Article

Nonlocal Inverse Square Law in Quantum Dynamics

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Abstract: Schrödinger dynamics is a nonlocal process. Not only does local perturbation affect instantaneously the entire space, but the effect decays slowly. When the wavefunction is spectrally bounded, the Schrödinger equation can be written as a universal set of ordinary differential equations, with universal coupling between them, which is related to Euler's formula. Since every variable represents a different local value of the wave equation, the coupling represents the dynamics' nonlocality. It is shown that the nonlocal coefficient is inversely proportional to the distance between the centers of these local areas. As far as we know, this is the first time that this inverse square law was formulated.

Keywords: quantum nonlocality, quantum decoding, inverse square law, Euler Formula, quantum causality

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1. Introduction

Nonlocality is a fundamental feature of quantum mechanics. It appears in many places of the quantum world. Most often, it is mentioned in the context of identical particles and entangled particles. The well-known EPR experiment [1], the Bell theorem [2] and its possible interpretations (see, for example, Ref.[3]) are classic examples. Another source of nonlocality arises from the nonlocal effect of potentials on the wavefunction (see, for example, the Aharonov-Bohm effect[4]). However, nonlocality appears in the single particle wavefunction as well. In fact, nonlocality is a fundamental property of Schrödinger dynamics.

Unlike in Maxwell's wave equation, where perturbations propagate at the speed of light, in Schrödinger dynamics, any local perturbation is instantaneously felt all over space, just as in the diffusion equation case[5]. However, unlike the diffusion equation where the nonlocal effect is exponentially small, in the Schrödinger equation, it decays much slower – as a power law.

In both cases, i.e., in the diffusion and the Schrödinger cases, the causality is violated due to the asymmetry between space and time.

In the Klein-Gordon's (KG), or similarly in the Dirac's, equation, due to the symmetry between space and time, causality reappears. In the KG case, the nonlinear dispersion relation distorts the wavefunction in high agreement with the Schrödinger equation only as far as causality allows, i.e., as far as the distance $x = ct$ from the local perturbation [6,7]. That is, any local perturbation has an effect over the entire $x = \pm ct$ domain. Clearly, in the non-relativistic regime (i.e., the Schrödinger case) this domain is the entire space. As a result, an initial discontinuous wavefunction can kindle currents all over space instantaneously [8,9].

Since the physical validity of discontinuous wavefunctions can be questioned, it is of interest to investigate the nonlocal effect of a local but smooth perturbation. We will see below that even in this case nonlocal behavior appears.

2. The Dynamics

The differential version of the free Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} \qquad (1)$$

sometimes conceals the nonlocal properties of the Schrödinger dynamics. However, its integral presentation

$$\psi(x,t)=\int_{-\infty}^{\infty}K(x-x',t)\psi(x',0)dx' \qquad (2)$$

with the free-space Schrödinger Kernel[10]

$$K(x-x',t)=\sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{im}{2\hbar}\frac{(x-x')^2}{t}\right] \qquad (3)$$

illustrates the nonlocality more vividly.

However, the oscillations' frequency increase so rapidly that their averages (which is equivalent to the integral operation) quickly converges to zero and the locality properties reappear. In particular, in some cases the latter nonlocal equation (2) was used to derive the former local one (1) [10].

Clearly, a local analysis of the Schrödinger equation is an excellent approximation in the quasi-classical regimes, which mathematically equivalent to the stationary phase approximation[11]. However, locality is questionable in the quantum regime. The problem is that Eq.(3) is the impulse response of the Schrödinger equation (1), i.e., it is the quantum system's response to the initial state of a delta function. However, a delta function can never be a physical state (it is based on infinite energies and it is not normalizable). To take a more physical initial state, it is usually accustomed to replacing the impulse response with a more physical, finite-width pulse –response, i.e.,

$$\psi(x,0)=\frac{1}{\rho^{1/2}}\left(\frac{2}{\pi}\right)^{1/4}\exp\left(-x^2/\rho^2\right) \qquad (4)$$

In which case the pulse response (after a period t)

$$\psi(x,t)=\frac{1}{\sqrt{1-i2\hbar t\rho^{-2}/m}\,\rho^{1/2}}\left(\frac{2}{\pi}\right)^{1/4}\exp\left(-\frac{x^2}{\left(1-i2\hbar t\rho^{-2}/m\right)\rho^2}\right) \qquad (5)$$

decays (in space) exponentially as well.

Therefore, it may seem as if the locality approximation is justified since in principle one can choose ρ to be arbitrarily small. However, this result is based on the premises that the spatial spectrum of the wavefunction is unbounded.

When the spatial spectrum of the wavefunction is bounded, i.e., when the spectral coefficient beyond a certain spatial frequency are all zero, then according to the Nyquist theorem, the wavefunction can be written as a superposition of sinc functions [12]. That is, all the information in the wavefunction can be written as an infinite discrete series of complex numbers $\psi_n=\Re\psi_n+i\Im\psi_n$ for $n=-\infty,\ldots-1,0,1,2,\ldots\infty$. In the spectral domain, the wavefunction occupies the spatial spectral bandwidth $1/\Delta x$, and therefore the initial wavefunction can be written as an infinite sequence of overlapping Nyquist-sinc functions (for applications in the optical communication sphere see Refs.[13-18]) (see Fig.1), i.e.,

$$\psi(x,t=0)=\sum_{n=-\infty}^{\infty}\psi_n\,\mathrm{sinc}(x/\Delta x-n), \qquad (6)$$

where $\mathrm{sinc}(\xi)\equiv\frac{\sin(\pi\xi)}{\pi\xi}$ is the "sinc" function.

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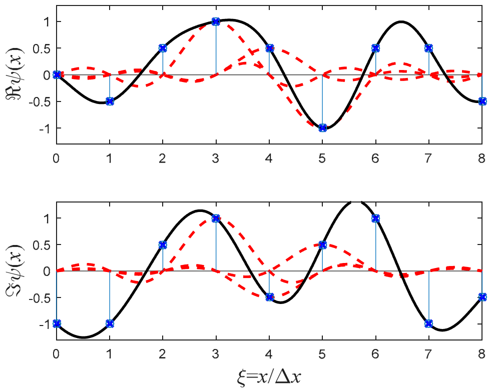

Figure 1. Illustration of the method, in which any spectrally bounded function can be written as an infinite series of sinc pulses. In the figure the sinc pulses, which are centered at $\xi = 3,4$ and 5 are plotted by dashed curves, while the final function is presented by solid curves (real/imaginary part in the upper/lower panel).

Due to the linear nature of the system, Eq.(6) can be solved directly

$$\psi(x,t>0) = \sum_{n=-\infty}^{\infty} \psi_n \,\mathrm{dsinc}\left(x/\Delta x - n, (\hbar/m)t/\Delta x^2\right) \tag{7}$$

where "dsinc" is the dynamic-sync function [19]

$$\mathrm{dsinc}(\xi,\tau) \equiv \frac{1}{2}\sqrt{\frac{i}{2\pi\tau}}\exp\left(-i\frac{\xi^2}{2\tau}\right)\left[\mathrm{erf}\left(-\frac{\xi-\pi\tau}{\sqrt{i2\tau}}\right) - \mathrm{erf}\left(-\frac{\xi+\pi\tau}{\sqrt{i2\tau}}\right)\right]. \tag{8}$$

Clearly, $\lim_{\tau\to 0}[\mathrm{dsinc}(\xi,\tau)] = \mathrm{sinc}(\xi)$.

To simplify the derivation we use the dimensionless variables

$$\tau \equiv (\hbar/m)t/\Delta x^2 \quad \text{and} \quad \xi \equiv x/\Delta x . \tag{9}$$

Some of the properties of the dsinc function are illustrated in Figs.2 and Fig.3. As can be seen, the distortions from the initial delta function $\mathrm{dsinc}(n,0) = \delta(n)$ gradually increase with time (τ).

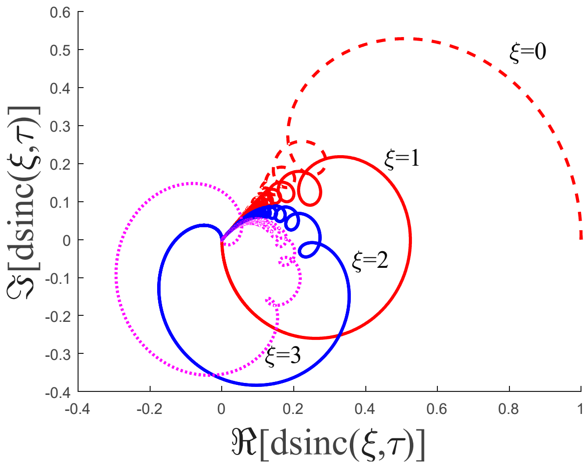

Figure 2: The relation between the real and imaginary parts of the dsinc function for the discrete values $\xi = 0,1,2,3$.

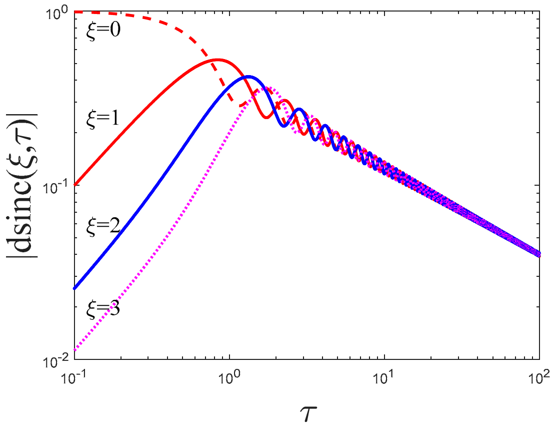

Figure 3: The temporal dependence of the absolute value of dsinc for the discrete values $\xi = 0,1,2,3$

With notations (9), Eq.(1) and (7) can be rewritten

$$i\frac{\partial\psi(\xi,\tau)}{\partial\tau} = -\frac{1}{2}\frac{\partial^2\psi(\xi,\tau)}{\partial\xi^2} \tag{10}$$

and

$$\psi(\xi,\tau>0) = \sum_{n=-\infty}^{\infty}\psi_n \,\mathrm{dsinc}(\xi-n,\tau) \tag{11}$$

Respectively.

3. Matrix Formulation and Nonlocality

When Δx is the spatial resolution of the problem, then the wavefunction at the center of the mth point, i.e., at $\xi = m$, is a simple discrete convolution

$$\psi(m,\tau) = \sum_{n=-\infty}^{\infty}\psi_n h(m-n) = \psi_m + \sum_{n=-\infty}^{\infty}\psi_n \delta h(m-n) \tag{12}$$

where

$$h(n) \equiv \mathrm{dsinc}(n,\tau) \text{ and } \delta h(n) \equiv \mathrm{dsinc}(n,\tau) - \delta(n). \tag{13}$$

Moreover, since

$$\left.\frac{\partial^2 \mathrm{sinc}(\xi)}{\partial\xi^2}\right|_{\tau=n\neq 0} = \frac{2}{n^2}(-1)^{n+1} \quad \text{and} \quad \left.\frac{\partial^2 \mathrm{sinc}(\xi)}{\partial\xi^2}\right|_{\tau=0} = -\frac{\pi^2}{3}, \tag{14}$$

then Eq.(10) can be written as a linear set of differential equations

$$\frac{d\psi(m,\tau)}{d\tau} = i\sum_n w(m-n)\psi(n,\tau) \equiv iw(m) * \psi(m,\tau) \tag{15}$$

with the universal and dimensionless vector

$$w(m) \equiv \begin{cases} (-1)^{m+1}/m^2 & m \neq 0 \\ -\pi^2/6 & m = 0 \end{cases}, \tag{16}$$

and the asterisk stands for discrete convolution.

Note that $\sum_{m=-\infty}^{\infty} w(m) = 0$ due to Euler's formula [20].

This equation is universal in the sense that the vector $w(m)$ is time independent. This is a unique property of the sinc pulses, which does not exist in other sets of orthogonal pulses (like rectangular pulses).

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Moreover, since the Schrödinger dynamics is a unitary operation, normalisation is kept and there is no change in the wavefunction spectrum. Therefore, Eq.(15) is valid for any given time.

In a matrix form, Eq.(15) can be written

$$\frac{d}{d\tau}\begin{pmatrix} \vdots \\ \psi(-2,\tau) \\ \psi(-1,\tau) \\ \psi(0,\tau) \\ \psi(1,\tau) \\ \psi(2,\tau) \\ \vdots \end{pmatrix} = i\begin{pmatrix} \ddots & & & \vdots & & & \iddots \\ & -\pi^2/6 & 1 & -2^{-2} & 3^{-2} & & \\ & 1 & -\pi^2/6 & 1 & -2^{-2} & 3^{-2} & \\ \cdots & -2^{-2} & 1 & -\pi^2/6 & 1 & -2^{-2} & \cdots \\ & 3^{-2} & -2^{-2} & 1 & -\pi^2/6 & 1 & \\ & & 3^{-2} & -2^{-2} & 1 & -\pi^2/6 & \\ \iddots & & & \vdots & & & \ddots \end{pmatrix}\begin{pmatrix} \vdots \\ \psi(-2,\tau) \\ \psi(-1,\tau) \\ \psi(0,\tau) \\ \psi(1,\tau) \\ \psi(2,\tau) \\ \vdots \end{pmatrix} \tag{17}$$

i.e.,

$$\frac{d}{d\tau}\boldsymbol{\psi} = iM\boldsymbol{\psi} \tag{18}$$

where M is a matrix with the coefficients

$$M(m,n) = \begin{cases} -\pi^2/6 & n = m \\ (-1)^{n-m+1}/(n-m)^2 & n \neq m \end{cases} \tag{19}$$

The nonlocality of this form is clearly emphasized, when compared to the ordinary numerical form of the Schrödinger equation with the ordinary 1D Cartesian local Laplacian

$$\frac{d}{d\tau}\begin{pmatrix} \vdots \\ \psi(-2,\tau) \\ \psi(-1,\tau) \\ \psi(0,\tau) \\ \psi(1,\tau) \\ \psi(2,\tau) \\ \vdots \end{pmatrix} = i\begin{pmatrix} \ddots & & & \vdots & & & \iddots \\ & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ \cdots & & 1 & -2 & 1 & & \cdots \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & \\ \iddots & & & \vdots & & & \ddots \end{pmatrix}\begin{pmatrix} \vdots \\ \psi(-2,\tau) \\ \psi(-1,\tau) \\ \psi(0,\tau) \\ \psi(1,\tau) \\ \psi(2,\tau) \\ \vdots \end{pmatrix} \tag{20}$$

In the presence of a non-zero potential, whose maximum spatial frequencies is also lower than $1/2\Delta x$ the Schrödinger equation can be rewritten in a matrix form

$$\frac{d}{d\tau}\boldsymbol{\psi} = i(M - V)\boldsymbol{\psi} \tag{21}$$

where

$$V = \begin{pmatrix} \ddots & & & \vdots & & & \iddots \\ & V(-2) & & & & & \\ & & V(-1) & & & & \\ \cdots & & & V(0) & & & \cdots \\ & & & & V(1) & & \\ & & & & & V(2) & \\ \iddots & & & \vdots & & & \ddots \end{pmatrix} \tag{22}$$

, or simply

$$V(n,m) = V(n)\delta(n-m). \tag{23}$$

Therefore, a pulse which is initially located at $x = 0$ has an instantaneous effect over the entire space, and its effect on any other point (say $n\Delta x$) is inversely proportional to the distance between them, i.e., $(n\Delta x)^{-2}$.

On the other hand, when the local Laplacian is used, then the pulse effect on a point $n\Delta x$ afar, would be felt only after n consecutive steps. Therefore, if there is a barrier between these points ($x = 0$ and $x = n\Delta x$) then with a local Laplacian it may seem that the effect of the one on the other (and vice versa) must take into account the barrier in between the two points. However, in fact, as the nonlocal form teaches, in the short time its effect is negligible since the Schrödinger equation can be approximated by

$$\frac{d}{d\tau}\begin{pmatrix}\psi(n,\tau)\\ \psi(m,\tau)\end{pmatrix}=-i\begin{pmatrix}\pi^2/6+V(n) & (-1)^{n-m}/(n-m)^2\\ (-1)^{n-m}/(n-m)^2 & \pi^2/6+V(m)\end{pmatrix}\begin{pmatrix}\psi(n,\tau)\\ \psi(m,\tau)\end{pmatrix}, \tag{24}$$

and its short-time solution

$$\begin{pmatrix}\psi(n,\tau)\\ \psi(m,\tau)\end{pmatrix}=\begin{pmatrix}1-i\tau\left[\pi^2/6+V(n)\right] & -i\tau(-1)^{n-m}/(n-m)^2\\ -i\tau(-1)^{n-m}/(n-m)^2 & 1-i\tau\left[\pi^2/6+V(m)\right]\end{pmatrix}\begin{pmatrix}\psi(n,0)\\ \psi(m,0)\end{pmatrix} \tag{25}$$

shows that in the short time the potential has only a local effect (provided it is a smooth function), i.e., $\psi(n,\tau)$ is affected only by $V(n)$ (and the effect is a simple phase change). However, the wavefunction has a nonlocal effect, i.e., $\psi(n,\tau)$ is affected by any non zero $\psi(m,\tau)$ (for any m). This result is consistent with Ref.[21], where it was demonstrated that in short time, singular wavefunction are unaffected by the barrier despite their nonlocal effect.

4. Inverse Square Law

By multiplying Eq.(15) by the complex conjugate of the wavefunction and taking the real part of the equation one finds a nonlocal equation for the probability density

$$\frac{d|\psi(m,\tau)|^2}{d\tau}=-2\sum_{n\neq m}\frac{(-1)^{m-n+1}}{(m-n)^2}\Im\{\psi(n,\tau)\psi^*(m,\tau)\} \tag{26}$$

Using the notation $\psi(n,\tau)\equiv A_n\exp(i\phi_n)$ then Eq.(26) is simply

$$\frac{dA_m^2}{d\tau}=2\sum_{n\neq m}\frac{(-1)^{m-n}A_nA_m\sin(\phi_n-\phi_m)}{(m-n)^2} \tag{27}$$

and the equivalent phase equation reads

$$\frac{d\phi_m}{d\tau}=-\sum_{n\neq m}\frac{(-1)^{m-n}}{(m-n)^2}\frac{A_n}{A_m}\cos(\phi_n-\phi_m)-\frac{\pi^2}{6} \tag{28}$$

If $\psi(n,\tau)$ is presented as a 2D vector in a 3D space

$$\boldsymbol{\psi}(n,\tau)\equiv\hat{x}\Re\,\psi(n,\tau)+\hat{y}\Im\,\psi(n,\tau) \tag{29}$$

instead of a complex number in a complex plane, then the numerator in the summation can be presented as the cross product of two vectors, i.e.,

$$\frac{d\|\boldsymbol{\psi}(m,\tau)\|^2}{d\tau}=2\sum_{n\neq m}\frac{(-1)^{m-n}\left[\boldsymbol{\psi}(n,\tau)\times\boldsymbol{\psi}(m,\tau)\right]\cdot\hat{z}}{(m-n)^2} \tag{30}$$

In this terminology, $\|\boldsymbol{\psi}(m,\tau)\|$ is the norm of the vecor $\boldsymbol{\psi}(m,\tau)$ and the cross represents cross product.

It is instructive to see the resemblance between this law and any other inverse square law.

Eq.(30) can also be written in terms of the derivative of the vector's norm $\|\boldsymbol{\psi}(m,\tau)\|$

$$\frac{d\|\boldsymbol{\psi}(m,\tau)\|}{d\tau}=\sum_{n\neq m}\frac{(-1)^{m-n}\left[\boldsymbol{\psi}(n,\tau)\times\hat{\boldsymbol{\psi}}_m\right]\cdot\hat{z}}{(m-n)^2} \tag{31}$$

where $\hat{\boldsymbol{\psi}}_m\equiv\boldsymbol{\psi}(m,\tau)/\|\boldsymbol{\psi}(m,\tau)\|$ is the unit vector in the $\boldsymbol{\psi}(m,\tau)$ direction.

It is therefore clear, that maximum probability density transfer occurs when the relative phase between the two points is $\pi/2$, i.e. when the "vectors" $\boldsymbol{\psi}(n,\tau)$ and $\boldsymbol{\psi}(m,\tau)$ are orthogonal.

In Fig. 4 such a density transfer is illustrated. In this case the wavefunction $\psi(\xi,\tau=0)=N[\mathrm{sinc}(\xi-8)+i\,\mathrm{sinc}(\xi-9)]$ was taken as the initial state (N is the normalisation constant). As can be seen, in the short time regime ($\tau=0.198$ in this case), probability was transferred from the pulse at $\xi=9$ to the one at $\xi=8$.

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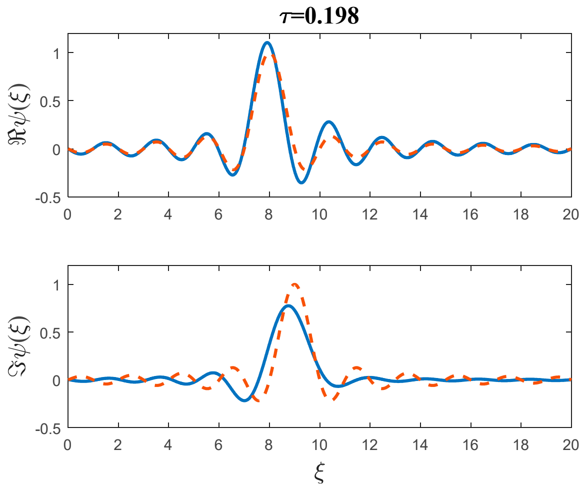

Figure 4: Illustration of probability transfer. The dashed curve represents the initial state $\psi(\xi,\tau=0)=\mathrm{sinc}(\xi-8)+i\,\mathrm{sinc}(\xi-9)$, while the solid curve stands for $\psi(\xi,\tau=0.198)=\mathrm{dsinc}(\xi-8,\tau)+i\,\mathrm{dsinc}(\xi-9,\tau)$. In both cases the real part is plotted in the upper panel, while the imaginary part is plotted in the lower one.

Moreover, it is clear from (27) and (30) that maximum probability transfer to a certain location (say $\xi=0$) occurs (provided the initial state is bounded) when the initial state oscillates in signs, i.e.,

$$\psi(\xi,\tau=0)=i\sum_{n\neq 0}(-1)^n\,\mathrm{sinc}(\xi-n)+\mathrm{sinc}(\xi) \qquad (32)$$

In this case the rate in which the probability increases (or decreases in the opposite case) is exactly $2\pi^2/3$ since (using Euler formula, see Ref.[20])

$$\frac{d\ln\left[\left|\psi(m,\tau)\right|^2\right]}{d\tau}=2\sum_{n\neq m}\frac{1}{(m-n)^2}=2\frac{\pi^2}{3} \qquad (33)$$

and the probability density at $\xi=0$ can increase almost four fold before it start to decay (see Fig.5).

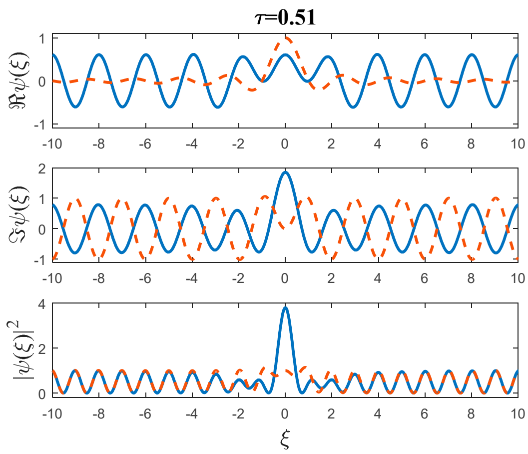

Figure 5: The short time dynamics of the wavefunction (32). The dashed curves represent the initial ($\tau=0$) state, while the solid curves represent the state after a period of $\tau=0.51$, where the probability density at $\xi=0$ increases by a factor of 4.

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Clearly, the source of this nonlocality is the fact that each one of the sinc is spread over the entire space. However, the important result is, that this nonlocal presentation of the Schrödinger equation is independent of Δx, which can be as short as the spatial measurement accuracy.

5. Conclusions

It has been shown that when a given wavefunction is spectrally bounded, then the Schrödinger dynamics can be formulated in a universal nonlocal form. Instead of a local partial differential equation, it can be formulated as an infinite set of ordinary differential equation, where the coupling are pure numbers, which are strongly related to Euler's formula $\sum_{n=1}^{\infty} n^2 = \pi^2/6$.

Therefore, the mutual effect of every two points on the wavefunction is instantaneous and can be formulated by an inverse square law.

The authors declare no conflict of interest.

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