On Total Vertex-Edge Domination

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Abstract. A novel domination invariant defined by Boutrig and Chellali in the recent: total vertex-edge domination. In this paper we obtain an improved upper bound of total vertex edge-dominion number of a tree. If $T$ is a connected tree with order $n$, then $\gamma_{ve}^t(T) \leq \frac{m}{3}$ with $m = 6\lceil n/6 \rceil$ and we characterize the trees attaining this upper bound. Furthermore we provide a characterization of trees $T$ with $\gamma_{ve}^t(T) = \gamma^t(T)$.

1. Introduction

Let $G = (V, E)$ be a simple connected graph whose vertex set $V$ and the edge set $E$. For the open neighbourhood of a vertex $v$ in a graph $G$, the notation $N_G(v)$ is used as $N_G(v) = \{u| (u, v) \in E(G) \}$ and the closed neighborhood of $v$ is used as $N_G[v] = N_G(v) \cup \{v\}$.

The degree of a vertex $v \in G$ is equal to the number of vertices adjacent to this vertex and denoted by $d_G(v)$. If degree of a vertex is one, we name it with a leaf. If a vertex is adjacent to a leaf, we name it with a support vertex. If degree of a support vertex is two, it is named with weak support vertex. If degree of a support vertex is at least three, it is named with strong support vertex.

In this paper, if a vertex adjacent to a support vertex different from a leaf, we name it with parent support vertex. We denote path and star of order $n$, with $P_n$ and $S_n$ respectively. The diameter of a tree is denoted with $diam(T)$.

A subset $S \subseteq V$ is a dominating set, if every vertex in $G$ either is element of $S$ or is adjacent to at least one vertex in $S$. The domination number of a graph $G$ is denoted with $\gamma(G)$ and it is equal to the minimum cardinality of a dominating set in $G$. By a similar definition, a subset $S \subseteq V$ is a total domination set if every vertex of a graph $G$ are ve-dominated by at least one vertex of $S ([2,4,5])$. The minimum cardinality of a ve-dominating set is named with ve-domination number and denoted with $\gamma_{ve}(G)$.

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A subset $S \subseteq V$ is a total vertex-edge dominating set (in simply, total ve-dominating set) of $G$, if $S$ is a ve-dominating set and every vertex of $S$ has a neighbor in $S$ ([1]). The total ve-domination number of a graph $G$ is denoted with $\gamma_{ve}(G)$ and it is equal to the minimum cardinality of a total ve-dominating set.

Let $T$ be a tree and $u$ be a vertex of $T$. If there exists a neighbor vertex $x$ of $u$ as one of the subtree of $T - ux$ is a path $P_n$ with $x$ a leaf, it is said that $u$ is adjacent to the $P_n$ ([7]).

In this paper, we attain a new upper bound for a connected tree with order $n$ such that $\gamma_{ve}(T) \leq m/3$ for $m = 6\lceil n/6 \rceil$ and we construct the family tree $F$ attaining the upper bound.

### 2. The Upper Bound

**Observation 2.1.** For every connected graph $G$, $\gamma_{ve}(G) \leq \gamma^t_{ve}(G) \leq \gamma^t(G)$ ([1]).

**Observation 2.2.** For every connected graph $G$ with diameter at least three, there is a $\gamma^t_{ve}(G)$-set that contains no leaf of $G$ ([1]).

**Theorem 2.3.** If $T$ is a tree with order $n \geq 4$ and $\text{diam}(T) \geq 3$ with $l$ leaves and $s$ support vertices, then

$$\gamma_{ve}(G) \leq (n - l + s)/2$$

with equality if and only if $T^* = H \circ P_3$ for some tree $H$ ([1]).

**Observation 2.4.** For every connected graph $G$, every support vertex is contained by every total domination set ([7]).

**Observation 2.5.** For every connected graph with diameter at least three, there is a total domination set contains no leaf ([7]).

**Observation 2.6.** For every connected graph with diameter at least four, every parent support vertex is contained by every total vertex-edge domination set.

**Lemma 2.7.** For $P_n$ the path graph with $n$ vertex, the total domination number is obtained by ([8])

$$\gamma^t(P_n) = \begin{cases} \frac{n}{2}, & n \equiv 0 \,(\text{mod}4) \\ \lfloor \frac{n}{2} \rfloor + 1, & \text{otherwise} \end{cases}$$

There is no total domination set in one vertex graph, so we interest the trees which has at least two vertices.

**Definition 2.8.** We introduce an integer value help us to obtain the upper bound of total vertex-edge domination number. Let $m$ is an integer which is calculated by least integer
value function such that \( m = 6[n/6] \) with \( n \) is order of a tree. It is clear that if \( n \equiv 0 \pmod{6} \), then \( m = n \).

Now we show that if \( T \) is a tree of order \( n \), then \( \gamma_{ve}(T) \leq m/3 \) where \( m \) is introduced in Definition 2.8. In order to characterize the trees attaining the upper bound, we construct a family tree \( F \) of trees \( T = T_k \). Let \( T_1 = P_6 \) and for a \( k \) positive integer, \( T_{k+1} \) is a tree recursively obtained from \( T_k \) by attaching a path \( P_6 \) by joining one of its leaves to a vertex of \( T_k \).

**Theorem 2.9.** If \( T \in F \), then \( \gamma_{ve}(T) = n/3 \).

**Proof.** We use induction by using \( k \) operations to obtain the tree \( T \). If \( T = T_1 = P_6 \), then \( \gamma_{ve}(P_6) = 6/3 = 2 \). Now let \( k \) is a positive integer. It is assumed that the result is true for every \( T' = T_k \) which is an element of \( F \) obtained by \( k - 1 \) operations. So \( n' = n - 6 \). Let \( x \) a leaf of \( T' = T_k \) which is a path \( P_6 \) \( v_1 v_2 v_3 v_4 v_5 v_6 \) is attached by joining one of its leaves to it. Let \( D' \) is \( \gamma_{ve}(T') \)-set. It is easy to see that \( D' \cup \{v_3, v_4\} \) is a TVEDS of \( T \). Thus,

\[
\gamma_{ve}(T) \leq \gamma_{ve}(T') + 2. \quad \text{Furtherly, if } D \text{ is a } \gamma_{ve}(T') \text{-set, } D \setminus \{v_3, v_4\} \text{ is a TVEDS of } T'. \text{ So that,}
\]

\[
\gamma_{ve}(T') \leq \gamma_{ve}(T) - 2. \quad \text{Consequently,}
\]

\[
\gamma_{ve}(T) = \gamma_{ve}(T') + 2 = \frac{n'}{3} + 2 = \frac{n-6}{3} + 2 = \frac{n}{3}.
\]

Now assume that a path \( P_6 \) \( v_1 v_2 v_3 v_4 v_5 v_6 \) is attached to a support vertex. Let \( D' \) is \( \gamma_{ve}(T') \)-set. It is clear that \( D' \cup \{v_3, v_4\} \) is a TVEDS of \( T \). Thus,

\[
\gamma_{ve}(T) \leq \gamma_{ve}(T') + 2. \quad \text{Inversely, } D \setminus \{v_3, v_4\} \text{ is a TVEDS of } T'. \text{ Therefore,}
\]

\[
\gamma_{ve}(T) = \gamma_{ve}(T') + 2 = \frac{n'}{3} + 2 = \frac{n-6}{3} + 2 = \frac{n}{3}.
\]

Now assume that a path \( P_6 \) \( v_1 v_2 v_3 v_4 v_5 v_6 \) is attached to a parent support vertex and this vertex is named with \( x \). \( x \) ve-dominates the edges \( xv_1, v_1 v_2 \). If \( D' \) is a \( \gamma_{ve}(T') \)-set, \( D' \cup \{v_4\} \) is a vertex-edge domination set of \( T \) but it is not total. Thus we add one of the vertex of \( \{v_3, v_5\} \) for obtaining the TVEDS of \( T \). Therefore, \( \gamma_{ve}(T) \leq \gamma_{ve}(T') + 2 \) and inversely, \( \gamma_{ve}(T') \leq \gamma_{ve}(T) - 2. \quad \text{Consequently,}
\]

\[
\gamma_{ve}(T) = \gamma_{ve}(T') + 2 = \frac{n'}{3} + 2 = \frac{n-6}{3} + 2 = \frac{n}{3}.
\]

**Theorem 2.10.** If \( T \) is a tree of order \( n \), then then \( \gamma_{ve}(T) \leq m/3 \) such that \( m = 6[n/6] \) with equality if and only if \( T \in F \).

**Proof.** Let diameter of \( T \) 2. So \( T \) is a star graph and \( \gamma_{ve}(S_n) = 2 \). It is clear that if diameter of \( T \) is smaller than 5, then \( \gamma_{ve}(T) = 2 \). We assume that \( diam(T) \geq 5 \). In this situation
number of the vertices is at least 6. We use induction and it is assumed that the result is true for every tree $T' = T_k$ with order $n' < n$ and $m' \leq m$.

First assume some support vertex of $T$, for example $x$, is strong. Let $y$ be a leaf adjacent to $x$ and $T' = T - y$. Let $D'$ is a $\gamma_{ve}^t(T')$-set and by Observation 2.2, $D'$ is also a TVEDS of $T$. Thus, $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$. So we can assume that every support vertex is weak.

We root $T$ at a vertex of maximum eccentricity $diam(T)$. Let $t$ be a leaf at maximum distance from $p$, $v$ be parent of $t$, $u$ be parent of $v$, $w$ be parent of $u$, $s$ be parent of $w$ and $r$ be parent of $s$ in the rooted tree. The subtree induced by a vertex $x$ and its descendants in the rooted tree $T$ is denoted by $T_x$.

Assume that some child of $u$ is a leaf and it is denoted with $x$. Let $T' = T - x$. If $D'$ is a $\gamma_{ve}^t(T')$-set, $D'$ is also a TVEDS of $T$ by Observation 2.2. Thus, $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$.

Now assume in the children of $u$ there is a support vertex other than $v$, for example $x$. We take $T' = T - T_v$. Let $D'$ is a $\gamma_{ve}^t(T')$-set. $D'$ must contain the vertex $u$ and $D'$ is also a TVEDS of $T$ by Observation 2.2. Thus $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$.

Now assume that $d_T(u) = 2$. First assume that $w$ is adjacent to a leaf, say $x$. Let $T' = T - x$ and $D'$ is a $\gamma_{ve}^t(T')$-set. $D'$ is also a TVEDS of $T$ by Observation 2.2. Thus $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$.

Now assume that a $P_2$ or $P_3$ is attached by joined one of its leaves to $w$. Let $T' = T - T_u$ and $n' = n - 3$. If $D'$ is a $\gamma_{ve}^t(T')$-set, $w$ must be contained by $D'$. So that $D' \cup \{u\}$ is a TVEDS of $T$. Thus

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + 1 \leq \frac{m'}{3} + 1 = \frac{6 \left[\frac{n}{6} - \frac{3}{6}\right]}{3} + 1 = \frac{6 \left[\frac{n}{6}\right]}{3} - 2 \left[\frac{3}{6}\right] + 1 < \frac{6 \left[\frac{n}{6}\right]}{3} = \frac{m}{3}.$$ 

Now assume that $d_T(w) = 2$. In this case first, let $s$ is adjacent to a leaf, say $x$. If $D'$ is a $\gamma_{ve}^t(T')$-set contains no leaf, it is a TVEDS of $T$. Therefore $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$.

Now assume that a path $P_2 , P_3$ or $P_4$ is attached to $w$ by an edge. Let $T' = T - T_w$ and $D'$ is a $\gamma_{ve}^t(T')$-set. Thus $n' = n - 4$ and

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + \gamma_{ve}^t(P_4) = \frac{m'}{3} + \gamma_{ve}^t(P_4) = \frac{6 \left[\frac{n}{6}\right]}{3} + \frac{6 \left[\frac{4}{6}\right]}{3} = \frac{6 \left[\frac{n - 4}{6}\right]}{3} + \frac{6 \left[\frac{4}{6}\right]}{3} < \frac{6 \left[\frac{n}{6}\right]}{3} = \frac{m}{3}.$$
Now assume that \( d_T(s) = 2 \). Let \( T' = T - T_s \). So we have \( n' = n - 5 \). If \( D' \) is a \( \gamma_{ve}(T') \)-set, the total vertex-edge domination number of \( T \) is

\[
\gamma_{ve}(T) \leq \gamma_{ve}(T') + \gamma_{ve}(P_5) = \frac{6\left\lfloor \frac{n'}{6} \right\rfloor}{3} + \frac{6\left\lfloor \frac{5}{6} \right\rfloor}{3} = \frac{6\left\lfloor \frac{n - 5}{6} \right\rfloor}{3} + \frac{6\left\lfloor \frac{5}{6} \right\rfloor}{3} < \frac{6\left\lfloor \frac{n}{6} \right\rfloor}{3} = \frac{m}{3}.
\]

Now assume \( d_T(r) = 2 \) and we take \( T'' = T - T_r \). We have \( n' = n - 6 \). If \( n' = 1 \), then \( T = P_7 \) and we obtain \( \gamma_{ve}(P_7) = 3 \leq 4 \). We assume that \( n' \geq 2 \). If \( D' \) is a \( \gamma_{ve}(T') \)-set, \( D' \cup \{w, u\} \) is be a \( \gamma_{ve}(T) \)-set. Thus,

\[
\gamma_{ve}(T) \leq \gamma_{ve}(T') + 2 = \frac{m'}{3} + 2 = \frac{6\left\lfloor \frac{n'}{6} \right\rfloor}{3} + 2 = \frac{6\left\lfloor \frac{n - 6}{6} \right\rfloor}{3} + 2 = \frac{m}{3}.
\]

Our upper bound is sharp and best possible not only trees but also other graphs. We use \( H \) graphs which are consisted from two \( n \) vertex paths connected with an edge, to see this fact with two cases.

In the first case, we use corona product of \( H \) graphs with \( P_2 \). For second case we use 2-corona of the \( H \) graphs for every \( x \in H \) we add two vertices \( u \) and \( v \) with the edges \( xu \) and \( uv \).

For the first case we obtain a polycyclic graph \( G \) has \( n \) triangular and \( 6n \) vertices. Thus \( \gamma_{ve}(G) = 6n/3 = 2n \). In the second case \( \gamma_{ve}(G) = 2n \). Furthermore the number of the leaves is equal to the number of support vertices. Thus \( \gamma_{ve}(G) = 6n/2 = 3n \) by upper bound defined in [1]. But \( \gamma_{ve}(G) = 6n/3 = 2n \) by our upper bound.

This fact is current for paths too. For the paths the number of the leaves is equal to the number of support vertices. Thus Thus \( \gamma_{ve}(P_n) = n/2 \) by upper bound defined in [1] and by our upper bound \( \gamma_{ve}(P_n) = 6\lfloor n/6 \rfloor /3 = 2\lfloor n/6 \rfloor \). If these bounds are checked, it was seen that our bound is efficient and best possible.

3. The trees with \( \gamma_{ve}(T) = \gamma^t(T) \)

Now we find a partial answer for trees which is mentioned in [1] with Problem 4.2 for the graphs which are characterized by the equation \( \gamma_{ve}(G) = \gamma^t(G) \).

First we found the paths \( P_n \) of order \( n \) attaining the equality \( \gamma_{ve}(P_n) = \gamma^t(P_n) \) and construct a family \( T \) of these paths. Because these paths are the first members of the family \( T \). We use Theorem 2.10 and Lemma 2.7. We have to look into four situations;

1) \( n \equiv 0 \pmod{4, mod6} \),
\( ii) \ n \equiv 0 \ (\text{only mod} \ 4), \)
\( iii) \ n \equiv 0 \ (\text{only mod} \ 6), \)
\( iv) \ n \) is not a multiple 4 and 6.

For the first situation,
\[
\frac{n}{3} = \frac{n}{2} \Rightarrow n = 0.
\]

For (ii) \( n \equiv 0 \ (\text{mod} \ 4), \)
\[
2 \left\lfloor \frac{n}{6} \right\rfloor = \frac{n}{2} \Rightarrow \left\lfloor \frac{n}{4} \right\rfloor \Rightarrow \frac{n}{4} - 1 < \frac{n}{6} \leq \frac{n}{4} \Rightarrow 0 \leq n < 12.
\]

\( n \) can be 4 and 8 for this situation and \( P_4 \) and \( P_8 \) attain the equality such that \( \gamma_{ve}^t(P_4) = \gamma^t(P_4) = 2, \gamma_{ve}^t(P_8) = \gamma^t(P_8) = 4. \)

For (iii) \( n \equiv 0 \ (\text{mod} \ 6), \)
\[
\frac{n}{3} = \left\lfloor \frac{n}{2} \right\rfloor + 1 \Rightarrow \frac{n}{3} - 1 = \left\lfloor \frac{n}{2} \right\rfloor \Rightarrow \frac{n}{3} - 1 < \frac{n}{2} < \frac{n}{3} \Rightarrow -6 \leq n < 0
\]
and there is no positive solve.

For the last situation,
\[
2 \left\lfloor \frac{n}{6} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor + 1
\]

If it is checked, this equation is attained for \( n = 2, 3, 7 \) by using the upper bound for total vertex-edge domination. But for \( n = 7 \) \( \gamma_{ve}^t(P_7) = 3 \neq \gamma^t(P_7) = 4. \)

Consequently the equation \( \gamma_{ve}^t(P_7) = \gamma^t(P_7) \) is attained for only the paths \( P_2, P_3, P_4, P_8. \)

Now we construct a family tree \( \mathcal{T} \) of trees \( T = T_k. \) Let \( T_1 \in \{P_2, P_3, P_4, P_8\} \) and for a \( k \) positive integer, \( T_{k+1} \) is a tree recursively obtained from \( T_k \) by one of the following two operations,

\textit{Operation} \( O_1: \) Add a vertex with an edge to any support vertex of \( T = T_k. \)

\textit{Operation} \( O_2: \) Add a vertex with an edge to a vertex of \( T = T_k \) adjacent to a path \( P_2. \)

\textbf{Theorem 3.1.} Let \( T \) be a tree. If \( T \in \mathcal{T} \), then \( \gamma_{ve}^t(T) = \gamma^t(T). \)

\textbf{Proof.} We use induction on the number of \( k \) operations which are used to construct the tree \( T. \) If \( T_1 \in \{P_2, P_3, P_4, P_8\}, \) then \( \gamma_{ve}^t(P_2) = \gamma^t(P_2) = 2, \gamma_{ve}^t(P_3) = \gamma^t(P_3) = 2, \gamma_{ve}^t(P_4) = \gamma^t(P_4) = 2 \) and \( \gamma_{ve}^t(P_8) = \gamma^t(P_8) = 4. \)

Assume that the argument is true for every \( T' = T_k \) of the family \( \mathcal{T} \) obtained by \( k - 1 \) operations and we want to show \( T = T_{k+1} \in \mathcal{T}. \)
First assume that $T$ is obtained from $T'$ by operation $O_1$. Let $D'$ be a TDS of $T'$. It is easy to see that $D'$ is also a TDS of $T$ by observation 2.5. Thus, $y^t(T) \leq y^t(T')$. Obviously, $y^t_{ve}(T') \leq y^t_{ve}(T)$. By induction hypothesis, $y^t(T) \leq y^t(T') = y^t_{ve}(T') \leq y^t_{ve}(T)$ and by Observation 2.1 $y^t(T) \geq y^t_{ve}(T)$ it is obtained $y^t_{ve}(T) = y^t(T)$.

Now assume that $T$ is obtained from $T'$ by operation $O_2$. Let $x$ be a vertex of $T' = T_k$ which is adjacent to a $P_2$ and the $P_2$ be $yz$ as $y$ a neighbor of $x$ with degree two. Let $D'$ be a TDS of $T'$. If we attach a vertex to $x, y \in D'. y$ has to be dominated, thus $x \in D'$. Therefore $D'$ is a TDS of $T$ and $y^t(T) \leq y^t(T')$. Obviously, $y^t_{ve}(T') \leq y^t_{ve}(T)$. Thus $y^t(T) \leq y^t(T') = y^t_{ve}(T') \leq y^t_{ve}(T)$ and using the fact $y^t(T) \geq y^t_{ve}(T)$ we obtain $y^t_{ve}(T) = y^t(T)$.

**Remark 3.2.** If $T \in \mathcal{T}$, then $T$ becomes a star graph, a bistar graph or a combination of two bistar graph by an edge between any two leaves of these bistars which we name it double bistar graph.

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**References**


