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On Total Vertex-Edge Domination

Bünyamin ŞAHİN*[1], Abdulgani ŞAHİN [2]

[1]Bayburt University, Education Faculty, 69000, Bayburt, TURKEY

[2] Ağrı İbrahim Çeçen University, Science Faculty 04000, Ağrı, TURKEY

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Abstract. A novel domination invariant defined by Boutrig and Chellali in the recent: total vertex-edge domination. In this paper we obtain an improved upper bound of total vertex edge-domination number of a tree. If T is a connected tree with order n, then $\gamma_{ve}^{t}(T) \leq m/3$ with $m = 6\lceil n/6 \rceil$ and we characterize the trees attaining this upper bound. Furthermore we provide a characterization of trees T with $\gamma_{ve}^{t}(T) = \gamma^{t}(T)$.

1.Introduction

Let $G = (V, E)$ be a simple connected graph whose vertex set V and the edge set E. For the open neighbourhood of a vertex v in a graph G, the notation $N_G(v)$ is used as $N_G(v) = \{u |\ (u, v) \in E(G)\ \}$ and the closed neighborhood of v is used as $N_G[v] = N_G(v) \cup \{v\}$.

The degree of a vertex $v \in G$ is equal to the number of vertices adjacent to this vertex and denoted by $d_G(v)$. If degree of a vertex is one, we name it with a leaf. If a vertex is adjacent to a leaf, we name it with a support vertex. If degree of a support vertex is two, it is named with weak support vertex. If degree of a support vertex is at least three, it is named with strong support vertex.

In this paper, if a vertex adjacent to a support vertex different from a leaf, we name it with parent support vertex. We denote path and star of order n, with P_n and S_n respectively. The diameter of a tree is denoted with $diam(T)$.

A subset $S \subseteq V$ is a dominating set, if every vertex in G either is element of S or is adjacent to at least one vertex in S. The domination number of a graph G is denoted with $\gamma(G)$ and it is equal to the minimum cardinality of a dominating set in G. By a similar definition, a subset $S \subseteq V$ is a total domination set if every vertex of S has a neighbor in S. The total domination number of a graph G is denoted with $\gamma^{t}(G)$ and it is equal to the minimum cardinality of a total dominating set in G. Fundamental notions of domination theory are outlined in the book [3] and studied in thesis [6].

A vertex v ve-dominates an edge e which is incident to v, as well as every edge adjacent to e. A set $S \subseteq V$ is a ve-dominating set if every edges of a graph G are ve-dominated by at least one vertex of S ([2,4,5]). The minimum cardinality of a ve-dominating set is named with ve-domination number and denoted with $\gamma_{ve}(G)$.

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A subset $S \subseteq V$ is a total vertex-edge dominating set (in simply, total ve-dominating set) of G, if S is a ve-dominating set and every vertex of S has a neighbor in S ([1]). The total ve-domination number of a graph G is denoted with $\gamma_{ve}^{t}(G)$ and it is equal to the minimum cardinality of a total ve-dominating set.

Let T be a tree and u be a vertex of T. If there exists a neighbor vertex x of u as one of the subtree of $T - ux$ is a path P_n with x a leaf, it is said that u is adjacent to the P_n ([7]).

In this paper, we attain a new upper bound for a connected tree with order n such taht $\gamma_{ve}^{t}(T) \leq m/3$ for $m = 6\lceil n/6 \rceil$ and we construct the family tree $\mathcal{F}$ attaining the upper bound.

2. The Upper Bound

Observation 2.1. For every connected graph G, $\gamma_{ve}(G) \leq \gamma_{ve}^{t}(G) \leq \gamma^{t}(G)$ ([1]).

Observation 2.2. For every connected graph G with diameter at least three, there is a $\gamma_{ve}^{t}(G)$-set that contains no leaf of G ([1]).

Theorem 2.3. If T is a tree with order $n \geq 4$ and $diam(T) \geq 3$ with l leaves and s support vertices, then

$$\gamma_{ve}^{t}(G) \leq (n - l + s)/2$$

with equalty if and only if $T^{*} = H \circ P_3$ for some tree H ([1]) .

Observation 2.4. For every connected graph G, every support vertex is contained by every total domination set ([7]).

Observation 2.5. For every connected graph with diameter at least three, there is a total domination set contains no leaf ([7]).

Observation 2.6. For every connected graph with diameter at least four, every parent support vertex is contained by every total vertex-edge domination set.

Lemma 2.7. For P_n the path graph with n vertex, the total domination number is obtained by ([8]) ,

$$\gamma^{t}(P_n) = \begin{cases} \frac{n}{2}, n \equiv 0 \ (mod 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 1, otherwise \end{cases}.$$

There is no total domination set in one vertex graph, so we interest the trees which has at least two vertices.

Definition 2.8. We introduce an integer value help us to obtain the upper bound of total vertex-edge domination number. Let m is an integer which is calculated by least integer value function such that $m = 6\lceil n/6 \rceil$ with n is order of a tree. It is clear that if $n \equiv 0\ (mod 6)$, then $m = n$.

Now we show that if T is a tree of order n, then $\gamma_{ve}^{t}(T) \leq m/3$ where m is introduced in Definition 2.8. In order to characterize the trees attaining the upper bound, we construct a family tree $\mathcal{F}$ of trees $T = T_k$. Let $T_1 = P_6$ and for a k positive integer, T_{k+1} is a tree recursively obtained from T_k by attaching a path P_6 by joining one of its leaves to a vertex of T_k.

Theorem 2.9. If $T \in \mathcal{F}$, then $\gamma_{ve}^{t}(T) = n/3$.

Proof. We use induction by using k operations to obtain the tree T. If $T = T_1 = P_6$, then $\gamma_{ve}^{t}(P_6) = 6/3 = 2$. Now let k is a positive integer. It is assumed that the result is true for every $T' = T_k$ which is an element of $\mathcal{F}$ obtained by $k-1$ operations. So $n' = n - 6$. Let x a leaf of $T' = T_k$ which is a path $P_6\ v_1v_2v_3v_4v_5v_6$ is attached by joining one of its leaves to it. Let D' is $\gamma_{ve}^{t}(T')$-set. It is easy to see that $D' \cup \{v_3, v_4\}$ is a TVEDS of T. Thus,

$\gamma_{ve}^{t}(T) \leq \gamma_{ve}^{t}(T') + 2$. Furtherly, if D is a $\gamma_{ve}^{t}(T)$-set, $D \setminus \{v_3, v_4\}$ is a TVEDS of T'. So that,

$\gamma_{ve}^{t}(T') \leq \gamma_{ve}^{t}(T) - 2$. Consequently,

$$\gamma_{ve}^{t}(T) = \gamma_{ve}^{t}(T') + 2 = \frac{n'}{3} + 2 = \frac{n-6}{3} + 2 = \frac{n}{3}\ .$$

Now assume that a path $P_6\ v_1v_2v_3v_4v_5v_6$ is attached to a support vertex. Let D' is $\gamma_{ve}^{t}(T')$-set. It is clear that $D' \cup \{v_3, v_4\}$ is a TVEDS of T. Thus,

$\gamma_{ve}^{t}(T) \leq \gamma_{ve}^{t}(T') + 2$. Inversely, $D \setminus \{v_3, v_4\}$ is a TVEDS of T'. Therefore,

$$\gamma_{ve}^{t}(T) = \gamma_{ve}^{t}(T') + 2 = \frac{n'}{3} + 2 = \frac{n-6}{3} + 2 = \frac{n}{3}\ .$$

Now assume that a path $P_6\ v_1v_2v_3v_4v_5v_6$ is attached to a parent support vertex and this vertex is named with x. x ve-dominates the edges xv_1, v_1v_2. If D' is a $\gamma_{ve}^{t}(T')$-set, $D' \cup \{v_4\}$ is a vertex-edge domination set of T but it is not total. Thus we add one of the vertex of $\{v_3, v_5\}$ for obtaining the TVEDS of T. Therefore, $\gamma_{ve}^{t}(T) \leq \gamma_{ve}^{t}(T') + 2$ and inversely, $\gamma_{ve}^{t}(T') \leq \gamma_{ve}^{t}(T) - 2$. Consequently,

$$\gamma_{ve}^{t}(T) = \gamma_{ve}^{t}(T') + 2 = \frac{n'}{3} + 2 = \frac{n-6}{3} + 2 = \frac{n}{3}\ .$$

Theorem 2.10. If T is a tree of order n, then then $\gamma_{ve}^{t}(T) \leq m/3$ such that $m = 6\lceil n/6 \rceil$ with equality if and only if $T \in \mathcal{F}$.

Proof. Let diameter of T 2. So T is a star graph and $\gamma_{ve}^{t}(S_n) = 2$. It is clear that if diameter of T is smaller than 5, then $\gamma_{ve}^{t}(T) = 2$. We assume that $diam(T) \geq 5$. In this situation

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number of the vertices is at least 6. We use induction and it is assumed that the result is true for every tree $T' = T_k$ with order $n' < n$ and $m' \leq m$.

First assume some support vertex of T, for example x, is strong. Let y be a leaf adjacent to x and $T' = T - y$. Let D' is a $\gamma_{ve}^t(T')$-set and by Observation 2.2 Let D' is also a TVEDS of T. Thus, $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$. So we can assume that every support vertex is weak.

We root T at a vertex of maximum eccentricity $diam(T)$. Let t be a leaf at maximum distance from p, v be parent of t, u be parent of v, w be parent of u, s be parent of w and r be parent of s in the rooted tree. The subtree induced by a vertex x and its descendants in the rooted tree T is denoted by T_x.

Assume that some child of u is a leaf and it is denoted with x. Let $T' = T - x$. If D' is a $\gamma_{ve}^t(T')$-set, D' is also a TVEDS of T by Observation 2.2. Thus, $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$.

Now assume in the children of u there is a support vertex other than v, for example x. We take $T' = T - T_v$. Let D' is a $\gamma_{ve}^t(T')$-set. D' must contain the vertex u and D' is also a TVEDS of T by Observation 2.2. Thus $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$.

Now assume that $d_T(u) = 2$. First assume that w is adjacent to a leaf, say x. Let $T' = T - x$ and D' is a $\gamma_{ve}^t(T')$-set. D' is also a TVEDS of T by Observation 2.2. Thus $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$.

Now assume that a P_2 or P_3 is attached by joined one of its leaves to w. Let $T' = T - T_u$ and $n' = n - 3$. If D' is a $\gamma_{ve}^t(T')$-set, w must be contained by D'. So that $D' \cup \{u\}$ is a TVEDS of T. Thus

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + 1 \leq \frac{m'}{3} + 1 = \frac{6\left\lceil \frac{n'}{6} \right\rceil}{3} + 1$$

$$= \frac{6\left\lceil \frac{n-3}{6} \right\rceil}{3} + 1 \leq \frac{6\left\lceil \frac{n}{6} \right\rceil}{3} - 2\left\lceil \frac{3}{6} \right\rceil + 1 < \frac{6\left\lceil \frac{n}{6} \right\rceil}{3} = \frac{m}{3} .$$

Now assume that $d_T(w) = 2$. In this case first, let s is adjacent to a leaf, say x. If D' is a $\gamma_{ve}^t(T')$-set contains no leaf, it is a TVEDS of T. Therefore $\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') \leq m'/3 \leq m/3$.

Now assume that a path P_2 ,P_3 or P_4 is attached to w by an edge. Let $T' = T - T_w$ and D' is a $\gamma_{ve}^t(T')$-set. Thus $n' = n - 4$ and

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + \gamma_{ve}^t(P_4) = \frac{m'}{3} + \gamma_{ve}^t(P_4) = \frac{6\left\lceil \frac{n'}{6} \right\rceil}{3} + \frac{6\left\lceil \frac{4}{6} \right\rceil}{3} = \frac{6\left\lceil \frac{n-4}{6} \right\rceil}{3} + \frac{6\left\lceil \frac{4}{6} \right\rceil}{3}$$

$$< \frac{6\left\lceil \frac{n}{6} \right\rceil}{3} = \frac{m}{3}.$$

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Now assume that $d_T(s) = 2$. Let $T' = T - T_s$. So we have $n' = n - 5$. If D' is a $\gamma_{ve}^t(T')$-set, the total vertex-edge domination number of T is

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + \gamma_{ve}^t(P_5) = \frac{6\left\lceil\frac{n'}{6}\right\rceil}{3} + \frac{6\left\lceil\frac{5}{6}\right\rceil}{3} = \frac{6\left\lceil\frac{n-5}{6}\right\rceil}{3} + \frac{6\left\lceil\frac{5}{6}\right\rceil}{3}$$

$$< \frac{6\left\lceil\frac{n}{6}\right\rceil}{3} = \frac{m}{3} .$$

Now assume $d_T(r) = 2$ and we take $T' = T - T_r$. We have $n' = n - 6$. If $n' = 1$, then $T = P_7$ and we obtain $\gamma_{ve}^t(P_7) = 3 \leq 4$. We assume that $n' \geq 2$. If D' is a $\gamma_{ve}^t(T')$-set, $D' \cup \{w, u\}$ is be a $\gamma_{ve}^t(T)$-set. Thus,

$$\gamma_{ve}^t(T) \leq \gamma_{ve}^t(T') + 2 = \frac{m'}{3} + 2 = \frac{6\left\lceil\frac{n'}{6}\right\rceil}{3} + 2 = \frac{6\left\lceil\frac{n-6}{6}\right\rceil}{3} + 2 = \frac{m}{3} .$$

Our upper bound is sharp and best possible not only trees but also other graphs. We use H graphs which are consisted from two n vertex paths connected with an edge, to see this fact with two cases.

In the first case, we use corona product of H graphs with P_2. For second case we use 2-corona of the H graphs for every $x \in H$ we add two vertices u and v with the edges xu and uv .

For the first case we obtain a polycyclic graph G has n triangular and $6n$ vertices. Thus $\gamma_{ve}^t(G) = 6n/3 = 2n$. In the second case $\gamma_{ve}^t(G) = 2n$. Furthermore the number of the leaves is equal to the number of support vertices. Thus $\gamma_{ve}^t(G) = 6n/2 = 3n$ by upper bound defined in [1]. But $\gamma_{ve}^t(G) = 6n/3 = 2n$ by our upper bound.

This fact is current for paths too. For the paths the number of the leaves is equal to the number of support vertices. Thus Thus $\gamma_{ve}^t(P_n) = n/2$ by upper bound defined in [1] and by our upper bound $\gamma_{ve}^t(P_n) = 6\lceil n/6\rceil /3 = 2\lceil n/6\rceil$. If these bounds are checked, it was seen that our bound is efficient and best possible.

3. The trees with $\gamma_{ve}^t(T) = \gamma^t(T)$

Now we find a partial answer for trees which is mentioned in [1] with Problem 4.2 for the graphs which are characterized by the equation $\gamma_{ve}^t(G) = \gamma^t(G)$.

First we found the paths P_n of order n attaining the equality $\gamma_{ve}^t(P_n) = \gamma^t(P_n)$ and construct a family $\mathcal{T}$ of these paths. Because these paths are the first members of the family $\mathcal{T}$. We use Theorem 2.10 and Lemma 2.7. We have to look into four situations;

$i)$ $n \equiv 0$ $(mod 4, mod 6)$,

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$ii) n \equiv 0\ (only\ mod4)$,

$iii) n \equiv 0\ (only\ mod6)$,

$iv)$ n is not a multiple 4 and 6.

For the first situation,

$$\frac{n}{3} = \frac{n}{2} \Rightarrow n = 0.$$

For (ii) $n \equiv 0\ (mod4)$,

$$2\left\lceil\frac{n}{6}\right\rceil = \frac{n}{2} \Rightarrow \frac{n}{4} = \left\lceil\frac{n}{6}\right\rceil \Rightarrow \frac{n}{4} - 1 < \frac{n}{6} \le \frac{n}{4} \Rightarrow 0 \le n < 12.$$

n can be 4 and 8 for this situation and P_4 and P_8 attain the equality such that $\gamma_{ve}^t(P_4) = \gamma^t(P_4) = 2,\ \gamma_{ve}^t(P_8) = \gamma^t(P_8) = 4$.

For (iii) $n \equiv 0\ (mod6)$,

$$\frac{n}{3} = \left\lfloor\frac{n}{2}\right\rfloor + 1 \Rightarrow \frac{n}{3} - 1 = \left\lfloor\frac{n}{2}\right\rfloor \Rightarrow \frac{n}{3} - 1 \le \frac{n}{2} < \frac{n}{3} \Rightarrow -6 \le n < 0$$

and there is no positive solve.

For the last situation,

$$2\left\lceil\frac{n}{6}\right\rceil = \left\lfloor\frac{n}{2}\right\rfloor + 1$$

If it is checked, this equation is attained for $n = 2,3,7$ by using the upper bound for total vertex-edge domination. But for $n = 7$ $\gamma_{ve}^t(P_7) = 3 \neq \gamma^t(P_7) = 4$.

Consequently the equation $\gamma_{ve}^t(P_n) = \gamma^t(P_n)$ is attained for only the paths P_2, P_3, P_4, P_8.

Now we construct a family tree $\mathcal{T}$ of trees $T = T_k$. Let $T_1 \in \{P_2, P_3, P_4, P_8\}$ and for a k positive integer, T_{k+1} is a tree recursively obtained from T_k by one of the following two operations,

Operation $\mathcal{O}_1$: Add a vertex with an edge to any support vertex of $T = T_k$.

Operation $\mathcal{O}_2$: Add a vertex with an edge to a vertex of $T = T_k$ adjacent to a path P_2.

Theorem 3.1. Let T be a tree. If $T \in \mathcal{T}$, then $\gamma_{ve}^t(T) = \gamma^t(T)$.

Proof. We use induction on the number of k operations which are used to construct the tree T. If $T_1 \in \{P_2, P_3, P_4, P_8\}$, then $\gamma_{ve}^t(P_2) = \gamma^t(P_2) = 2, \gamma_{ve}^t(P_3) = \gamma^t(P_3) = 2, \gamma_{ve}^t(P_4) = \gamma^t(P_4) = 2$ and $\gamma_{ve}^t(P_8) = \gamma^t(P_8) = 4$.

Assume that the argument is true for every $T' = T_k$ of the family $\mathcal{T}$ obtained by $k - 1$ operations and we want to show $T = T_{k+1} \in \mathcal{T}$.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. Let D' is a TDS of T'. It is easy to see that D' is also a TDS of T by observation 2.5. Thus, $\gamma^t(T) \leq \gamma^t(T')$. Obviously, $\gamma_{ve}^t(T') \leq \gamma_{ve}^t(T)$. By induction hypothesis , $\gamma^t(T) \leq \gamma^t(T') = \gamma_{ve}^t(T') \leq \gamma_{ve}^t(T)$ and by Observation 2.1 $\gamma^t(T) \geq \gamma_{ve}^t(T)$ it is obtained $\gamma_{ve}^t(T) = \gamma^t(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. Let x be a vertex of $T' = T_k$ which is adjacent to a P_2 and the P_2 be yz as y a neighbor of x with degree two. Let D' is a TDS of T'. If we attach a vertex to $x, y \in D'$. y has to be dominated, thus $x \in D'$. Therefore D' is a TDS of T and $\gamma^t(T) \leq \gamma^t(T')$. Obviously, $\gamma_{ve}^t(T') \leq \gamma_{ve}^t(T)$. Thus $\gamma^t(T) \leq \gamma^t(T') = \gamma_{ve}^t(T') \leq \gamma_{ve}^t(T)$ and using the fact $\gamma^t(T) \geq \gamma_{ve}^t(T)$ we obtain $\gamma_{ve}^t(T) = \gamma^t(T)$.

Remark 3.2. If $T \in \mathcal{T}$, then T becomes a star graph, a bistar graph or a combination of two bistar graph by an edge between any two leaves of these bistars which we name it double bistar graph.

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