 doi:10.20944/preprints201806.0461.v1

The Reynolds Navier-Stokes turbulence equations of incompressible flow are closed rather than unclosed

Bohua Sun[1]

[1]Cape Peninsula University of Technology, Cape Town, South Africa*

(Dated: June 26, 2018)

This paper shown that turbulence closure problem is not an issue at all. All mistakes in the literature regarding the numbers of unknown quantities in the Reynolds turbulence equations stem from the misunderstandings of physics of the Reynolds stress tensor, i.e., all literatures have stated that the symmetric Reynolds stress tensor has six unknowns; however, it actually has only three unknowns, i.e., the three components of fluctuation velocity. We shown the integral-differential equations of the Reynolds mean and fluctuation equations have exactly eight equations, which equal to the numbers of quantities in total, namely, three components of mean velocity, three components of fluctuation velocity, one mean pressure and one fluctuation pressure. That is why we claim in this paper, that the Reynolds Navier-Stokes turbulence equations of incompressible flow are closed rather than unclosed. This study may help to solve the puzzle that has eluded scientists and mathematicians for centuries.

PACS numbers: 47.27.-i,47.27.Ak,47.27.E-
Keywords: Turbulence, the Reynolds stress tensor, turbulence closure problem

Turbulence is everywhere, controlling the drag on cars, airplanes, and trains, whilst dictating the weather through its influence on large-scale atmospheric and oceanic flows. Even solar flares are a manifestation of turbulence, since they are triggered by vigorous motions on the surface of the sun. It is easy to be intrigued by a subject that pervades so many aspects of peoples' daily lives [1–16].

The study of turbulence is not simple owing to its complex and forbidding mathematical descriptions, as well as the profound difficulties of inherent instabilities and even chaotic processes. People believe that turbulence prediction can be attained by understanding solutions to Navier-Stokes equations. However, understanding of the Navier-Stokes equations remains minimal, while there is still surprisingly little that can be predicted with relative certainty [15, 17, 18]. In respect of the turbulence problem, a myriad of tentative theories have been proposed, each with its own doctrines and beliefs, whilst often focused on particular experiments; however, there is not much in the way of a coherent theoretical framework [1–8, 10–16]. Turbulence is a unique subject that engineers, mathematicians, and physicists tend to view in rather different ways. Many engineers promote the use of semi-empirical models of turbulence, while mathematicians advocate the use of purely statistical models [19–24], and the formulism of chaos theory and fractals [25–27].

In 1972 a new chapter was launched in turbulence theory: Orszag and Patterson demonstrated that it was possible to perform direct numerical simulation (DNS) of a fully turbulent flow [28]. It is important to understand that DNS does not require any turbulence model to parameterize influence of the turbulent eddies. Rather, every eddy, from the largest to the smallest, is computed. Technically speaking, the turbulence can be solved by DNS if computers have infinite speed. However, a huge chasm remains between what the engineer needs to know, and what can be realized by DNS, using current computers. Even if DNS can assist to solve turbulence issues and problems, one still requires turbulence modelling to acquire a physical understanding of it.

Although there are different views about turbulence, there is a consensus that the deterministic Navier-Stokes equation probably contains all information relevant to turbulence [11]. It is believed that turbulence can be figured out once the Navier-Stokes equation is solved. Hence, scholars have been critical of the Navier-Stokes equation, and numerous works have been published as a result [1–8, 10–16].

In 1895 Reynolds published a seminal work on turbulence [29], in which he proposed that flow velocity $\boldsymbol{u}$ and pressure p are decomposed into its time-averaged quantities, $\bar{\boldsymbol{u}, t}$, $\bar{p}$, and fluctuating quantities,

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 June 2018 doi:10.20944/preprints201806.0461.v1

$\boldsymbol{u}'$, p'; thus, the Reynolds decompositions are: $\boldsymbol{u} = \bar{\boldsymbol{u}}(\boldsymbol{x},t) + \boldsymbol{u}'(\boldsymbol{x},t)$ and $p(\boldsymbol{x},t) = \bar{p}(\boldsymbol{x},t) + p'(\boldsymbol{x},t)$, where coordinates and times are $(\boldsymbol{x},t)$. With decomposition the Navier-Stokes equation is then transformed into Reynolds-averaged Navier - Stokes equations (or RANS equations), where the Reynolds stress tensor $\boldsymbol{\tau} = -\rho\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'} = -\rho\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}(\boldsymbol{u}'\otimes\boldsymbol{u}')dt$ is introduced, where T is the period of time over which the averaging takes place and must be sufficiently large to give meaningful averages. Reynolds stress is apparent stress owing to the fluctuating velocity field $\boldsymbol{u}'$. Since introduction of the Reynolds stress tensor, the closure problem of turbulence, namely the Reynolds equations are unclosed, has eluded scientists and mathematicians for centuries. The Reynolds equations can not be solved unless some additional restrictions are somehow determined.

As we know, the Navier-Stokes momentum equation is $\rho\boldsymbol{u}_{,t} + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = 0$, continuity equation of incompressible flow is $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$, where the energy-moementum tensor given by $\boldsymbol{\Pi} = p\boldsymbol{I} + \rho\boldsymbol{u}\otimes\boldsymbol{u} - \mu(\boldsymbol{\nabla u} + \boldsymbol{u\nabla})$, dynamic viscosity μ, gradient operator $\boldsymbol{\nabla} = \boldsymbol{e}_i\partial_i$, base vector in the i-coordinate $\boldsymbol{e}_i$, and tensor product $\otimes$. By introducing the Reynolds decomposition and averaging operation, we have the Reynolds equations and continuity equation of the mean velocity as follows, respectively: $\rho\bar{\boldsymbol{u}}_{,t} + \rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}}) + \boldsymbol{\nabla}\bar{p} = \mu\boldsymbol{\nabla}^2\bar{\boldsymbol{u}} - \rho\boldsymbol{\nabla}\cdot(\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'})$ and $\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0$.

For a general three-dimensional flow, there are four independent equations governing the mean velocity field; namely three components of the Reynolds equations together with one mean continuity equation. However, these four equations contain more than four unknowns. In addition to $\bar{\boldsymbol{u}}$ and $\bar{p}$ (four quantities), there are also the Reynolds stresses. The Reynolds equations are unclosed. This is a manifestation of the closure problem.

In 1940, P.-Y. Chou presented another approach [30, 31], who pointed out that because the Navier-Stokes equations are the basic dynamical equations of fluid motion, it is insufficient to consider only the mean turbulent motion. The turbulent fluctuations are as important as the mean motion and the equations for turbulent fluctuations also need to be considered.

Subtracting the mean motions from the Navier-Stokes equation and continuity equation, Chou [30, 31] obtained the equations of the turbulence fluctuations $\rho\bar{\boldsymbol{u}'}_{,t} + \rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\boldsymbol{u}' + \boldsymbol{u}'\otimes\bar{\boldsymbol{u}}) + \boldsymbol{\nabla}p' = \mu\boldsymbol{\nabla}^2\boldsymbol{u}' + \rho\boldsymbol{\nabla}\cdot(\overline{\boldsymbol{u}'\otimes\boldsymbol{u}'})$ and $\boldsymbol{\nabla}\cdot\boldsymbol{u}' = 0$. After having the above fluctuation equations, Chou [30, 31] introduced hierarchy of equations for velocity fluctuation correlations, however, any velocity correlation equation of a given order always obtains an unknown velocity correlation of one higher order, i.e. all hierarchy are still not closed.

Although Chou [31] mentioned that the rigorous way of treating the turbulence problem is probably to solve the Reynolds' equations of mean motion and the equations of turbulent fluctuation simultaneously. However, from the presentation of [31] and all his forthcoming publications [32–36], we noticed that Chou and all other researchers [1–8, 10–16] still believe that the Reynolds mean and fluctuation equations are unclosed, which has led to the creation of many different turbulence models [2, 7, 8, 11, 14]. A number of turbulence models on the Reynolds stress modelling have been proposed [1–8, 10–16].

Regarding the closure issue, here we have a completely different perspectives, namely, the Reynolds mean and fluctuation turbulence equations are closed and no additional restrictions are needed. The so-called turbulence closure problem is not an issue at all.

To support this claim, let's re-writing the equations of mean and fluctuation motions as follows

$$\rho\bar{\boldsymbol{u}}_{,t} + \rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\bar{\boldsymbol{u}}) + \boldsymbol{\nabla}\bar{p} = \mu\boldsymbol{\nabla}^2\bar{\boldsymbol{u}} - \rho\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\boldsymbol{\nabla}\cdot(\boldsymbol{u}'\otimes\boldsymbol{u}')dt, \tag{1}$$

$$\rho\bar{\boldsymbol{u}'}_{,t} + \rho\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\otimes\boldsymbol{u}' + \boldsymbol{u}'\otimes\bar{\boldsymbol{u}}) + \boldsymbol{\nabla}p' = \mu\boldsymbol{\nabla}^2\boldsymbol{u}' + \rho\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}\boldsymbol{\nabla}\cdot(\boldsymbol{u}'\otimes\boldsymbol{u}')dt, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \tag{3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}' = 0. \tag{4}$$

denoting kinematic viscosity $\nu = \mu/\rho$, the above integral-differential equations can be expressed equivalently as

doi:10.20944/preprints201806.0461.v1

follows

$$\bar{\boldsymbol{u}}_{,t} + \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}} + \frac{1}{\rho} \boldsymbol{\nabla} \bar{p} = \nu \boldsymbol{\nabla}^2 \bar{\boldsymbol{u}} - \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} (\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}') dt, \tag{5}$$

$$\bar{\boldsymbol{u}'}_{,t} + \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \boldsymbol{u}' + \boldsymbol{u}' \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}} + \frac{1}{\rho} \boldsymbol{\nabla} p' = \nu \boldsymbol{\nabla}^2 \boldsymbol{u}' + \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} (\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}') dt, \tag{6}$$

$$\boldsymbol{\nabla}^2 \bar{p} = -\rho \boldsymbol{\nabla} \cdot (\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}}) - \rho \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \boldsymbol{\nabla} \cdot (\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}') dt, \tag{7}$$

$$\boldsymbol{\nabla}^2 p' = -\rho \boldsymbol{\nabla} \cdot [\bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} \boldsymbol{u}' + \boldsymbol{u}' \cdot \boldsymbol{\nabla} \bar{\boldsymbol{u}}] + \rho \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \boldsymbol{\nabla} \cdot (\boldsymbol{u}' \cdot \boldsymbol{\nabla} \boldsymbol{u}') dt. \tag{8}$$

Amazingly the integral-differential equations in Eqs.(1,2,3,4) and or Eqs.(5,6,7,8) have exactly eight equations, which equal to the numbers of unknowns, namely, three components of mean velocity $\bar{\boldsymbol{u}}$, three components of fluctuation velocity $\boldsymbol{u}'$, one mean pressure $\bar{p}$ and one fluctuation pressure p'. That is why the integral-differential equations system of Eqs.(1,2,3,4)and or Eqs.(5,6,7,8) are closed !

All mistakes regarding the numbers of unknown quantities stem from misunderstandings of the formulation of Reynolds stress tensor, i.e., all literatures report the symmetric Reynolds stress tensor has six unknowns; however, the Reynolds stress tensor actually is not a general tensor but the product of fluctuation velocity, has only three unknowns, namely the three components of fluctuation velocity $\boldsymbol{u}'$.

This can be proved easily as follows: the Reynolds stress tensor can be defined by

$$\begin{aligned}\boldsymbol{\tau} &= -\rho\overline{\boldsymbol{u}' \otimes \boldsymbol{u}'} = -\rho\overline{u_i'\boldsymbol{e}_i \otimes u_j'\boldsymbol{e}_j} \\ &= -\rho\overline{u_i'u_j'}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= -\rho\left(\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} u_i'u_j'dt\right)\boldsymbol{e}_i \otimes \boldsymbol{e}_j,\end{aligned} \tag{9}$$

and fluctuation velocity convective terms

$$\begin{aligned}&\overline{\boldsymbol{u}' \cdot \boldsymbol{\nabla}\boldsymbol{u}'} = \overline{u_i'\boldsymbol{e}_i \cdot [\boldsymbol{e}_k\partial_k \otimes (u_j'\boldsymbol{e}_j)]} \\ &= \overline{u_i'u_{j,k}'}\boldsymbol{e}_i \cdot (\boldsymbol{e}_k \otimes \boldsymbol{e}_j) = \overline{u_i'u_{j,k}'}(\boldsymbol{e}_i \cdot \boldsymbol{e}_k)\boldsymbol{e}_j \\ &= \overline{u_i'u_{j,k}'}\delta_{ik}\boldsymbol{e}_j = \overline{u_i'u_{j,i}'}\boldsymbol{e}_j \\ &= \left(\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T} u_i'u_{j,i}'dt\right)\boldsymbol{e}_j.\end{aligned} \tag{10}$$

The formulations in Eq.(9,10) reveal that the Reynolds stress tensor $\boldsymbol{\tau} = -\rho\overline{\boldsymbol{u}' \otimes \boldsymbol{u}'}$ and fluctuation velocity convective terms can be calculated by the three components of fluctuation velocity u_1', u_2', u_3', hence the Reynolds stress tensor has only three unknowns.

Therefore, our statement can be written as follows: The Reynolds Navier-Stokes turbulence equations of incompressible flow in Eqs.(1,2,3,4) are closed rather than unclosed.

With the closed integral differential equations systems in Eqs.(1,2,3,4) and or their equivalent Eqs.(5,6,7,8), there is no need to propose any additional restriction for a given turbulence problem. Therefore, in the future, all turbulence modelling using the Reynolds velocity decomposition becomes to solve a mathematical assignments listed in Eqs.(1,2,3,4) and or Eqs.(1,2,3,4) numerically.

It is my great pleasure to have shared and discussed some of the above with Michael Sun from Bishops Diocesan College, whose pure and direct scientific sense inspired me.

* Electronic address: sunb@cput.ac.za

[1] Bradshaw P. *An Introduction to Turbulence and Its Measurement.* Pergamon Press, New York (1971)

[2] Tennekes, H. and Lumley, J.L. *A First Course in Turbulence.* Cambridge: The MIT Press (1972).

[3] Lesilie D.C. *Developments in the Theory of Turbulence.* Clarendon Press, Oxford (1973).

[4] Townsend A.A. *The Structure of Turbulent Shear Flow* 2nd ed., Cambridge University Press, New York (1976).

[5] Lesieur M. *Turbulence in Fluids.* 2nd ed. Kluwer, Dordrecht (1990).

[6] Wilcox D.C. *Turbulence modeling for CFD.* D C W Industries (1993).

[7] Pope S.B. *Turbulent Flows*. Cambridge University Press, Cambridge (2000).

[8] Davidson, P.A. *Turbulence*. Oxford University Press, Oxford (2004).

[9] Hof B. Experimental Observation of Nonlinear Traveling Waves in Turbulent Pipe Flow. Science **305**, 1594 (2004)

[10] Falkovich, G. and Sreenivasan K.R. Lessons from hydrodynamic turbulence. Physics Today, 43-49 (April 2006).

[11] Frisch, U. *Turbulence: The Legacy of A.N. Kolmogorov*. Cambridge University Press,Cambridge (2008).

[12] Marusic I. Mathis R. and Hutchins N. Predictive model for wall-bounded turbulent Flow. Science **329**, 193 (2010).

[13] Smits A.J., McKeon B.J. and Marusic I. High – Reynolds Number Wall Turbulence, Annu. Rev. Fluid Mech. **43**, 353 (2011).

[14] Davison, P.A. *et al. A Voyage Through Turbulence*. Cambridge: Cambridge University Press (2011)

[15] Suri B., Tithof J. R., Grigoriev R.O. and Schatz M.F. Forecasting Fluid Flows Using the Geometry of Turbulence. Phys. Rev. Lett. **118**, 114501 (2017).

[16] Cardesa et al. The turbulent cascade in five dimensions. Science 357, 782 – 784 (2017) 25

[17] http://www.claymath.org/millennium-problems/navier-stokes-equation.

[18] Castelvecchi, D. On the trial of turbulence. Nature, **548**:382 (2017).

[19] Kolmogorov, A.N. The local structure of turbulence in incompressible viscous fluid for very large Reynolds number. Dokl. Akad. Nauk SSSR, **30**:299-303 (1941a) (reprinted in *Proc.R.Soc.Lond.* A, 434,9-13, 1991).

[20] Kolmogorov, A.N. On degeneration (decay) of isotropic turbulence in an incompressible visous liquid. Dokl. Akad. Nauk SSSR, **31**:538-540 (1941b).

[21] Kolmogorov, A.N. Dissipation of energy in locally isotropic turbulence. Dokl.Akad. Nauk SSSR, **32**:16-18 (1941c).(reprinted in Proc.R.Soc.Lond. A, 434,15-17, 1991).

[22] Sreenivasan K.R. On the scaling of the turbulence energy dissipation rate. Physics of Fluids, **27**,5:1048-1051 (1984).

[23] Sun, B. The temporal scaling laws of compressible turbulence. Modern Physics Letters B. **30**,(23) 1650297 (2016).

[24] Sun, B. Scaling laws of compressible turbulence. Appl. Math. Mech.-Engl. Ed. **38**: 765(2017).

[25] She Z. S. and Lévêque E. Universal scaling laws in fully developed turbulence. Phys. Rev. Lett. **72**,336(1994) .

[26] Lorenz E.N. Deterministic non-periodic flow. J. Atmos Sci. **20**:130 – 41(1963).

[27] Benzi R, Paladin P., Parisis G. and Vulpiani A. On the multifractal nature of fully developed turbulence and chaotic systems. J. Phys. A: Math. Gen. **17**:3521-3531(1984).

[28] Orszag S. A. and Patterson G. S. Jr. Numerical simulation of three-dimensional homogeneous isotropic turbulence. Phys. Rev. Lett. **28**,76(1972).

[29] Reynolds O. On the dynamical theory of incompressible viscous fluids and the determination of the criterion. Philos. Trans. R. Soc. 186:123 – 164(1895).

[30] Chou P-Y. On an extension of Reynolds' method of finding apparent stress and the nature of turbulence. Chin. J. Phys. 1-53 (1940).

[31] Chou P-Y. On velocity correlations and the solutions of the equations of turbulent fluctuation. Q. Appl. Math. **111**(1):38-54(1945).

[32] Chou, P-Y, Cai S-T. The vorticity structure of homogeneous isotropic turbulence in its final period of decay. Acta Mech. Sinica **1**(1): 3-14(1957).

[33] Chou, P-Y, Huang Y-N. On the statistical vorticity structure theory of homogeneous isotropic turbulence. Sci. Sinica XVIII(2): 199-222(1975).

[34] Chou P-Y. On the condition of pseudo-similarity and the theory of turbulence. Sci. Sinica XXVIII(4): 405(1985)

[35] Chou, P-Y, Chen S-Y. On the theory of turbulence for incompressible fluid. Sci. Sinica (4): 369-80 (1987).

[36] Chou, P-Y. and Ru-Lin Chou, 50 years of turbulence research in China. Annu. Rev. Fluid Mech. **27**:1-15(1995).