threshold regression model:

$$y_i = \alpha(x_i) + \beta I\{q_i > \gamma\} + \varepsilon_i, \tag{1}$$

Where, for $i = 1, .., n$, $\alpha(.)$ is unknown smooth function, x_i is a vector of d regressors, q_i is the threshold variable, γ is the threshold level, $I(.)$ is the indicator function, and β measures the jump size of the regression function at $q > \gamma$. Also, x_i and q_i may have common variable.

2.1 Semiparametric M-estimator

If γ is known *a priori*, model (1) is known as a partially linear model. The conventional method to estimate the unknown γ is minimizing the sum of squared errors, which can be iterated by the grid search. Therefore, Henderson et al. (2017) suggest the semi-parametric M-estimator of the nonparametric threshold model, which can be obtained in three steps.

In step one, given (β, γ), model (1) becomes a standard nonparametric model. Therefore, we can obtain the Nadaraya-Watson (NW) estimator of α

$$\hat{\alpha}(x; \beta, \gamma) = arg \min_{\alpha \in \Theta_\alpha} \, n^{-1} \sum_{i=1}^{n} [y_i - \alpha - \beta I\{q_i > \gamma\}]^2 K_h(X_i - x), \tag{2}$$

Where $K_h(X_i - x) = h^{-d} \prod_{j=1}^{d} k(\frac{X_{ij} - x_j}{h})$, $X_i = [X_{i1}, ..., X_{id}]'$, $x = [x_1, .., x_d]'$, $k(.)$ is a second order kernel function, h is the bandwidth, and d is the dimension of x.

In step two, given γ, model (1) becomes a partially linear model. Then, β can be estimated as

$$\hat{\beta}(\gamma) = arg \min_{\beta \in \Theta_\beta} \, n^{-1} \sum_{i=1}^{n} [y_i - \hat{\alpha}(X_i; \beta, \gamma) - \beta I\{q_i > \gamma\}]^2 \hat{f}_h^2(X_i), \tag{3}$$

Where $\hat{f}_h(X_i) = n^{-1} \sum_{i=1}^{n} K_h(X_i - x)$ works as the weighting function.

Henderson et al. (2017) show the minimizer is given as follows,

$$\hat{\beta}(\gamma) = \Big[n^{-1} \sum_{i=1}^{n} \big[\sum_{j=1}^{n} K_h(X_i - X_j)(I_i - I_j) \big]^2 \Big]^{-1} n^{-1} \sum_{i=1}^{n} \big[\sum_{j=1}^{n} K_h(X_i - X_j)(I_i - I_j) \sum_{j=1}^{n} K_h(X_i - X_j)(y_i - y_j) \big], \tag{4}$$

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where we denote $I_i = I(q_i > \gamma)$.

In step three, we can estimate the threshold level γ by solving the following sample minimization problem,

$$\hat{\gamma} = arg \min_{\gamma \in \Theta_\gamma} |n^{-1} \sum_{i=1}^{n} [y_i - \hat{\alpha}(X_i; \beta(\gamma), \gamma) - \hat{\beta}(\gamma) I\{q_i > \gamma\}] w(X_i)|, \tag{5}$$

where the $w(X_i)$ is a weighting function and is application-dependent.

As mentioned in section one, the convergence rate of the threshold estimator of Henderson et al. (2017) is $\sqrt{n/h}$, which is faster than the usual $\sqrt{n}$ rate. However, the unknown function α and the jump size β converge at standard nonparametric rate of $\sqrt{nh^d}$ and $\sqrt{nh}$ respectively.

2.2 DKE and IDKE

Instead of using the absolute value of the weighted average of the sum of errors as the objective function, Delgado and Hidalgo (2000) consider using the difference between $\hat{E}[y|x_0, q = \gamma-]$ and $\hat{E}[y|x_0, q = \gamma+]$ as the objective function. Ideally, the closer γ approaches to the true value, the larger the absolute value of the above difference should be. As a result, we are able to estimate the threshold level by choosing γ that gives the most considerable gap between the two one-sided expectations. Therefore, the difference kernel estimator (DKE) can be obtained by

$$\hat{\gamma}^{DKE} = arg \max_{\gamma \in \Theta_\gamma} [\frac{1}{n} \sum_{i=1}^{n} y_i K_{h,i}^{\gamma-} - \frac{1}{n} \sum_{i=1}^{n} y_i K_{h,i}^{\gamma+}]^2, \tag{6}$$

where, if q_i is not part of X_i,

$$K_{h,i}^{\gamma+} = K_h(X_i - x_0) \cdot k_h^+(q_i - \gamma),$$

$$K_{h,i}^{\gamma-} = K_h(X_i - x_0) \cdot k_h^-(q_i - \gamma),$$

if q_i is part of X_i, $X_i = [X_{1i}, q_i]$, and $x_0 = [x_{10}, q_0]$,

$$K_{h,i}^{\gamma+} = K_h(X_{1i} - x_{10}) \cdot k_h^+(q_i - \gamma),$$

$$K_{h,i}^{\gamma-} = K_h(X_{1i} - x_{10}) \cdot k_h^-(q_i - \gamma),$$

and $k_h^{+/-}(.)$ is the one-sided kernel function with

$$k_h^+(q_i - \gamma) = k(\frac{q_i - \gamma}{h}) I(q_i > \gamma),$$

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$$k_h^-(q_i-\gamma)=k(\frac{q_i-\gamma}{h})I(q_i\leq\gamma),$$

and $k(.)$ is a second order kernel function.

Obviously, it is reasonable to expect that the DKE estimator is sensitive to the choice of x_0. Furthermore, as the convergence rate of the DKE, nh^{d-1}, slows for a high d, the DKE may suffer from the curse of high-dimensionality problem. To fix these potential weaknesses, Yu and Philips (2018) propose an integrated difference kernel estimator, which allows $\hat{\gamma}$ to not rely on a single choice in x_0 but the expectation of all x. The $\hat{\gamma}^{IDKE}$ can be derived as follows:

$$\hat{\gamma}^{IDKE}=arg\ \underset{\gamma\in\Theta_\gamma}{max} n^{-1}\sum_{i=1}^{n}[\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}y_jK_{h,ij}^{\gamma-}-\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}y_jK_{h,ij}^{\gamma+}]^2, \quad (7)$$

where, if q_i is not part of X_i,

$$K_{h,ij}^{\gamma+}=K_h(X_i-x_j)\cdot k_h^+(q_i-\gamma),$$

$$K_{h,i}^{\gamma-}=K_h(X_i-x_j)\cdot k_h^-(q_i-\gamma),$$

if q_i is part of X_i, $X_i=[X_{1i},q_i]$, and $x_j=[x_{1j},q_j]$,

$$K_{h,i}^{\gamma+}=K_h(X_{1i}-x_{1j})\cdot k_h^+(q_i-\gamma),$$

$$K_{h,i}^{\gamma-}=K_h(X_{1i}-x_{1j})\cdot k_h^-(q_i-\gamma),$$

and $k_h^{+/-}(.)$ is defined the same as above.

The IDKE is super-consistent with convergent rate n. Yu and Philips (2018) show that IDKE is consistent even if the threshold variable is endogenous. They explain that the role of the instruments of the endogenous regressors and the endogenous threshold variable is improving the efficiency of the IDKE.

3 Estimation Difficulties in Difference Kernel Type Estimator with Near Boundary γ_0

In this section, we use a simple version of model (1) to explain the estimation difficulties of the difference-kernel type estimators when γ_0 lies at the tails of the threshold variable distribution.

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This estimation difficulty motivates us to investigate the position effect of the true threshold level on the finite sample performance. Specifically, we consider the true model as

$$y_i = I(x_i \geq \gamma_0), \tag{8}$$

where x_i is randomly drawn from uniform distribution over interval of $[-0.5, 0.5]$ for all $i = 1, ..., n$.

Above model can be regarded as model (1) with $\alpha(x_i) = 0$, $\beta = 1$, and $\varepsilon_i = 0$ for all $i = 1, ..., n$. Therefore, the DKE is based on the objective function:

$$\hat{Q}_n(\gamma)^{DKE} = [\frac{1}{n}\sum_{i=0}^{n} k(\frac{x_i - \gamma}{h})I(x_i < \gamma)y_i - \frac{1}{n}\sum_{i=0}^{n} k(\frac{x_i - \gamma}{h})I(x_i \geq \gamma)y_i]^2. \tag{9}$$

The probability limit of $\hat{Q}_n(\gamma)$ is

$$Q_n(\gamma)^{DKE} = h^2[\int_{\frac{-0.5-\gamma}{h}}^{\frac{0.5-\gamma}{h}} k(u)I(u < 0)I(u \geq \frac{\gamma_0 - \gamma}{h})du - \int_{\frac{-0.5-\gamma}{h}}^{\frac{0.5-\gamma}{h}} k(u)I(u \geq 0)I(u \geq \frac{\gamma_0 - \gamma}{h})du]^2, \tag{10}$$

where $u = \frac{x_i - \gamma}{h}$ and h is the bandwidth.

If $\gamma < \gamma_0$, we obtain

$$Q_n(\gamma)^{DKE} = h^2[\int_{\frac{-0.5-\gamma}{h}}^{\frac{0.5-\gamma}{h}} k(u_x)du_x]^2, \tag{11}$$

and

$$\frac{\partial Q_n(\gamma)^{DKE}}{\partial \gamma} = 2h(\int_{\frac{\gamma_0-\gamma}{h}}^{\frac{0.5-\gamma}{h}} k(u_x)du_x)[k(\frac{\gamma_0 - \gamma}{h}) - k(\frac{0.5 - \gamma}{h})] > 0, \tag{12}$$

where the positive sign follows the bell-shaped second order kernel for all $\gamma_0 < 0.5$.

It is worth noting that as γ_0 approaches to 0.5 from the left side, the difference between $k(\frac{\gamma_0-\gamma}{h})-k(\frac{0.5-\gamma}{h})$ become smaller. As a result, for all γ, the above derivative goes to zero , which makes the objective function flat and leads to the estimation difficulty.

Similarly, if $\gamma > \gamma_0$, we have

$$Q_n(\gamma)^{DKE} = h^2(\int_{\frac{\gamma_0-\gamma}{h}}^{0} k(u_x)du_x - \int_{0}^{\frac{0.5-\gamma}{h}} k(u_x)du_x)^2, \tag{13}$$

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and

$$\frac{\partial Q_n(\gamma)^{DKE}}{\partial\gamma} = 2h(\int_{\frac{\gamma_0-\gamma}{h}}^{0} k(u_x)du_x - \int_{0}^{\frac{0.5-\gamma}{h}} k(u_x)du_x)[k(\frac{\gamma_0-\gamma}{h}) + k(\frac{0.5-\gamma}{h})] < 0, \tag{14}$$

where the negative sign follows the bell-shaped second order kernel for all $\gamma_0 > -0.5$.

Therefore, we observe that as γ_0 approaches to -0.5 from the right side, for all γ, the difference between $\int_{\frac{\gamma_0-\gamma}{h}}^{0} k(u_x)du_x - \int_{0}^{\frac{0.5-\gamma}{h}} k(u_x)du_x$ become smaller, which makes the derivative goes to zero and results in a flat objective function.

In summary, the DKE is asymptotically consistent with $\gamma_0 \in (-0.5, 0.5)$. However, it is reasonable to suspect that DKE may have poor finite performance with true threshold level lies at the tails of threshold variable due to the estimation difficulty of the flat objective function.

Next, we assume that there are additionally possible covariates, z_i, which is randomly drawn from uniform distribution over interval of $[-0.5, 0.5]$, for all $i = 1, ..., n$, and $\{x_i\}$ and $\{z_i\}$ are independent. Therefore, the probability limit of the objective function of the IDKE is (with the same bandwidth)

$$Q_n(\gamma)^{IDKE} = h^4 \int_{-0.5}^{0.5} [\int_{\frac{-0.5-z_0}{h}}^{\frac{0.5-z_0}{h}} \int_{\frac{-0.5-\gamma}{h}}^{\frac{0.5-\gamma}{h}} k(u_z)k(u_x)I(u_x < 0)I(u_x \geq \frac{\gamma_0-\gamma}{h})du_x du_z \tag{15}$$

$$- \int_{\frac{-0.5-z_0}{h}}^{\frac{0.5-z_0}{h}} \int_{\frac{-0.5-\gamma}{h}}^{\frac{0.5-\gamma}{h}} k(u_z)k(u_x)I(u_x \geq 0)I(u_x \geq \frac{\gamma_0-\gamma}{h})du_x du_z]^2 dz_0, \tag{16}$$

where $u_z = \frac{z_i - z_0}{h}$.

Note that

$$\frac{\partial Q_n(\gamma)^{IDKE}}{\partial\gamma} = h^2 \int_{-0.5}^{0.5} [\int_{\frac{-0.5-z_0}{h}}^{\frac{0.5-z_0}{h}} k(u_z)du_z]^2 dz_0 \frac{\partial Q_n(\gamma)^{DKE}}{\partial\gamma}. \tag{17}$$

As a result, in this typical example, $\frac{\partial Q_n(\gamma)^{IDKE}}{\partial\gamma}$ can be interpreted as a rescaled $\frac{\partial Q_n(\gamma)^{DKE}}{\partial\gamma}$, which implies the IDKE may have the same boundary problem as DKE estimator.

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4 Monte Carlo Designs

To evaluate the finite sample performance of the three nonparametric threshold estimators, we consider seven data generating mechanisms that are similar to those studied in Henderson et al. (2017) and Yu and Phillips (2018).

- DGP 1:
$$y_i = 2I(x_i \geq \gamma_0) + \varepsilon_i \tag{18}$$
- DGP 2:
$$y_i = x_i + 2I(x_i \geq \gamma_0) + \varepsilon_i \tag{19}$$
- DGP 3:
$$y_i = sin(x_i) + 2I(x_i \geq \gamma_0) + \varepsilon_i \tag{20}$$
- DGP 4:
$$y_i = x_i^2 + 2I(x_i \geq \gamma_0) + \varepsilon_i \tag{21}$$
- DGP 5:
$$y_i = x_{1i} + x_{2i} + x_{3i} + 2I(x_{1i} \geq \gamma_0) + \varepsilon_i \tag{22}$$
- DGP 6:
$$y_i = x_{1i}^2 + x_{2i}x_{3i} + 2I(x_{1i} \geq \gamma_0) + \varepsilon_i \tag{23}$$
- DGP 7:
$$y_i = sin(x_{1i}) + cos(x_{2i}) + sin(x_{3i}) + 2I(x_{1i} \geq \gamma_0) + \varepsilon_i \tag{24}$$

where x_i is randomly drawn from uniform distribution over interval of $[-0.5, 0.5]$ for all $i = 1, .., n$. [1]. ε_i is independently and identically distributed with $\varepsilon_i \sim N(0, 1)$.

All DGPs are based on the fixed threshold effect framework of Chan (1993). DGP 1-4 are univariate threshold models. More specifically, DGP 1-2 are typical linear threshold models. DGP 3-4 are nonlinear threshold models with modeling the periodicity and the quadraticity, respectively. DGP 5-7 are multivariate threshold models. DGP 5 characterizes the multivariate linear threshold

[1] With the uniform distribution, the intensity of the Poisson process would not change with the change in the true threshold location. Therefore, the limiting distribution of both the DKE and the IDKE are not affected given γ_0 is not on the boundary of Θ_γ.

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model. DGP 6-7 are nonlinear threshold models extending DGP 3-4 to multivariate specifications.

To examine the position effect of the true threshold level on the finite sample performance, we set γ_0 at different segment of threshold variable distribution. We set the true threshold, γ_0 as the p^{th} quantile of threshold variable . We vary $p = 25$, 50, and 75 to place the true threshold level to the left tail, medium, and the right tail of the threshold variable respectively.

We set $x_0 = x^{max}$ for the DKE estimate of Delgado and Hidalgo (2000), where x^{max} is the data with the greatest empirical density among all generated x for each simulation of each DGP. [2] We use the rule of thumb bandwidth, $h = C\hat{\sigma}_x n^{-1/(d+4)}$, where $C = \frac{4}{d+2}^{\frac{1}{d+4}}$, d is the dimension of x_i, and $\hat{\sigma}_x$ is the sample standard deviation of $\{x_i\}$. We use Gaussian kernel function. As suggested by Yu and Phillips (2018), we use the one-sided rescaled Epanechnikov kernel with $k^-(q,0) = \frac{3}{4}(1-q^2)I(q<0)$ and $k^+(q,0) = k^-(-q,0)$ to estimate the DKE and the IDKE.

We repeat 2,000 times for each simulation. [3] And, we set the sample size $n = 100$, 300 and 500. For each simulation, we report the average Bias, mean squared error (or MSE) and the standard deviation (or stdev) of the threshold estimates. Table 1 -7 contain the details of the simulation results.

5 Monte Carlo Results

We first show the performance of each estimator and compare the results with the theoretical expectations. Then, we concentrate on discussing the main issue: the position effect of the true threshold on finite sample performance.

For the semi-parametric M-estimator introduced by Henderson et al. (2017), our results show that the performance is slightly affected by the position of the true threshold level. Meanwhile, as sample size increases, this position effect gradually vanishes [4] . Additionally, we observe that the bias is smaller for multivariate models than univariate models. Using the bandwidth as defined in

[2]The theoretical density should be the same for all x due to the fact of uniform distribution. The reason we use the data-driven of choosing x_0 is because we are unknown about the data true density in reality.

[3]All programming is finished in Matlab

[4]with n=100, all bias, MSE and standard deviation are larger with γ_0 placing at two tails than γ_0 placing at the median point. However, with n=500, there is no apparent difference between tail position γ_0 estimator and the medium position γ_0 estimator

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section 4, which behaves roughly as $O(n^{-1/5})$ for univariate models and $O(n^{-1/7})$ for multivariate models, the theoretical convergence rates are $O(n^{-1.2})$ and $O(n^{-1.14})$ accordingly. From Table 8, the super consistency is confirmed with the estimated convergence rate. Consistent with the theory, the realized convergence rate decreases as the dimension increases. It is quite interesting that, for almost all univariate models, the realized convergence rate of the left-tailed or the right-tailed γ_0 is faster than that of the medium γ_0. However, for multivariate models, the realized rates seem to be stable with the position of γ_0.

For the DKE, as we conjectured, it is severely affected by the place of the true threshold value for all DGPs, which may come from the estimation difficulties problem as we argue in section 3. Furthermore, even with the middle γ_0, the bias still shows non-decreasing with the sample size under some multivariate specifications. [5] Intuitively, this may result from the choice of x_0, which distorts the result by providing useless information. According to the comment in the supplementary material of Yu and Phillips (2015), the choice in x_0 is crucial in identifying the DKE estimator. On the one hand, the optimal x_0 should make $[E(y|x_0, q = \gamma_0^-) - E(y|x_0, q = \gamma_0^+)]^2$ as large as possible. On the other hand, we need the conditional density $f(x_0|q = \gamma_0)$ to be large enough to provide sufficient information. Therefore, theoretically, with uniform distribution and univariate linear threshold model as DGP2, the ideal x_0 should be at the middle of its distribution with the value of zero. However, in the simulation, we set x_0 equal to the value with the largest empirical density, which may appear at the two tails. This may lead to $[E(y|x_0, q = \gamma_0^-) - E(y|x_0, q = \gamma_0^+)]^2$ approaches to 0. Moreover, with the multivariate and nonlinear specification, we can expect more distortion involved. As a result, the DKE performs the worst among all three competitors for all DGPs.

For the IDKE, our results show several features. Firstly, the IDKE is affected by the position of the actual threshold value. The influence is not as substantial as the DKE. Indeed, the integration allows more local information to be used and alleviate the possible distortion due to the choice of x_0. Surprisingly, unlike the DKE, this position effect seems to be asymmetric for the IDKE. For most DGPs, we observe that the average bias and MSE are larger with the left tailed γ_0 than the right-tailed γ_0. The theoretical convergence rate of the IDKE estimator, n, is unrelated to both the bandwidth and the dimension, which is faster than the semi-parametric M-estimator of Henderson et al. (2017). Strikingly, this is inconsistent with our realized convergence rates with the middle γ_0, which is shown in table 8. Moreover, for all DGPs, the realized convergence rates are larger

[5] For example, in Table 6, the bias monotonically increases with the in sample size.

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with two sided tailed γ_0 than the median γ_0.

In summary, the simulation results give some evidence that the finite sample performances are affected by the place of the true threshold level for all three nonparametric threshold estimators. However, this effect is heterogeneous. The position effect least influences the semi-M estimator of Henderson et al., (2017). Meanwhile, the difference kernel type estimators are severely distorted by the tailed γ_0, which confirms our conjecture suggested in section 3. Furthermore, our results show that the position of the true threshold level also affects the realized convergence rate. We also find, for the semi-M estimator of Henderson et al. (2017) and the IDKE estimator, the tail distortion tend to be reduced with multivariate models.

6 Conclusion

In this paper, we evaluate the finite sample performance of three non-parametric threshold estimators and identify the relationship between the performances of different estimators and the position of the true threshold level with Monte Carlo methods.

The study shows, with all three estimators affected by the tail position of the true threshold value, the semi-M estimator of Henderson et al.(2017) outperforms DKE and IDKE with roughly all DGPs considered in the paper. Interestingly, there appears to be some evidence that the distortion can be reduced if there are other covariates besides the threshold variable for the semi-M estimator and the IDKE. Consistent with theory, we find that the realized convergence rates support the super consistency in threshold estimate for all three estimators. However, we find the realized converge rates are also affected by the position of the true threshold value. We therefore conclude that, in applied work, using the difference kernel type estimation, researchers must be careful when the threshold estimate is at the left-tail or the right-tail of values of the threshold variable.

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References

[1] Caner, M., Hansen, B.E., 2004. Instrumental variable estimation of a threshold model, *Econometric Theory*, Vol. 20, pp. 813-843.

[2] Chan, K. S., 1993. Consistency and Limiting Distribution of the Least Squares Estimator of a Threshold Autoregressive Model, *Annals of Statistics*, Vol. 21, pp. 520-533.

[3] Hansen, B.E., 2000. Sample splitting and threshold estimation, *Econometrica*, Vol. 68, pp. 575-603.

[4] Hansen, B.E., 2011. Threshold Autoregression in Economics, *Statistics and Its Interface*, Vol.4, pp.123-127.

[5] Heckman, J.J., 1979. Sample Selection Bias as a Specification Error, *Econometrica*, vol. 47, pp. 153-161.

[6] Henderson, D.J., Parmeter, C.F., Su, L., 2017. Nonparametric Threshold Regression: Estimation and Inference, *Working paper*

[7] Horowitz, J.L, 1992. A smoothed maximum score estimator for the binary response model. *Econometrica*, Vol. 60, pp.505531.

[8] Kourtellos, A., Stengos, T., Tan, C.M., 2016. Structural Threshold Regression, *Econometric Theory*, Vol.32, pp. 827-860.

[9] Kourtellos, A., T. Stengos, Y. Sun., 2017. Endogeneity in Semiparametric Threshold Regression, *Working paper, University of Cyprus and University of Guelph.*

[10] Delgado, M.A., Hidalgo, J., 2000. Nonparametric inference on structural breaks, *Journal of Econometrics*, Vol. 96, pp. 113-144.

[11] Potter, S.M., 1995. A Nonlinear Approach to US GNP, *Journal of Applied Econometrics*, Vol. 10, pp.109-125.

[12] Seo, M. H., Linton, O., 2007. A smoothed least squares estimator for threshold regression models, *Journal of Econometrics*, Vol.141, pp. 704-735.

[13] Seo, M. H., Y. Shin, 2016. Dynamic Panels with Threshold Effect and Endogeneity, *Journal of Econometrics*, Vol.195, pp. 169-186.

Peer-reviewed version available at J. Risk Financial Manag. 2018, 11, 49; doi:10.3390/jrfm11030049

[14] Yu, Ping, Peter C. B. Phillips, 2018. Threshold Regression with Endogeneity, *Journal of Econometrics*, Vol.203, pp. 50-68.

Table 1: Simulation Results of Nonparametric Threshold Estimators, DGP 1

γ_0 is the 25^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0336	0.2705	0.0679	0.0144	0.0913	0.0225	0.1152	0.1345	0.1338
300	0.0015	0.2929	0.0870	0.0006	0.0986	0.0308	0.0241	0.1133	0.1525
500	0.0002	0.2632	0.1530	0.0001	0.0920	0.0544	0.0097	0.1509	0.1760

γ_0 is the 50^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0056	-0.0346	-0.0183	0.0084	0.0154	0.0012	0.0916	0.1191	0.0288
300	0.0007	-0.0346	-0.0083	0.0009	0.0209	0.0002	0.0302	0.1406	0.0126
500	0.0008	-0.0347	-0.0055	0.0003	0.0233	0.0001	0.0166	0.1488	0.0080

γ_0 is the 75^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0397	-0.2485	-0.0666	0.0163	0.1082	0.0087	0.1215	0.2156	0.0650
300	-0.0028	-0.2590	-0.0377	0.0009	0.1143	0.0029	0.0299	0.2174	0.0391
500	-0.0004	-0.2841	-0.0287	0.0001	0.1288	0.0018	0.0118	0.2193	0.0308

This table reports the simulation results of three estimators, semiparametric M-estomator of Henderson et al. (2017), the DKE of Delgado and Hidalgo (2000) and the IDKE of Yu and Phillips (2018) for the simple jump function defined as equation (18). The first column gives the sample size that the simulation used. The third to the fifth columns report the average bias. Sixth to eighth columns give the mean squared errors of the threshold estimates. The last three columns present the standard deviations.

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Table 2: Simulation Results of Nonparametric Threshold Estimators, DGP 2

γ_0 is the 25^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0359	0.2272	0.0813	0.0154	0.0823	0.0250	0.1190	0.1752	0.1357
300	0.0053	0.2680	0.1019	0.0020	0.0954	0.0324	0.0442	0.1536	0.1485
500	0.0002	0.2632	0.1530	0.0001	0.0920	0.0544	0.0097	0.1509	0.1760

γ_0 is the 50^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0008	-0.0246	-0.0151	0.0082	0.0122	0.0009	0.0907	0.1077	0.0257
300	0.0002	-0.0147	-0.0067	0.0009	0.0130	0.0002	0.0306	0.1130	0.0107
500	0.0002	-0.0131	-0.0044	0.0000	0.0154	0.0001	0.0068	0.1233	0.0073

γ_0 is the 75^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0307	-0.2465	-0.1031	0.0119	0.1049	0.0159	0.1048	0.2101	0.0730
300	-0.0059	-0.2564	-0.0786	0.0023	0.1009	0.0086	0.0477	0.1876	0.0494
500	-0.0008	-0.2651	-0.0699	0.0003	0.1060	0.0065	0.0177	0.1891	0.0397

This table reports the simulation results of three estimators, semiparametric M-estimator of Henderson et al. (2017), the DKE of Delgado and Hidalgo (2000) and the IDKE of Yu and Phillips (2018) for the univariate linear threshold model defined as equation (19). The first column gives the sample size that the simulation used. The third to the fifth columns report the average bias. Sixth to eighth columns give the mean squared errors of the threshold estimates. The last three columns present the standard deviations.

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Table 3: Simulation Results of Nonparametric Threshold Estimators, DGP 3

γ_0 is the 25^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0303	0.2211	0.0785	0.0128	0.0791	0.0233	0.1092	0.1739	0.1310
300	0.0022	0.2725	0.1137	0.0014	0.0980	0.0373	0.0376	0.1541	0.1561
500	0.0005	0.2694	0.1570	0.0002	0.0961	0.0546	0.0131	0.1535	0.1730

γ_0 is the 50^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0017	-0.0236	-0.0137	0.0073	0.0111	0.0008	0.0852	0.1027	0.0257
300	0.0002	-0.0220	-0.0061	0.0004	0.0132	0.0001	0.0196	0.1128	0.0101
500	-0.0003	-0.0114	-0.0041	0.0001	0.0149	0.0001	0.0112	0.1215	0.0067

γ_0 is the 75^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0358	-0.2471	-0.1036	0.0160	0.1031	0.0160	0.1212	0.2051	0.0725
300	-0.0027	-0.2592	-0.0822	0.0013	0.1041	0.0091	0.0360	0.1924	0.0482
500	-0.0007	-0.2637	-0.0686	0.0004	0.1031	0.0065	0.0203	0.1832	0.0422

This table reports the simulation results of three estimators, semiparametric M-estimator of Henderson et al. (2017), the DKE of Delgado and Hidalgo (2000) and the IDKE of Yu and Phillips (2018) for the univariate threshold periodic model defined as equation (20). The first column gives the sample size that the simulation used. The third to the fifth report propose the average bias. Sixth to eighth columns give the mean squared errors of the threshold estimates. The last three columns present the standard deviations.

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Table 4: Simulation Results of Nonparametric Threshold Estimators, DGP 4

γ_0 is the 25^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0371	0.2754	0.1038	0.0168	0.0922	0.0348	0.1242	0.1278	0.1551
300	0.0065	0.2817	0.1479	0.0030	0.0921	0.0526	0.0545	0.1131	0.1754
500	0.0010	0.2884	0.2146	0.0005	0.0974	0.0794	0.0221	0.1196	0.1826

γ_0 is the 50^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0050	-0.0324	-0.0173	0.0086	0.0156	0.0016	0.0930	0.1205	0.0355
300	-0.0010	-0.0408	-0.0071	0.0012	0.0212	0.0002	0.0341	0.1400	0.0135
500	0.0000	-0.0340	-0.0051	0.0000	0.0222	0.0001	0.0038	0.1451	0.0086

γ_0 is the 75^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0378	-0.2562	-0.0694	0.0157	0.1105	0.0089	0.1196	0.2120	0.0640
300	-0.0025	-0.2622	-0.0445	0.0007	0.1131	0.0037	0.0266	0.2107	0.0411
500	-0.0007	-0.2709	-0.0358	0.0004	0.1162	0.0024	0.0203	0.2070	0.0334

This table reports the simulation results of three estimators, semiparametric M-estimator of Henderson et al. (2017), the DKE of Delgado and Hidalgo (2000) and the IDKE of Yu and Phillips (2018) for the univariate threshold quadratic model defined as equation (21). The first column gives the sample size that the simulation used. The third to the fifth report propose the average bias. Sixth to eighth columns give the mean squared errors of the threshold estimates. The last three columns present the standard deviations.

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Table 5: Simulation Results of Nonparametric Threshold Estimators, DGP 5

γ_0 is the 25^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0141	0.2560	0.0751	0.0060	0.1005	0.0213	0.0762	0.1871	0.1253
300	0.0005	0.2587	0.0421	0.0006	0.0970	0.0104	0.0253	0.1733	0.0931
500	0.0000	0.2696	0.0333	0.0000	0.0977	0.0085	0.0038	0.1583	0.0862

γ_0 is the 50^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0035	-0.0232	-0.0167	0.0050	0.0248	0.0014	0.0710	0.1559	0.0335
300	0.0000	-0.0176	-0.0082	0.0001	0.0205	0.0003	0.0118	0.1420	0.0136
500	0.0001	-0.0330	-0.0057	0.0000	0.0222	0.0001	0.0041	0.1452	0.0106

γ_0 is the 75^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0203	-0.2778	-0.1173	0.0085	0.1239	0.0212	0.0900	0.2161	0.0864
300	-0.0007	-0.2878	-0.0958	0.0002	0.1256	0.0133	0.0154	0.2069	0.0639
500	0.0000	-0.2883	-0.0944	0.0000	0.1253	0.0119	0.0035	0.2056	0.0544

This table reports the simulation results of three estimators, semiparametric M-estimator of Henderson et al. (2017), the DKE of Delgado and Hidalgo (2000) and the IDKE of Yu and Phillips (2018) for the multivariate linear threshold model defined as equation (22). The first column gives the sample size that the simulation used. The third to the fifth report propose the average bias. Sixth to eighth columns give the mean squared errors of the threshold estimates. The last three columns present the standard deviations.

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Table 6: Simulation Results of Nonparametric Threshold Estimators, DGP 6

γ_0 is the 25^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0197	0.2495	0.0704	0.0082	0.0972	0.0188	0.0882	0.1871	0.1177
300	0.0002	0.2652	0.0364	0.0001	0.0997	0.0094	0.0114	0.1714	0.0898
500	0.0000	0.2738	0.0297	0.0000	0.1003	0.0074	0.0032	0.1594	0.0807

γ_0 is the 50^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0019	-0.0107	-0.0158	0.0051	0.0242	0.0013	0.0711	0.1553	0.0323
300	-0.0004	-0.0251	-0.0074	0.0002	0.0216	0.0002	0.0138	0.1450	0.0125
500	0.0001	-0.0280	-0.0054	0.0000	0.0210	0.0001	0.0036	0.1422	0.0094

γ_0 is the 75^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0184	-0.2709	-0.1164	0.0082	0.1177	0.0207	0.0886	0.2105	0.0846
300	-0.0007	-0.2717	-0.0975	0.0004	0.1157	0.0131	0.0194	0.2048	0.0600
500	0.0002	-0.2647	-0.0889	0.0000	0.1080	0.0104	0.0042	0.1949	0.0497

This table reports the simulation results of three estimators, semiparametric M-estimator of Henderson et al. (2017), the DKE of Delgado and Hidalgo (2000) and the IDKE of Yu and Phillips (2018) for the multivariate threshold quadratic model defined as equation (23). The first column gives the sample size that the simulation used. The third to the fifth columns report the average bias. Sixth to eighth columns give the mean squared errors of the threshold estimates. The last three columns present the standard deviations.

Table 7: Simulation Results of Nonparametric Threshold Estimators, DGP 7

γ_0 is the 25^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	0.0207	0.2936	0.1292	0.0097	0.1086	0.0419	0.0964	0.1498	0.1588
300	0.0005	0.2915	0.1275	0.0003	0.1031	0.0393	0.0168	0.1347	0.1517
500	0.0003	0.2947	0.1378	0.0001	0.1048	0.0427	0.0105	0.1341	0.1542

γ_0 is the 50^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0034	0.0004	-0.0373	0.0051	0.0265	0.0074	0.0716	0.1630	0.0778
300	0.0013	0.0049	-0.0366	0.0003	0.0229	0.0029	0.0178	0.1514	0.0398
500	0.0003	0.0077	-0.0315	0.0001	0.0180	0.0019	0.0081	0.1339	0.0294

γ_0 is the 75^{th} quantile of the threshold variable

	Bias			MSE			Stdev		
n	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE	Semi-M	DKE	IDKE
100	-0.0244	-0.2830	-0.2242	0.0106	0.1137	0.0575	0.0998	0.1834	0.0849
300	0.0000	-0.2798	-0.2068	0.0001	0.1074	0.0457	0.0084	0.1708	0.0539
500	0.0000	-0.2823	-0.1963	0.0000	0.1039	0.0403	0.0036	0.1558	0.0424

This table reports the simulation results of three estimators, semiparametric M-estimator of Henderson et al. (2017), the DKE of Delgado and Hidalgo (2000) and the IDKE of Yu and Phillips (2018) for the multivariate threshold periodic model defined as equation (24). The first column gives the sample size that the simulation used. The third to the fifth columns report the average bias. Sixth to eighth columns give the mean squared errors of the threshold estimates. The last three columns present the standard deviations.

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Table 8: Estimated Convergence Rate of the Nonparametric Threshold Estimators

Semiparametric M-estimator of Henderson et al. (2017)

	DGP 1	DGP 2	DGP 3	DGP 4	DGP 5	DGP 6	DGP 7
p=25	-1.137	-1.582	-1.623	-1.831	-1.316	-1.228	-1.455
p=50	-1.624	-1.447	-1.478	-1.345	-1.260	-1.255	-1.404
p=75	-1.575	-1.730	-1.728	-1.669	-1.269	-1.319	-1.218

IDKE of Yu and Phillips (2018)

	DGP 1	DGP 2	DGP 3	DGP 4	DGP 5	DGP 6	DGP 7
p=25	-3.123	-3.229	-3.242	-3.642	-2.516	-2.446	-3.455
p=50	-1.352	-1.305	-1.288	-1.370	-1.395	-1.368	-1.945
p=75	-1.955	-2.379	-2.389	-2.024	-2.630	-2.595	-3.623

This table reports the realized convergence rates of the semiparametric M-estimator of Henderson et al. (2017) and the IDKE of Yu and Phillips (2018). The realized convergence rates are shown as the coefficient estimate by regressing the logarithm of $RMSE$ on the logarithm of the sample size for each DGP.