Article

Conformable Laplace Transform of Fractional Differential Equations

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Abstract: In this paper we use the conformable fractional derivative to discuss some fractional linear differential equations with constant coefficients. By applying some similar arguments to the theory of ordinary differential equations, we establish a sufficient condition to guarantee the reliability of solving constant coefficient fractional differential equations by the conformable Laplace transform method. Finally, we analyze the analytical solution for a class of fractional models associated with Logistic model, Von Foerster model and Bertalanffy model is presented graphically for various fractional orders and solution of corresponding classical model is recovered as a particular case.

Keywords: Fractional Differential Equations; Conformable Derivative; Bernoulli Equation; Exact Solution

MSC: 26A33; 34A08; 34A12; 91B62.

1. Introduction

Fractional calculus is a generalization of ordinary calculus, where derivatives and integrals of arbitrary (non-integer) order are defined. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The idea of these operators appeared first appeared in a letter between L'Hopital and Leibniz in which the question of a half-order derivative was posed [1–3]. Some important contributions to science, engineering, applied mathematics, economics and biomechanics have been reported in the literature. There are good textbooks for the fractional calculus [4–10].

Several types of fractional derivatives have been introduced to date, among which the Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard, Erdélyi-Kober, Weyl, Marchaud and Riesz are just a few to name [11]. All of them also satisfy the following important properties: fractional operators are linear, that is, if L is a fractional derivative, then

$$L(f + kg) = L(f) + kL(g)$$

for any functions $f, g \in C^n[a, b]$ and $k \in \mathbb{R}$. Unfortunately all these fractional derivatives have a lot of unusual properties [12], for example:

1. All fractional derivatives do not obey the familiar Product Rule for two functions:

$$L(fg) = fL(g) + gL(f).$$

2. All fractional derivatives do not obey the Chain Rule:

$$L(f \circ g)(t) = L(f)(g(t))\, L(g)(t).$$

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These properties lead to some difficulties in application of fractional derivatives in physics and engineering. To overcome some of these and other difficulties, Khalil et al. [13] proposed the so-called conformable fractional derivative of order α, $0 < \alpha < 1$, in order to generalize classical properties of integer-order calculus and proved the conformable fractional Leibniz rule. Also, the author in [14], generalizing the conformable operators to higher orders, presents for instance the chain rule, integration by parts and Taylor series expansion. Consequently, the conformable derivative satisfies almost all the classical properties of the derivative hold. This suggests that one may try to solve conformable fractional differential equations using the same techniques for solving ordinary differential equations.

Real-world phenomena often are modelled by the nonlinear fractional differential equations. Its applications are rapidly increasing in remodelling different dynamical models and emerging variety of methods with this definition [15–18]. Adding to this, integral transforms are also ground-breaking inventions in fractional calculus. In general, most of the fractional differential equations do not have exact solutions. An effective and convenient method for solving fractional differential equations is needed. Abdeljawad [14] introduced a method based on the conformable Laplace transform technique, it is suitable for a large class of initial value problems for fractional differential equations. On the other hand, not every function has a conformable Laplace transform, because the defining integral can fail to converge. Then the interest arose to sort it out and to be able to use them properly.

Integral inequalities are very useful in the study of ordinary differential and integral equations. For example the Gronwall inequality and its generalizations play an important role in the discussion of existence, uniqueness, and qualitative behavior of solutions (see [19,20]). Motivated by applications of fractional integral inequalities (see [21–23]), we study the reliability of conformable Laplace transform method for solving linear fractional differential equations with constant coefficients

$$\mathfrak{D}_t^{(\alpha)} x(t) + Bx(t) = A(t),\ \forall t > 0,\ x(0) = x_0, \tag{1}$$

where $\mathfrak{D}_t^{(\alpha)}$ is the conformable fractional derivative operator of order $\alpha \in (0, 1]$, $B \in \mathbb{R}$, and $A : [0.\infty) \to \mathbb{R}$ is a continuous function, which is called the forcing term or driving term.

This paper is organized as follows. In Section 2, some basic properties of conformable fractional calculus are given. In Section 3, we enunciate and probe the reliability of the conformable Laplace transform method for solving linear fractional differential equations with constant coefficients. In section 4, analytical solutions of the fractional models are obtained. Initial value problems are considered and a few concluding remarks in section 5.

2. Brief of conformable fractional calculus

Let us review the conformable calculus [13,14]. The interest for this new approach was born from the notion that makes a dependency just on the basic limit definition of the derivative.

Definition 1. *([13]). Let $f : [0, \infty) \to \mathbb{R}$ be a function. Then, the conformable fractional derivative of f of order α, $0 < \alpha \leq 1$, is defined by,*

$$T_\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \tag{2}$$

for all $t > 0$.

Every real function which satisfy in Eq. 2 and corresponding limit exist, is called as the α-differentiable function. In addition, if f is α-differentiable in some $t \in (0, b)$, and $\lim_{t\to 0^+} T_{(\alpha)}(f)(t)$ exists, then we define

$$T_\alpha(f)(0) = \lim_{t \to 0^+} T_\alpha(f)(t).$$

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The relationship between the conformable derivative and the first derivative can be represented

$$T_\alpha f(t) = t^{1-\alpha} f'(t),\ f \in C^1. \tag{3}$$

Consider the limit $\alpha \to 1^-$. In this case, for $t > 0$, we obtain the classical definition for derivative of a function, $T_1 f(t) = f'(t)$. This shows that the Conformable derivative is a generalization of the integer-order derivative. Moreover, the physical interpretation of the conformable derivative is a modification of classical derivative in direction and magnitude [15].

Remark 1. *Differentiability implies α-differentiability but the contrary is not true: a nondifferentiable function can be α-differentiable. For a discussion of this issue see [13].*

Notation 1. *We can write $\mathfrak{D}_t^{(\alpha)} f(t)$ for $T_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α.*

This new definition of fractional derivative satisfies the following properties:

Theorem 1 ([13]). *Let $\alpha \in (0,1]$ and $a \in \mathbb{R}$, then*

(1) Linearity: $\mathfrak{D}_t^{(\alpha)}(f + kg) = \mathfrak{D}_t^{(\alpha)} f + k\,\mathfrak{D}_t^{(\alpha)} g$, for all $a \in \mathbb{R}$,
(2) Leibniz rule: $\mathfrak{D}_t^{(\alpha)}(fg) = f\,\mathfrak{D}_t^{(\alpha)} g + g\,\mathfrak{D}_t^{(\alpha)} f$,
(3) Quotient Rule: $\mathfrak{D}_t^{(\alpha)}(f/g) = \left(g\,\mathfrak{D}_t^{(\alpha)} f - f\,\mathfrak{D}_t^{(\alpha)} g\right)/g^2$.

Moreover, in [14] demonstrated that the chain rule is valid for conformable fractional derivatives.

Theorem 2. *Let f be a differentiable at $g(t)$, and g α-differentiable function defined in the range of f at $t > 0$. Then*

$$\mathfrak{D}_t^{(\alpha)}(f \circ g)(t) = f'(g(t))\,\mathfrak{D}_t^{(\alpha)} g(t). \tag{4}$$

Remark 2. *In general, other fractional derivatives do not obey satisfy the properties as shown in theorems 1 and 2.*

The analogous definition of the integral operator corresponding to derivative operator is given by the following definition.

Definition 2 (Conformable fractional integral). *Let $\alpha \in (0,1]$ and $f : [0,\infty) \to \mathbb{R}$. The conformable fractional integral of f of order α from 0 to t, is defined by*

$$\mathcal{I}_\alpha f(t) = \int_0^t f(s)\, d_\alpha s := \int_0^t f(s) s^{\alpha-1} ds = \mathcal{I}_1(t^{\alpha-1} f)(t), \quad t \geq 0,$$

where the above integral is the usual improper Riemann integral.

Lemma 1. *Assume that f is a continuous function on an $(0,\infty)$ and $0 < \alpha \leq 1$. Then for all $t > 0$ we have $\mathfrak{D}_t^{(\alpha)}[\mathcal{I}_\alpha f(t)] = f(t)$.*

Definition 3. *[13] The conformable fractional exponential function is defined for every $t \leq 0$ by*

$$E_\alpha(c,t) = \exp\left(c\frac{t^\alpha}{\alpha}\right), \tag{5}$$

where $c \in \mathbb{R}$ and $0 < \alpha \leq 1$.

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Resulting from Eq. 3

$$\mathfrak{D}_t^{(\alpha)} E_\alpha(c,t) = c\, E_\alpha(c,t), \tag{6}$$

that is, the famous stretched exponential function $E_\alpha(1,t)$ [24], is an engenfunction of $\mathfrak{D}_t^{(\alpha)}$ with an eigenvalue of 1.

Integral inequalities play an important role in the qualitative analysis of the solutions to differential and integral equations [20]. As follows a conformable fractional version of Gronwall theorem which plays an important role in stability analysis of the conformable fractional systems.

Lemma 2 (Conformable Gronwall inequality [14]). *Let r be a continuous, nonnegative function on $0 \le t < T$ (some $T \le \infty$) and a and b be nonnegative constants such that*

$$r(t) \le a + b \int_a^t r(s)\, d_\alpha s$$

on this interval. Then

$$r(t) \le a E_\alpha(b,t).$$

Here, we deal with the fractional Laplace transform which was first defined by Abdeljawad [14].

Definition 4 (Abdeljawad [14]). *Let $0 < \alpha \le 1$ and $f : [0,\infty) \to$ be real valued function. Then the fractional Laplace transform of order α starting from a of f is defined by*

$$\mathfrak{L}_\alpha[f(t)](s) = \int_0^\infty E_\alpha(-s,t) f(t)\, d_\alpha t. \tag{7}$$

From the definition 4 we can write the conformable Laplace transform of conformable fractional derivative according to:

$$\mathfrak{L}_\alpha[\mathfrak{D}_t^{(\alpha)} f(t)] = s\mathfrak{L}_\alpha[f(t)] - f(0). \tag{8}$$

The relation between the usual and the fractional Laplace transforms is given below.

Theorem 3 (Abdeljawad [14]). *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $\mathfrak{L}_\alpha\{f(t)\}(s) = F_\alpha(s)$ exists. Then*

$$F_\alpha(s) = \mathcal{L}[f((\alpha t)^{1/\alpha})](s), \tag{9}$$

where $\mathcal{L}[g(t)](s) = \int_0^\infty e^{-st} g(t) dt$.

From 7, we note that

$$\mathfrak{L}_\alpha[af(t) + bg(t)] = a\mathfrak{L}_\alpha[f(t)] + b\mathfrak{L}_\alpha[g(t)],$$

where a and b are constant real numbers.

It is easy to show that

Theorem 4 ([30]). *If $F_\alpha(s) = \mathfrak{L}[f(t)]$ exists for $s > 0$, then*

1. *If c is a constant then*
$$\mathfrak{L}[c] = \frac{c}{s}. \tag{10}$$
2. *Let q is a constant*
$$\mathfrak{L}_\alpha[t^q](s) = \alpha^{q/\alpha} \frac{\Gamma(1+\frac{q}{\alpha})}{s^{1+q/\alpha}} \tag{11}$$

3. *If c, q are arbitrary constants*

$$\mathfrak{L}_\alpha[t^q E_\alpha(c,t)](s) = \alpha^{q/\alpha}\frac{\Gamma(1+\frac{q}{\alpha})}{(s-c)^{1+q/\alpha}} \tag{12}$$

Proof. One can easily see the proof by using the definition of conformable Laplace transform and Theorem 3. □

Theorem 5 ([16]). *Let $f, g : [0,\infty] \to \mathbb{R}$ be real valued functions and $0 < \alpha \leq 1$. Then if $F_\alpha(s) = \mathfrak{L}_\alpha[f(t^\alpha)]$ and $G_\alpha(s) = \mathfrak{L}_\alpha[g(t)]$ both exist for $s \geq 0$ then*

$$\mathfrak{L}_\alpha[f * g](t) := \int_0^t f(t^\alpha - s^\alpha)g(s)\, d_\alpha s = F_\alpha(s)G_\alpha(s). \tag{13}$$

Definition 5. *A function f is said to be conformable exponentially bounded if it satisfies an inequality of the form*

$$||f|| \leq ME_\alpha(c,t), \tag{14}$$

where M, c are positive real constants and $0 < \alpha \leq 1$, for all sufficiently large t.

3. Validity of conformable Laplace transform for linear fractional-order equations

Theorem 6. *Assume that the Eq. 1 has a unique continuous solutions $x(t)$, if $A(t)$ is continuous on $[0,\infty)$ and conformable exponentially bounded, then derivative $x(t)$ and its $\mathfrak{D}_t^{(\alpha)}x(t)$ are both conformable exponentially bounded, thus their conformable Laplace transforms exist.*

Proof. From 5, we have that there exist positive constants M, σ and enough large T such that $||A(t)|| \leq ME_\alpha(\sigma, t)$ for all $t \geq T$.

Every solution of 1 is also a solution of the Volterra integral equation given below and vice versa.

$$x(t) = x_0 + \int_0^t (-Bx(s) + A(s))\, d_\alpha s, \quad t \geq 0. \tag{15}$$

For $t \geq T$, 15 can be rewritten as

$$x(t) = x_0 + \int_0^T s^{\alpha-1}[-Bx(s) + A(s)]\, ds + \int_T^t s^{\alpha-1}[-Bx(s) + A(s)]\, ds. \tag{16}$$

By the continuity of $x(t)$, then $-Bx(t) + A(t)$ is bounded on $[0, T]$, i.e., there exists a constant $K > 0$ such that $||-Bx(t) + A(t)|| \leq K$. We have

$$||x(t)|| \leq ||x_0|| + K\int_0^T s^{\alpha-1}\, ds + |B|\int_T^t s^{\alpha-1}||x(s)||\, ds + \int_T^t s^{\alpha-1}||A(s)||\, ds. \tag{17}$$

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Multiply this inequality by $E_\alpha(-\sigma,t)$ and note that $E_\alpha(-\sigma,t) \leq E_\alpha(-\sigma,T)$ and $||A(t)|| \leq ME_\alpha(\sigma,t)$ $(t \geq T)$ to obtain

$$\begin{aligned}
||x(t)||e^{-\sigma\frac{t^\alpha}{\alpha}} &\leq ||x_0||E_\alpha(-\sigma,t) + KE_\alpha(-\sigma,t)\int_0^T s^{\alpha-1}\,ds + |B|E_\alpha(-\sigma,t)\int_T^t s^{\alpha-1}||x(s)||\,ds \\
&\quad + E_\alpha(-\sigma,t)\int_T^t s^{\alpha-1}||A(s)||\,ds. \\
&\leq ||x_0||E_\alpha(-\sigma,T) + \frac{KT^\alpha}{\alpha}E_\alpha(-\sigma,T) + |B|\int_T^t s^{\alpha-1}||x(s)||E_\alpha(-\sigma,s)\,ds \\
&\quad + E_\alpha(-\sigma,t)\int_T^t s^{\alpha-1}||A(s)||\,ds. \\
&\leq ||x_0||E_\alpha(-\sigma,T) + \frac{KT^\alpha}{\alpha}E_\alpha(-\sigma,T) + |B|\int_0^t s^{\alpha-1}||x(s)||E_\alpha(-\sigma,s)\,ds \\
&\quad + M\int_0^t s^{\alpha-1}e^{\sigma\frac{s^\alpha-t^\alpha}{\alpha}}\,ds. \\
&\leq ||x_0||E_\alpha(-\sigma,T) + \frac{KT^\alpha}{\alpha}E(-\sigma,T) + |B|\int_0^t s^{\alpha-1}||x(s)||E_\alpha(-\sigma,s)\,ds \\
&\quad + M\int_0^t e^{-\sigma u}\,du. \\
&\leq ||x_0||E_\alpha(-\sigma,T) + \frac{KT^\alpha}{\alpha}E_\alpha(-\sigma,T) + |B|\int_0^t s^{\alpha-1}||x(s)||E_\alpha(-\sigma,s)\,ds \\
&\quad + M\int_0^\infty e^{-\sigma u}\,du. \\
&\leq ||x_0||E_\alpha(-\sigma,T) + \frac{KT^\alpha}{\alpha}E_\alpha(-\sigma,T) + \frac{M}{\sigma} + |B|\int_0^t s^{\alpha-1}||x(s)||E_\alpha(-\sigma,s)\,ds, \quad t \geq T.
\end{aligned}$$

Denote

$$a = ||x_0||E_\alpha(-\sigma,T) + \frac{KT^\alpha E_\alpha(-\sigma,T)}{\alpha} + \frac{M}{\sigma}, \quad b = |B|, \quad r(t) = ||x(t)||E_\alpha(-\sigma,t),$$

we get

$$r(t) \leq a + b\int_0^t s^{\alpha-1}r(s)ds, \quad t > T. \tag{18}$$

By Lemma 2,

$$r(t) \leq aE_\alpha(b,t),$$

then

$$||x(t)|| \leq aE_\alpha(b+\sigma,t), \quad t \geq T. \tag{19}$$

From Eq. 1, we obtain

$$\begin{aligned}
||\mathfrak{D}_t^{(\alpha)}x(t)|| &\leq |B|\,||x(t)|| + ||A(t)|| \\
&\leq a|B|\,E_\alpha(b+\sigma,t) + ME_\alpha(\sigma,t), \quad t \geq T.
\end{aligned}$$

This concludes the proof of Theorem 6. □

4. Illustrative examples

In this section, we present three examples, which indicate how our theorem can be applied to concrete problems. We consider the scalar fractional differential equation nonlinear of the form

$$\mathfrak{D}_t^{(\alpha)}x(t) + B\,x(t) = A(t)[x(t)]^q, \quad x(0) = x_0 \geq 0, \quad t > 0, \tag{20}$$

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with order $\alpha \in (0, 1)$. We remark that Pospíšil [17] has given necessary and sufficient conditions for the existence and uniqueness of the solution of 20.

Remark 3. *Considering the limit when $\alpha \to 1^-$ in the Eq. 20, it becomes a Bernoulli type differential equation.*

If $q \neq 0, 1$, we make the change of variables

$$z(t) = [x(t)]^{1-q}. \tag{21}$$

Consequently, from 2, we have the linear fractional differential equations of order α

$$\mathfrak{D}_t^{(\alpha)} z(t) = (1-q)[z(t)]^{-q} \mathfrak{D}_t^{\alpha} u(t). \tag{22}$$

The linearised fractional conformable form of Eq. (20) is

$$\mathfrak{D}_t^{(\alpha)} z(t) + (1-q) B\, z(t) = (1-q) A(t), \tag{23}$$

whose exact closed form solution (see [16]) can be found efficiently by Theorem 6.

Example 1. Regard the given conformable differential equation of Logistic type (see [25]) below

$$\mathfrak{D}_t^{(\alpha)} x(t) = x(t)[1 - E_\alpha(-1,t)\, x(t)], \quad x_0 = 1/2. \tag{24}$$

Using the change of variable 21 for $q = 2$, we have

$$\mathfrak{D}_t^{(\alpha)} z(t) = E_\alpha(-1,t) - z(t). \tag{25}$$

Applying conformable Laplace transform to the both sides of Eq. 25

$$\begin{aligned} \mathfrak{L}_\alpha[\mathfrak{D}_t^{(\alpha)} z(t)] &= \mathfrak{L}_\alpha[E_\alpha(-1,t) - z(t)], \quad (26)\\ s Z_\alpha(s) - 2 &= \frac{1}{s+1} - Z_\alpha(s),\\ Z_\alpha(s) &= \frac{1}{(s+1)^2} + \frac{2}{s+1}. \quad (27) \end{aligned}$$

Applying the inverse conformable Laplace transform to Eq. 27 with the help of convolution theorem 5, we obtain (Fig. 1)

$$x(t) = \frac{E_\alpha(-1,t)}{\dfrac{t^\alpha}{\alpha} + 2} \tag{28}$$

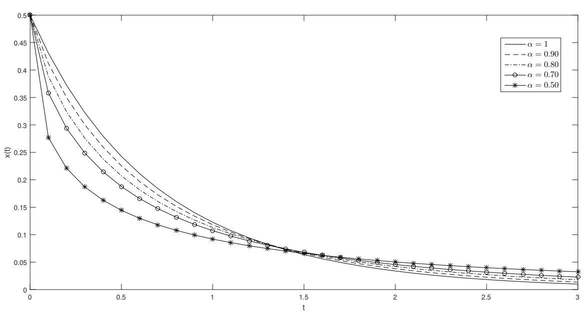

Figure 1. The solution of Eq. 24 considering several values of α.

Example 2 (Von Foerster model)**.** Equation 6 can however be viewed as a special case of a more general equation

$$\mathfrak{D}_t^{(\alpha)} x(t) = x^q(t). \tag{29}$$

Let the initial condition be $x(0) = 1$, by using 21 into 29 we find

$$\mathfrak{D}_t^{(\alpha)} z(t) = 1 - q, \tag{30}$$

subject to the initial condition $z(0) = 1$. After an algebraic manipulation

$$z(t) = 1 + (1-q)\frac{t^\alpha}{\alpha}. \tag{31}$$

Finally, we get the solution as

$$x(t) = \left[1 + (1-q)\frac{t^\alpha}{\alpha}\right]_+^{\frac{1}{1-q}} = \exp_q\left(\frac{t^\alpha}{\alpha}\right), \tag{32}$$

with $[y]_+ = \max\{y, 0\}$.

Obviously, one has

$$\lim_{q\to 1} x(t) = \lim_{q\to 1} \exp_q\left(\frac{t^\alpha}{\alpha}\right) = E_\alpha(1,t). \tag{33}$$

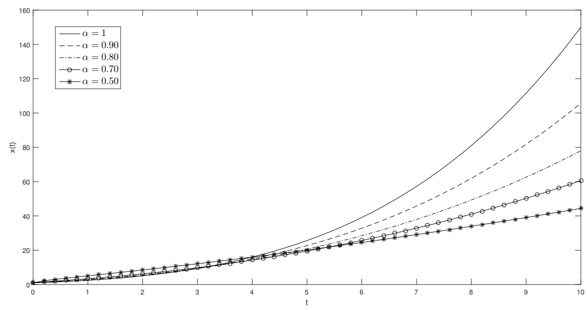

Figure 2. The physical behavior of $x(t)$ with $q = 0.5$ and for different values of α.

Remark 4.

$$\lim_{\alpha\to 1^-} \mathfrak{D}_t^{(\alpha)} x(t) = \frac{dx}{dt} = x^q, \tag{34}$$

subject to the initial condition $x(0) = 1$. The exact solution generalizes the standard exponential function as

$$x(t) = \exp_q(t) = [1 + (1-q)t]_+^{1/(1-q)}, \tag{35}$$

where q is a real parameter, the entropic index. It has become common to call the corresponding statistics 'q-statistics' [26].

Example 3. Consider the following fractional Bertalanffy-Logistic differential equation:

$$\mathfrak{D}_t^{(\alpha)} x(t) = [x(t)]^{2/3} - x(t), \quad 0 < \alpha < 1, \tag{36}$$

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subject to the initial condition $x(0) = x_0$. For $\alpha = 1$, Eq. 36 is the standard Bertalanffy-Logistic equation

$$x'(t) = [x(t)]^{2/3} - x(t), \quad x(0) = x_0.$$

The exact solution to this problem is

$$x(t) = \left[1 + (x_0^{\frac{1}{3}} - 1)e^{-\frac{t}{3}}\right]^3. \tag{37}$$

The Von Bertalanffy equation is a logistic model widely applied to describe growth of different types of populations [27–29].

By using 21 into 36 we find

$$\mathfrak{D}_t^{(\alpha)} z(t) = \frac{1}{3}(1 - z(t)), \quad z_0 = x_0^{\frac{1}{3}}, \quad 0 < \alpha < 1. \tag{38}$$

Applying conformable Laplace transform to the both sides of Eq. 38

$$Z_\alpha(s) = \frac{1}{s} - \frac{1}{s + \frac{1}{3}} + z_0. \tag{39}$$

Applying the inverse conformable Laplace transform to Eq. 39, we obtain (Fig. 1)

$$x(t) = \left[1 + (x_0^{\frac{1}{3}} - 1)e^{-\frac{t^\alpha}{3\alpha}}\right]^3. \tag{40}$$

It can be seen when $\alpha \to 1$ in Eq. 40, we get the classical solution given by Eq. 37

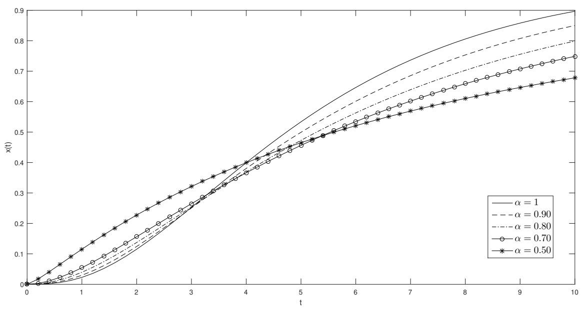

Figure 3. The solution of Eq. 36 considering $x_0 = 0$ and several values of α.

5. Concluding Remark

The exact solutions of fractional differential equations play a crucial role in the mathematical physics. Similarly to integer-order derivatives, by conformable Gronwall inequality, solutions of fractional-order equations are showed to be of conformable exponentially bounded. So the validity of Laplace transform of fractional-order equations is justified, but it demands for forcing terms, so not every constant coefficient fractional differential equation can be solved by the conformable Laplace transform method. We apply conformable fractional Laplace transform to the conformable fractional-order Bernoulli equation. Differences between the solutions of the model with integer derivatives and conformable factional derivatives are graphically investigated. The present study confirms previous findings in case of $\alpha = 1$. Some illustrative examples are given to show the effectiveness of the contributed results.

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