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A NOTE ON DEGENERATE BERNSTEIN AND DEGENERATE EULER POLYNOMIALS

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ABSTRACT. In this paper, we investigate the recently introduced degenerate Bernstein polynomials and operators and derive some of their properties. Also, we give some properties of the degenerate Euler numbers and polynomials and their connection with the degenerate Euler polynomials.

1. Introduction

Let $C[0,1]$ denote the set of continuous function on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well known linear operator (see [3,20]):

$$\begin{aligned} \mathbb{B}_n(f|x) &= \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k} x^k(1-x)^{n-k} \\ &= \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k} B_{k,n}(x), \quad (\text{see } [3,8-27]), \end{aligned} \tag{1.1}$$

where

$$B_{k,n}(x) = \binom{n}{k} x^k(1-x)^{n-k}, \ (n,k \in \mathbb{Z}_{\geq 0}) \tag{1.2}$$

are called Bernstein polynomials of degree n or Bernstein basis polynomials.

Here $\mathbb{B}_n$, $(n \geq 0)$, is called Bernstein operator of order n.

A Bernoulli trial involves performing an experiment once and noting whether a particular event A occurs. The outcome of Bernoulli trial is said to be "Success" if A occurs and a "failure" otherwise. The probability $P_n(k)$ of k successes in n independent Bernoulli trials with the probability of success p is given by the binomial probability law

$$P_n(k) = \binom{n}{k} p^k(1-p)^{n-k}, \text{for } k = 0,1,2,3,\cdots,$$

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From the definition of Bernstein polynomials, we note that Bernstein basis is the probability mass function of binomial distribution. The Bernstein polynomials of degree n can be defined by blending together two Bernstein polynomials of degree $n-1$. That is, the k-th Bernstein polynomial of degree n can be written as

$$B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x), \ (k,n \in \mathbb{N}). \tag{1.3}$$

For example, $B_{0,1}(x) = 1-x$, $B_{1,1}(x) = x$, $B_{0,2}(x) = (1-x)^2$, $B_{1,2}(x) = 2x(1-x)$, $B_{2,2}(x) = x^2$, $B_{0,3}(x) = (1-x)^3$, $B_{1,3}(x) = 3x(1-x)^2$, $B_{2,3}(x) = 3x^2(1-x)$, $B_{3,3}(x) = x^3, \cdots$.

Thus, we note that

$$\begin{aligned} x^k = x(x^{k-1}) &= x\sum_{i=k-1}^{n} \frac{\binom{i}{k-1}}{\binom{n}{k-1}} B_{i,n-1}(x) = \sum_{i=k}^{n} \frac{\binom{i-1}{k-1}}{\binom{n-1}{k-1}} xB_{i-1,n-1}(x) \\ &= \sum_{i=k-1}^{n-1} \frac{\binom{i}{k-1}}{\binom{n}{k-1}} \frac{i}{n} B_{i,n}(x) = \sum_{i=k-1}^{n-1} \frac{\binom{i}{k}}{\binom{n}{k}} B_{i,n}(x). \end{aligned} \tag{1.4}$$

For $\lambda \in \mathbb{R}$, L. Carlitz introduced the degenerate Euler poynomials given by the generating function

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}, \quad \text{(see [4])}. \tag{1.5}$$

When $x = 0$, $\mathcal{E}_{n,\lambda} = \mathcal{E}_{n,\lambda}(0)$ are called the degenerate Euler numbers. It is easy to show that $\lim_{\lambda \to 0} \mathcal{E}_{n,\lambda}(x) = E_n(x)$, where $E_n(x)$ are the Euler polynomials given by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad \text{(see [1 - 7])}. \tag{1.6}$$

For $n \geq 0$, we define the λ-product as follows:

$$(x)_{0,\lambda} = 1, \ (x)_{n,\lambda} = x(x-\lambda)(x-2\lambda)\cdots(x-(n-1)\lambda), \ (n \geq 1), \quad \text{(see [9])}. \tag{1.7}$$

Note that $\lim_{\lambda \to 0}(x)_{n,\lambda} = x^n, \ (n \geq 1)$.

Recently, the degenerate Bernstein polynomials of degree n are introduced as

$$B_{k,n}(x|\lambda) = \binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}, \ (x \in [0,1], \ n,k \geq 0), \quad \text{(see [9])}. \tag{1.8}$$

From (1.8), we note that the generating function for $B_{k,n}(x|\lambda)$ is given by

$$\frac{1}{k!}(x)_{k,\lambda}t^k(1+\lambda t)^{\frac{1-x}{\lambda}} = \sum_{n=k}^{\infty} B_{k,n}(x|\lambda)\frac{t^n}{n!}, \quad \text{(see [9])}. \tag{1.9}$$

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By (1.9), we easily get $\lim_{\lambda\to 0} B_{k,n}(x|\lambda) = B_{k,n}(x)$, $(n, k \geq 0)$.

In this paper, we investigate the recently introduced degenerate Bernstein polynomials and operators and derive some of their properties (see [9]). Also, we give some properties of the degenerate Euler numbers and polynomials and their connection with the degenerate Euler polynomials.

2. Degenerate Bernstein polynomials and operators

Let $f(x)$ be a continuous function on $[0, 1]$. Then the degenerate Bernstein operator of order n is defined as

$$\mathbb{B}_{n,\lambda}(f|x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} = \sum_{k=0}^{n} f(\frac{k}{n})B_{k,n}(x|\lambda), \tag{2.1}$$

where $x \in [0, 1]$ and $n, k \in \mathbb{Z}_{\geq 0}$. From (2.1), we note that

$$\mathbb{B}_{n,\lambda}(1|x) = \sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}. \tag{2.2}$$

Now, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{t^n}{n!} &= (1+\lambda t)^{\frac{1}{\lambda}} = (1+\lambda t)^{\frac{x}{\lambda}}(1+\lambda t)^{\frac{1-x}{\lambda}}\\ &= \left(\sum_{l=0}^{\infty}(x)_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(1-x)_{m,\lambda}\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(x)_{l,\lambda}(1-x)_{n-l,\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By comparing the coefficients on the both sides of (2.3), we get

$$(1)_{n,\lambda} = \sum_{l=0}^{n}\binom{n}{l}(x)_{l,\lambda}(1-x)_{n-l,\lambda}. \tag{2.4}$$

From (2.2) and (2.4), we have

$$\mathbb{B}_{n,\lambda}(1|x) = \sum_{k=0}^{n}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda} = (1)_{n,\lambda}, \ (n \geq 0). \tag{2.5}$$

Also, we get from (2.1) that for $f(x)=x$,

$$\begin{aligned}\mathbb{B}_{n,\lambda}(x|x)&=\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\&=\sum_{k=1}^{n}\binom{n-1}{k-1}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\&=\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_{k,\lambda}(1-x)_{n-1-k,\lambda}(x-k\lambda)\\&=x(1)_{n-1,\lambda}-(n-1)\lambda\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_{k,\lambda}(1-x)_{n-1-k,\lambda}\frac{k}{n-1}\\&=x(1)_{n-1,\lambda}-(n-1)\lambda B_{n-1,\lambda}(x|x).\end{aligned}\tag{2.6}$$

From (2.6), we can derive the following equation (2.7).

$$\begin{aligned}\mathbb{B}_{n,\lambda}(x|x)&=x(1)_{n-1,\lambda}-(n-1)\lambda\{x(1)_{n-2,\lambda}-(n-2)\lambda B_{n-2,\lambda}(x|x)\}\\&=x(1)_{n-1,\lambda}-x(n-1)\lambda(1)_{n-2,\lambda}+(-1)^2(n-1)(n-2)\lambda^2B_{n-2,\lambda}(x|x)\\&=x(1)_{n-1,\lambda}-x(n-1)\lambda(1)_{n-2,\lambda}\\&\quad+(-1)^2(n-1)(n-2)\lambda^2\{x(1)_{n-3,\lambda}-(n-3)\lambda B_{n-3,\lambda}(x|x)\}\\&=x(1)_{n-1,\lambda}-x(n-1)\lambda(1)_{n-2,\lambda}+(-1)^2(n-1)(n-2)\lambda^2x(1)_{n-3,\lambda}\\&\quad+(-1)^3(n-1)(n-2)(n-3)\lambda^3B_{n-3,\lambda}(x|x)\\&=\cdots\\&=x\sum_{k=0}^{n-1}(-1)^k\lambda^k(n-1)_k(1)_{n-1-k,\lambda},\end{aligned}\tag{2.7}$$

where $(x)_k=x(x-1)\cdots(x-k+1)$, $(k\geq 1)$, $(x)_0=1$.

Therefore, we obtain the following theorem.

Theorem 2.1. *For $n\geq 0$, we have*

$$\mathbb{B}_{n,\lambda}(f|0)=f(0)(1)_{n,\lambda},\ \mathbb{B}_{n,\lambda}(f|1)=f(1)(1)_{n,\lambda},$$

and

$$\mathbb{B}_{n,\lambda}(1|x)=(1)_{n,\lambda},\ \mathbb{B}_{n,\lambda}(x|x)=x\sum_{k=0}^{n-1}(-1)^k\lambda^k(n-1)_k(1)_{n-1-k,\lambda},\ (n\geq 1).$$

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Let f, g be continuous functions defined on $[0,1]$. Then we note that

$$\mathbb{B}_{n,\lambda}(\alpha f+\beta g|x)=\alpha\mathbb{B}_{n,\lambda}(f|x)+\beta\mathbb{B}_{n,\lambda}(g|x),\ (n\geq 0), \tag{2.8}$$

where α, β are constants.

So, the degenerate Bernstein operator is linear. From (1.8), we note that

$$\mathbb{B}_{0,1}(x|\lambda)=1-x,\ \mathbb{B}_{1,1}(x|\lambda)=x,\ \mathbb{B}_{0,2}(x|\lambda)=(1-x)^2-\lambda(1-x),$$
$$\mathbb{B}_{1,2}(x|\lambda)=2x(1-x),\ \mathbb{B}_{2,2}(x|\lambda)=x^2-\lambda x.$$

It is easy to show that

$$\begin{aligned}\sum_{n=0}^{\infty}(1-x)_{n,\lambda}\frac{t^n}{n!}&=(1+\lambda t)^{\frac{1-x}{\lambda}}=(1+\lambda t)^{\frac{1}{\lambda}}(1+\lambda t)^{-\frac{x}{\lambda}}\\&=\left(\sum_{l=0}^{\infty}(1)_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(-x)_{m,\lambda}\frac{t^m}{m!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}(-1)^l(x)_{l,-\lambda}\right)\frac{t^n}{n!}.\end{aligned}$$

Thus we get

$$(1-x)_{n,\lambda}=\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}(-1)^l(x)_{l,-\lambda},\ (n\geq 0). \tag{2.9}$$

From (2.1), we have

$$\begin{aligned}\mathbb{B}_{n,\lambda}(f|x)&=\sum_{k=0}^{n}f(\frac{k}{n})B_{k,n}(x|\lambda)=\sum_{k=0}^{n}f(\frac{k}{n})\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\&=\sum_{k=0}^{n}f(\frac{k}{n})\binom{n}{k}(x)_{k,\lambda}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j(1)_{n-k-j,\lambda}(x)_{j,-\lambda}.\end{aligned} \tag{2.10}$$

It is easy to show that

$$\binom{n}{k}\binom{n-k}{j}=\binom{n}{k+j}\binom{k+j}{k},\ (n,k\geq 0), \tag{2.11}$$

and

$$(x)_{j,-\lambda}=\frac{(x)_{k+j,-\lambda}}{(x+(j+k-1)\lambda)_{k,\lambda}}. \tag{2.12}$$

Let $k+j=m$. Then, by (2.11), we get

$$\binom{n}{k}\binom{n-k}{j}=\binom{n}{m}\binom{m}{k}.$$

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From (2.10) and (2.12), we have

$$\mathbb{B}_{n,\lambda}(f|x)=\sum_{m=0}^{n}\binom{n}{m}(x)_{m,-\lambda}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(1)_{n-m,\lambda}\frac{(x)_{k,\lambda}}{(x+(m-1)\lambda)_{k,\lambda}}f(\frac{k}{n}). \tag{2.13}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.2. *For $f\in C[0,1]$ and $n\in\mathbb{Z}_{\geq 0}$, we have*

$$\mathbb{B}_{n,\lambda}(f|x)=\sum_{m=0}^{n}\binom{n}{m}(x)_{m,-\lambda}\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(1)_{n-m,\lambda}\frac{(x)_{k,\lambda}}{(x+(m-1)\lambda)_{k,\lambda}}f(\frac{k}{n}).$$

From (1.8), (2.9) and (2.12), we note that

$$\begin{aligned}B_{k,n}(x|\lambda)&=\binom{n}{k}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\&=\binom{n}{k}(x)_{k,\lambda}\sum_{i=0}^{n-k}\binom{n-k}{i}(-1)^{i}(x)_{i,-\lambda}(1)_{n-k-i,\lambda}\\&=\sum_{i=0}^{n-k}(-1)^{i}\binom{n}{k}\binom{n-k}{i}(x)_{k+i,-\lambda}\frac{(x)_{k,\lambda}}{(x+(k+i-1)\lambda)_{k,\lambda}}(1)_{n-k-i,\lambda}\\&=\sum_{i=k}^{n}(-1)^{i-k}\binom{n}{k}\binom{n-k}{i-k}(x)_{i,-\lambda}\frac{(x)_{k,\lambda}}{(x+(i-1)\lambda)_{k,\lambda}}(1)_{n-i,\lambda}\\&=\sum_{i=k}^{n}(-1)^{i-k}\binom{n}{i}\binom{i}{k}(x)_{i,-\lambda}\frac{(x)_{k,\lambda}}{(x+(i-1)\lambda)_{k,\lambda}}(1)_{n-i,\lambda}.\end{aligned}\tag{2.14}$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 2.3. *For $n,k\in\mathbb{Z}_{\geq 0}$ and $x\in[0,1]$, we have*

$$B_{k,n}(x|\lambda)=\sum_{i=k}^{n}(-1)^{i-k}\binom{n}{i}\binom{i}{k}(x)_{i,-\lambda}\frac{(x)_{k,\lambda}}{(x+(i-1)\lambda)_{k,\lambda}}(1)_{n-i,\lambda}.$$

From (1.8) and (2.5), it is to see that

$$(x-k\lambda)_{i,\lambda}=\frac{1}{(1)_{n-i,\lambda}}\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x|\lambda), \tag{2.15}$$

where $n,i\in\mathbb{N}$ with $i\geq n$ and $x\in[0,1]$.

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Indeed,

$$\begin{aligned}\sum_{k=1}^{n}\frac{k}{n}B_{k,n}(x|\lambda)&=\sum_{k=1}^{n}\binom{n-1}{k-1}(x)_{k,\lambda}(1-x)_{n-k,\lambda}\\&=\sum_{k=0}^{n-1}\binom{n-1}{k}(x)_{k+1,\lambda}(1-x)_{n-1-k,\lambda}\\&=(x-k\lambda)(1)_{n-1,\lambda}.\end{aligned}$$

In the same manner, we can show that

$$\sum_{k=i}^{n}\frac{\binom{k}{i}}{\binom{n}{i}}B_{k,n}(x|\lambda)=(x-k\lambda)_{i,\lambda}(1)_{n-i,\lambda}.$$

3. Degenerate Euler polynomials associated with degenerate Bernstein polynomials

From (1.5), we note that

$$\begin{aligned}2&=\left(\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}\frac{t^l}{l!}\right)\left((1+\lambda t)^{\frac{1}{\lambda}}+1\right)=\left(\sum_{l=0}^{\infty}\mathcal{E}_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(1)_{m,\lambda}\frac{t^m}{m!}+1\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,\lambda}(1)_{n-l,\lambda}\right)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\mathcal{E}_{l,\lambda}+\mathbb{E}_{n,\lambda}\right)\frac{t^n}{n!}.\end{aligned}\tag{3.1}$$

By comparing the coefficients on both sides of (3.1), we obtain the following theorem.

Theorem 3.1. *For $n\geq 0$, we have*

$$\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\mathcal{E}_{l,\lambda}+\mathbb{E}_{n,\lambda}=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n>0.\end{cases}$$

From Theorem 4, we note that

$$\mathcal{E}_{0,\lambda}=1,\ \mathcal{E}_{n,\lambda}=-\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\mathcal{E}_{l,\lambda}=-\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}$$

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By (1.5), we get

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(1-x)\frac{t^n}{n!}=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1-x}{\lambda}}=\frac{2}{(1++\lambda t)^{-\frac{1}{\lambda}}+1}(1+\lambda t)^{-\frac{x}{\lambda}}$$
$$=\sum_{n=0}^{\infty}\mathcal{E}_{n,-\lambda}(x)(-1)^n\frac{t^n}{n!}. \tag{3.2}$$

By comparing the coefficients on the sides of (3.2), we have

$$\mathcal{E}_{n,\lambda}(1-x)=(-1)^n\mathcal{E}_{n,-}(x),\ \ (n\geq 0). \tag{3.3}$$

In particular, if we take $x=-1$, we get

$$\mathcal{E}_{n,\lambda}(2)=(-1)^n\mathcal{E}_{n,-\lambda}(-1),\ \ (n\geq 0). \tag{3.4}$$

Therefore, we obtain the following theorem.

Theorem 3.2. *For $n\geq 0$, we have*

$$\mathcal{E}_{n,\lambda}=-\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\mathcal{E}_{l,\lambda},\ \ \mathcal{E}_{n,\lambda}(1-x)=(-1)^n\mathcal{E}_{n,-\lambda}(x).$$

In particular

$$\mathcal{E}_{n,\lambda}(2)=(-1)^n\mathbb{E}_{n,-\lambda}(-1),\ \ (n\geq 0).$$

From (1.5), we have

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(2)\frac{t^n}{n!}=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{2}{\lambda}}=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1}{\lambda}}((1+\lambda t)^{\frac{1}{\lambda}}+1-1)$$
$$=2(1+\lambda t)^{\frac{1}{\lambda}}-\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1}{\lambda}}$$
$$=\sum_{n=0}^{\infty}\left(2(1)_{n,\lambda}-\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\mathcal{E}_{l,\lambda}\right)\frac{t^n}{n!}. \tag{3.5}$$

Thus, by (3.5), we get

$$\mathcal{E}_{n,\lambda}(2)=2(1)_{n,\lambda}-\sum_{l=0}^{n}\binom{n}{l}(1)_{n-l,\lambda}\mathcal{E}_{l,\lambda}=2(1)_{n,\lambda}+\mathcal{E}_{n,\lambda},\ \ (n\geq 0).$$

Corollary 3.3. *For $n\geq 0$, we have*

$$\mathcal{E}_{n,\lambda}(2)=2(1)_{n,\lambda}+\mathcal{E}_{n,\lambda}.$$

It is easy to show that

$$\mathcal{E}_{n,\lambda}(x) = \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_{l,\lambda}(x)_{n-l,\lambda}, \ (n \geq 0).$$

Now, we observe that

$$\begin{aligned} \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}((1+\lambda t)^{\frac{1}{\lambda}}+1-1)(1+\lambda t)^{\frac{x-1}{\lambda}} \\ &= 2(1+\lambda t)^{\frac{x-1}{\lambda}} - \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-1}{\lambda}} \\ &= 2(1+\lambda t)^{\frac{x-1}{\lambda}} - 2(1+\lambda t)^{\frac{x-2}{\lambda}} + \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-2}{\lambda}}. \end{aligned} \tag{3.6}$$

Continuing the process in (3.6), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(2\sum_{i=1}^{k}(x-i)_{n,\lambda}(-1)^{i-1}\right)\frac{t^n}{n!} \\ &\quad + (-1)^k \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x-k}{\lambda}}. \end{aligned} \tag{3.7}$$

Therefore, by (1.5) and (3.7), we obtain the following theorem.

Theorem 3.4. *For $n \geq 0, k \geq 1$, we have*

$$\mathcal{E}_{n,\lambda}(x) = 2\sum_{i=1}^{k}(x-i)_{n,\lambda}(-1)^{i-1} + (-1)^k \mathcal{E}_{n,\lambda}(x-k).$$

By (1.9), we get

$$(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}} = \frac{k!}{t^k}\sum_{n=k}^{\infty} B_{k,n}(x|\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty} B_{k,n+k}(x|\lambda)\frac{1}{\binom{n+k}{n}}\frac{t^n}{n!}. \tag{3.8}$$

On the other hand

$$(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}} = \sum_{n=0}^{\infty}(x)_{k,\lambda}(1-x)_{n,\lambda}\frac{t^n}{n!}. \tag{3.9}$$

From (3.8) and (3.9), we have

$$(x)_{k,\lambda}(1-x)_{n,\lambda} = \frac{1}{\binom{n+k}{n}} B_{k,n+k}(x|\lambda), \ (n,k \geq 0). \tag{3.10}$$

Now, we observe that

$$\begin{aligned}(x)_{k,\lambda}(1+\lambda t)^{\frac{1-x}{\lambda}} &= \frac{(x)_{k,\lambda}}{2}\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1-x}{\lambda}}((1+\lambda t)^{\frac{1}{\lambda}}+1)\\ &= \frac{(x)_{k,\lambda}}{2}\left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{2-x}{\lambda}}+\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{1-x}{\lambda}}\right)\\ &= \frac{(x)_{k,\lambda}}{2}\left(\sum_{n=0}^{\infty}(\mathcal{E}_{n,\lambda}(2-x)+\mathcal{E}_{n,\lambda}(1-x))\frac{t^n}{n!}\right).\end{aligned} \tag{3.11}$$

By (3.9) and (3.11), we get

$$(x)_{k,\lambda}(1-x)_{n,\lambda} = \frac{(x)_{k,\lambda}}{2}(\mathcal{E}_{n,\lambda}(2-x)+\mathcal{E}_{n,\lambda}(1-x)),\ (n\geq 0). \tag{3.12}$$

Therefore, by (3.10) and (3.12), we obtain the following theorem.

Theorem 3.5. *For $n,k\geq 0$, we have*

$$B_{k,n+k}(x|\lambda) = \frac{1}{2}(x)_{k,\lambda}\binom{n+k}{k}(\mathcal{E}_{n,\lambda}(2-x)+\mathcal{E}_{n,\lambda}(1-x)).$$

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