INEQUALITIES OF HERMITE-HADAMARD TYPE FOR COMPOSITE h -CONVEX FUNCTIONS

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ABSTRACT. In this paper we obtain some inequalities of Hermite-Hadamard type for composite convex functions. Applications for AG , AH - h -convex functions, GA , GG , GH - h -convex functions and HA , HG , HH - h -convex function are given.

1. INTRODUCTION

We recall here some concepts of convexity that are well known in the literature. Let I be an interval in $\mathbb{R}$.

Definition 1 ([52]). We say that $f : I \rightarrow \mathbb{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is non-negative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$(1.1) \quad f(tx + (1-t)y) \leq \frac{1}{t}f(x) + \frac{1}{1-t}f(y).$$

Some further properties of this class of functions can be found in [42], [43], [45], [58], [64] and [65]. Among others, it has been noted that non-negative monotone and non-negative convex functions belong to this class of functions.

Definition 2 ([45]). We say that a function $f : I \rightarrow \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and for all $x, y \in I$ and $t \in [0, 1]$ we have

$$(1.2) \quad f(tx + (1-t)y) \leq f(x) + f(y).$$

Obviously $Q(I)$ contains $P(I)$ and for applications it is important to note that also $P(I)$ contain all nonnegative monotone, convex and *quasi convex functions*, i. e. nonnegative functions satisfying

$$(1.3) \quad f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0, 1]$.

For some results on P -functions see [45] and [62] while for quasi convex functions, the reader can consult [44].

Definition 3 ([10]). Let s be a real number, $s \in (0, 1]$. A function $f : [0, \infty) \rightarrow [0, \infty)$ is said to be s -convex (in the second sense) or Breckner s -convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Convex functions, AG , AH - h -convex functions, GA , GG , GH - h -convex functions and HA , HG , HH - h -convex function, Integral inequalities.

For some properties of this class of functions see [2], [3], [10], [11], [40], [41], [53], [55] and [67].

In order to unify the above concepts for functions of real variable, S. Varošanec introduced the concept of h -convex functions as follows.

Assume that I and J are intervals in $\mathbb{R}$, $(0, 1) \subseteq J$ and functions h and f are real non-negative functions defined in J and I , respectively.

Definition 4 ([70]). Let $h : J \rightarrow [0, \infty)$ with h not identical to 0. We say that $f : I \rightarrow [0, \infty)$ is an h -convex function if for all $x, y \in I$ we have

$$(1.4) \quad f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y)$$

for all $t \in (0, 1)$.

For some results concerning this class of functions see [70], [9], [56], [68], [66] and [69].

We can introduce now another class of functions.

Definition 5. We say that the function $f : I \rightarrow [0, \infty)$ is of s -Godunova-Levin type, with $s \in [0, 1]$, if

$$(1.5) \quad f(tx + (1-t)y) \leq \frac{1}{t^s}f(x) + \frac{1}{(1-t)^s}f(y),$$

for all $t \in (0, 1)$ and $x, y \in I$.

We observe that for $s = 0$ we obtain the class of P -functions while for $s = 1$ we obtain the class of Godunova-Levin. If we denote by $Q_s(C)$ the class of s -Godunova-Levin functions defined on C , then we obviously have

$$P(C) = Q_0(C) \subseteq Q_{s_1}(C) \subseteq Q_{s_2}(C) \subseteq Q_1(C) = Q(C)$$

for $0 \leq s_1 \leq s_2 \leq 1$.

If $f : I \rightarrow [0, \infty)$ is an h -convex function on an interval I of real numbers with $h \in L[0, 1]$ and $f \in L[a, b]$ with $a, b \in I$, $a < b$, then we have the Hermite-Hadamard type inequality obtained by Sarikaya et al. in [66]

$$(1.6) \quad \frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(u) du \\ = \int_0^1 f((1-\lambda)a + \lambda b) d\lambda \leq [f(a) + f(b)] \int_0^1 h(t) dt.$$

For an extension of this result to functions defined on convex subsets of linear spaces and refinements, see [31].

In order to extend this result for other classes of functions, we need the following preparations.

Let $g : [a, b] \rightarrow [g(a), g(b)]$ be a continuous strictly increasing function that is differentiable on (a, b) .

Definition 6. A function $f : [a, b] \rightarrow \mathbb{R}$ will be called composite- g^{-1} h -convex (concave) on $[a, b]$ if the composite function $f \circ g^{-1} : [g(a), g(b)] \rightarrow \mathbb{R}$ is h -convex (concave) in the usual sense on $[g(a), g(b)]$.

If $f : [a, b] \rightarrow \mathbb{R}$ is composite- g^{-1} h -convex on $[a, b]$ then we have the inequality

$$(1.7) \quad f \circ g^{-1}((1-\lambda)u + \lambda v) \leq h(1-\lambda)f \circ g^{-1}(u) + h(\lambda)f \circ g^{-1}(v)$$

for any $u, v \in [g(a), g(b)]$ and $\lambda \in [0, 1]$.

This is equivalent to the condition

$$(1.8) \quad f \circ g^{-1}((1-\lambda)g(t) + \lambda g(s)) \leq h(1-\lambda)f(t) + h(\lambda)f(s)$$

for any $t, s \in [a, b]$ and $\lambda \in [0, 1]$.

If we take $g(t) = \ln t$, $t \in [a, b] \subset (0, \infty)$, then the condition (1.8) becomes

$$(1.9) \quad f(t^{1-\lambda}s^\lambda) \leq h(1-\lambda)f(t) + h(\lambda)f(s)$$

for any $t, s \in [a, b]$ and $\lambda \in [0, 1]$, which is the concept of *GA-h-convexity* as considered in [1].

If we take $g(t) = -\frac{1}{t}$, $t \in [a, b] \subset (0, \infty)$, then (1.8) becomes

$$(1.10) \quad f\left(\frac{ts}{(1-\lambda)s + \lambda t}\right) \leq h(1-\lambda)f(t) + h(\lambda)f(s)$$

for any $t, s \in [a, b]$ and $\lambda \in [0, 1]$, which is the concept of *HA-h-convexity* as considered in [5].

If $p > 0$ and we consider $g(t) = t^p$, $t \in [a, b] \subset (0, \infty)$, then the condition (1.8) becomes

$$(1.11) \quad f\left[\left((1-\lambda)t^p + \lambda s^p\right)^{1/p}\right] \leq h(1-\lambda)f(t) + h(\lambda)f(s)$$

for any $t, s \in [a, b]$ and $\lambda \in [0, 1]$. For $h(t) = t$ the concept of *p-convexity* was considered in [71].

If we take $g(t) = \exp t$, $t \in [a, b]$, then the condition (1.8) becomes

$$(1.12) \quad f[\ln((1-\lambda)\exp(t) + \exp g(s))] \leq (1-\lambda)f(t) + \lambda f(s),$$

which is the concept of *LogExp h-convex function* on $[a, b]$. For $h(t) = t$, the concept was considered in [28].

Further, assume that $f : [a, b] \rightarrow \mathcal{J}$, $\mathcal{J}$ an interval of real numbers and $k : \mathcal{J} \rightarrow \mathbb{R}$ a continuous function on $\mathcal{J}$ that is *strictly increasing (decreasing)* on $\mathcal{J}$.

Definition 7. We say that the function $f : [a, b] \rightarrow \mathcal{J}$ is *k-composite h-convex (concave)* on $[a, b]$, if $k \circ f$ is *h-convex (concave)* on $[a, b]$.

With $g : [a, b] \rightarrow [g(a), g(b)]$ a *continuous strictly increasing function* that is *differentiable* on (a, b) , $f : [a, b] \rightarrow \mathcal{J}$, $\mathcal{J}$ an interval of real numbers and $k : \mathcal{J} \rightarrow \mathbb{R}$ a continuous function on $\mathcal{J}$ that is *strictly increasing (decreasing)* on $\mathcal{J}$, we can also consider the following concept:

Definition 8. We say that the function $f : [a, b] \rightarrow \mathcal{J}$ is *k-composite- g^{-1} h-convex (concave)* on $[a, b]$, if $k \circ f \circ g^{-1}$ is *h-convex (concave)* on $[g(a), g(b)]$.

This definition is equivalent to the condition

$$(1.13) \quad k \circ f \circ g^{-1}((1-\lambda)g(t) + \lambda g(s)) \leq h(1-\lambda)(k \circ f)(t) + h(\lambda)(k \circ f)(s)$$

for any $t, s \in [a, b]$ and $\lambda \in [0, 1]$.

If $k : \mathcal{J} \rightarrow \mathbb{R}$ is *strictly increasing (decreasing)* on $\mathcal{J}$, then the condition (1.13) is equivalent to:

$$(1.14) \quad f \circ g^{-1}((1-\lambda)g(t) + \lambda g(s)) \leq (\geq) k^{-1}[h(1-\lambda)(k \circ f)(t) + h(\lambda)(k \circ f)(s)]$$

for any $t, s \in [a, b]$ and $\lambda \in [0, 1]$.

If $k(t) = \ln t$, $t > 0$ and $f : [a, b] \rightarrow (0, \infty)$, then the fact that f is k -composite h -convex on $[a, b]$ is equivalent to the fact that f is *log-convex* or *multiplicatively convex* or *AG-h-convex*, namely, for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality:

$$(1.15) \quad f(tx + (1-t)y) \leq [f(x)]^{h(t)} [f(y)]^{h(1-t)}.$$

A function $f : I \rightarrow \mathbb{R} \setminus \{0\}$ is called *AH-h-convex (concave)* on the interval I if the following inequality holds [1]

$$(1.16) \quad f((1-\lambda)x + \lambda y) \leq (\geq) \frac{f(x)f(y)}{h(1-\lambda)f(y) + h(\lambda)f(x)}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$.

An important case that provides many examples is that one in which the function is assumed to be positive for any $x \in I$. In that situation the inequality (1.16) is equivalent to

$$h(1-\lambda) \frac{1}{f(x)} + h(\lambda) \frac{1}{f(y)} \leq (\geq) \frac{1}{f((1-\lambda)x + \lambda y)}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$.

Taking into account this fact, we can conclude that the function $f : I \rightarrow (0, \infty)$ is *AH-h-convex (concave)* on I if and only if f is k -composite h -concave (convex) on I with $k : (0, \infty) \rightarrow (0, \infty)$, $k(t) = \frac{1}{t}$.

Following [1], we can introduce the concept of *GH-h-convex (concave)* function $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ on an interval of positive numbers I as satisfying the condition

$$(1.17) \quad f(x^{1-\lambda}y^\lambda) \leq (\geq) \frac{f(x)f(y)}{h(1-\lambda)f(y) + h(\lambda)f(x)}.$$

Since

$$f(x^{1-\lambda}y^\lambda) = f \circ \exp[(1-\lambda)\ln x + \lambda\ln y]$$

and

$$\frac{f(x)f(y)}{h(1-\lambda)f(y) + h(\lambda)f(x)} = \frac{f \circ \exp(\ln x) f \circ \exp(\ln y)}{h(1-\lambda)f \circ \exp(y) + h(\lambda)f \circ \exp(x)}$$

then $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ is *GH-convex (concave)* on I if and only if $f \circ \exp$ is *AH-convex (concave)* on $\ln I := \{x | x = \ln t, t \in I\}$. This is equivalent to the fact that f is k -composite- g^{-1} h -concave (convex) on I with $k : (0, \infty) \rightarrow (0, \infty)$, $k(t) = \frac{1}{t}$ and $g(t) = \ln t$, $t \in I$.

Following [1], we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow (0, \infty)$ is *HH-h-convex* if

$$(1.18) \quad f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{f(x)f(y)}{h(1-t)f(y) + h(t)f(x)}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.18) is reversed, then f is said to be *HH-h-concave*.

We observe that the inequality (1.18) is equivalent to

$$(1.19) \quad h(1-t) \frac{1}{f(x)} + h(t) \frac{1}{f(y)} \leq \frac{1}{f\left(\frac{xy}{tx + (1-t)y}\right)}$$

for all $x, y \in I$ and $t \in [0, 1]$.

This is equivalent to the fact that f is k -composite- g^{-1} h -concave on $[a, b]$ with $k : (0, \infty) \rightarrow (0, \infty)$, $k(t) = \frac{1}{t}$ and $g(t) = -\frac{1}{t}$, $t \in [a, b]$.

The function $f : I \subset (0, \infty) \rightarrow (0, \infty)$ is called *GG-h-convex* on the interval I of real umbers $\mathbb{R}$ if [5]

$$(1.20) \quad f(x^{1-\lambda}y^\lambda) \leq [f(x)]^{h(1-\lambda)} [f(y)]^{h(\lambda)}$$

for any $x, y \in I$ and $\lambda \in [0, 1]$. If the inequality is reversed in (1.20) then the function is called *GG-h-concave*.

For $h(t) = t$, this concept was introduced in 1928 by P. Montel [59], however, the roots of the research in this area can be traced long before him [60]. It is easy to see that [60], the function $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ is *GG-h-convex* if and only if the the function $g : [\ln a, \ln b] \rightarrow \mathbb{R}$, $g = \ln \circ f \circ \exp$ is *h-convex* on $[\ln a, \ln b]$. This is equivalent to the fact that f is *k-composite- g^{-1} h-convex* on $[a, b]$ with $k : (0, \infty) \rightarrow \mathbb{R}$, $k(t) = \ln t$ and $g(t) = \ln t$, $t \in [a, b]$.

Following [1] we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \rightarrow (0, \infty)$ is *HG-h-convex* if

$$(1.21) \quad f\left(\frac{xy}{tx + (1-t)y}\right) \leq [f(x)]^{h(1-t)} [f(y)]^{h(t)}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.8) is reversed, then f is said to be *HG-h-concave*.

Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ and define the associated functions $G_f : [\frac{1}{b}, \frac{1}{a}] \rightarrow \mathbb{R}$ defined by $G_f(t) = \ln f(\frac{1}{t})$. Then f is *HG-h-convex* on $[a, b]$ iff G_f is *h-convex* on $[\frac{1}{b}, \frac{1}{a}]$. This is equivalent to the fact that f is *k-composite- g^{-1} h-convex* on $[a, b]$ with $k : (0, \infty) \rightarrow \mathbb{R}$, $k(t) = \ln t$ and $g(t) = -\frac{1}{t}$, $t \in [a, b]$.

We say that the function $f : [a, b] \rightarrow (0, \infty)$ is *r-h-convex*, for $r \neq 0$, if

$$(1.22) \quad f((1-\lambda)x + \lambda y) \leq [h(1-\lambda)f^r(y) + h(\lambda)f^r(x)]^{1/r}$$

for any $x, y \in [a, b]$ and $\lambda \in [0, 1]$. For $h(t) = t$, the concept was considered in [61],

If $r > 0$, then the condition (1.22) is equivalent to

$$f^r((1-\lambda)x + \lambda y) \leq h(1-\lambda)f^r(y) + h(\lambda)f^r(x)$$

namely f is *k-composite convex* on $[a, b]$ where $k(t) = t^r$, $t \geq 0$.

If $r < 0$, then the condition (1.22) is equivalent to

$$f^r((1-\lambda)x + \lambda y) \geq h(1-\lambda)f^r(y) + h(\lambda)f^r(x)$$

namely f is *k-composite h-concave* on $[a, b]$ where $k(t) = t^r$, $t > 0$.

In this paper we obtain some inequalities of Hermite-Hadamard type for composite convex functions. Applications for *AG*, *AH-h-convex* functions, *GA*, *GG*, *GH-h-convex* functions and *HA*, *HG*, *HH-h-convex* function are given.

2. REFINEMENTS OF HH-INEQUALITY

The following representation result holds.

Lemma 1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{C}$ where I is an interval of the real numbers $\mathbb{R}$. Let $y, x \in I$ with $y \neq x$ and assume that the mapping $[0, 1] \ni t \mapsto f \circ g^{-1}[(1-t)g(x) + tg(y)]$ is Lebesgue integrable on $[0, 1]$. Then for any $\lambda \in [0, 1]$

we have the representation

$$\begin{aligned}
 (2.1) \quad & \int_0^1 f \circ g^{-1} [(1-t)g(x) + tg(y)] dt \\
 &= (1-\lambda) \int_0^1 f \circ g^{-1} [(1-t)((1-\lambda)g(x) + \lambda g(y)) + tg(y)] dt \\
 &\quad + \lambda \int_0^1 f \circ g^{-1} [(1-t)g(x) + t((1-\lambda)g(x) + \lambda g(y))] dt.
 \end{aligned}$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (2.1) is obvious.

Let $\lambda \in (0, 1)$. Observe that

$$\begin{aligned}
 & \int_0^1 f \circ g^{-1} [(1-t)(\lambda g(y) + (1-\lambda)g(x)) + tg(y)] dt \\
 &= \int_0^1 f \circ g^{-1} [((1-t)\lambda + t)g(y) + (1-t)(1-\lambda)g(x)] dt
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^1 f \circ g^{-1} [t(\lambda g(y) + (1-\lambda)g(x)) + (1-t)g(x)] dt \\
 &= \int_0^1 f \circ g^{-1} [t\lambda g(y) + (1-\lambda t)g(x)] dt.
 \end{aligned}$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1-u = (1-t)(1-\lambda)$ and $du = (1-\lambda)dt$. Then

$$\begin{aligned}
 & \int_0^1 f \circ g^{-1} [((1-t)\lambda + t)g(y) + (1-t)(1-\lambda)g(x)] dt \\
 &= \frac{1}{1-\lambda} \int_{\lambda}^1 f \circ g^{-1} [ug(y) + (1-u)g(x)] du.
 \end{aligned}$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 f \circ g^{-1} [t\lambda g(y) + (1-\lambda t)g(x)] dt = \frac{1}{\lambda} \int_0^{\lambda} f \circ g^{-1} [ug(y) + (1-u)g(x)] du.$$

Therefore

$$\begin{aligned}
 & (1-\lambda) \int_0^1 f \circ g^{-1} [(1-t)(\lambda g(y) + (1-\lambda)g(x)) + tg(y)] dt \\
 &+ \lambda \int_0^1 f \circ g^{-1} [t(\lambda g(y) + (1-\lambda)g(x)) + (1-t)g(x)] dt \\
 &= \int_{\lambda}^1 f \circ g^{-1} [ug(y) + (1-u)g(x)] du + \int_0^{\lambda} f \circ g^{-1} [ug(y) + (1-u)g(x)] du \\
 &= \int_0^1 f \circ g^{-1} [ug(y) + (1-u)g(x)] du
 \end{aligned}$$

and the identity (2.1) is proved. $\square$

Theorem 1. Assume that the function $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ is a composite- g^{-1} h -convex function with $h \in L[0, 1]$. Let $y, x \in I$ with $y \neq x$, then for any $\lambda \in [0, 1]$ we have the inequalities

$$\begin{aligned}
 (2.2) \quad & \frac{1}{2h\left(\frac{1}{2}\right)} \left\{ (1-\lambda) f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \right. \\
 & \quad \left. + \lambda f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right] \right\} \\
 & \leq \frac{1}{g(y) - g(x)} \int_x^y f(t) g'(t) dt \\
 & \leq [f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y)) + (1-\lambda)f(y) + \lambda f(x)] \int_0^1 h(t) dt \\
 & \leq \{[h(1-\lambda) + \lambda]f(x) + [h(\lambda) + 1-\lambda]f(y)\} \int_0^1 h(t) dt.
 \end{aligned}$$

If $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ is a composite- g^{-1} h -concave function, then the inequalities reverse in (2.2).

Proof. Since $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ is a composite- g^{-1} h -convex function, then by Hermite-Hadamard type inequality (1.6) we have

$$\begin{aligned}
 (2.3) \quad & \frac{1}{2h\left(\frac{1}{2}\right)} f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \\
 & \leq \int_0^1 f \circ g^{-1}[(1-t)((1-\lambda)g(x) + \lambda g(y)) + tg(y)] dt \\
 & \leq [f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y)) + f(y)] \int_0^1 h(t) dt
 \end{aligned}$$

and

$$\begin{aligned}
 (2.4) \quad & \frac{1}{2h\left(\frac{1}{2}\right)} f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right] \\
 & \leq \int_0^1 f \circ g^{-1}[(1-t)g(x) + t((1-\lambda)g(x) + \lambda g(y))] dt \\
 & \leq [f(x) + f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y))] \int_0^1 h(t) dt.
 \end{aligned}$$

Now, if we multiply the inequality (2.3) by $1-\lambda \geq 0$ and (2.4) by $\lambda \geq 0$ and add the obtained inequalities, then we get

$$\begin{aligned}
 (2.5) \quad & \frac{1-\lambda}{2h\left(\frac{1}{2}\right)} f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \\
 & + \frac{\lambda}{2h\left(\frac{1}{2}\right)} f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right]
 \end{aligned}$$

$$\begin{aligned}
&\leq (1-\lambda) \int_0^1 f \circ g^{-1} [(1-t)((1-\lambda)g(x) + \lambda g(y)) + tg(y)] dt \\
&+ \lambda \int_0^1 f \circ g^{-1} [(1-t)g(x) + t((1-\lambda)g(x) + \lambda g(y))] dt \\
&\leq (1-\lambda) [f \circ g^{-1} ((1-\lambda)g(x) + \lambda g(y)) + f(y)] \int_0^1 h(t) dt \\
&+ \lambda [f(x) + f \circ g^{-1} ((1-\lambda)g(x) + \lambda g(y))] \int_0^1 h(t) dt
\end{aligned}$$

and by (2.1) we obtain

$$\begin{aligned}
(2.6) \quad &\frac{1}{2h(\frac{1}{2})} \left\{ (1-\lambda) f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \right. \\
&\quad \left. + \lambda f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right] \right\} \\
&\leq \int_0^1 f \circ g^{-1} [(1-t)g(x) + tg(y)] dt \\
&\leq [f \circ g^{-1} ((1-\lambda)g(x) + \lambda g(y)) + (1-\lambda)f(y) + \lambda f(x)] \int_0^1 h(t) dt \\
&\leq \{[h(1-\lambda) + \lambda]f(x) + [h(\lambda) + 1 - \lambda]f(y)\} \int_0^1 h(t) dt,
\end{aligned}$$

where the last inequality follows by the definition of composite- g^{-1} h -convexity and performing the required calculation.

By using the change of variable $u = (1-t)g(x) + tg(y)$, we have $du = (g(y) - g(x))dt$ and then

$$\int_0^1 f \circ g^{-1} [(1-t)g(x) + tg(y)] dt = \frac{1}{g(y) - g(x)} \int_{g(x)}^{g(y)} f \circ g^{-1}(u) du.$$

If we change the variable $t = g^{-1}(u)$, then $u = g(t)$, which gives that $du = g'(t)dt$ and then

$$\int_{g(x)}^{g(y)} f \circ g^{-1}(u) du = \int_x^y f(t) g'(t) dt$$

and the inequality (2.2) is obtained. $\square$

Remark 1. With the assumptions from Theorem 1, we observe that if we take either $\lambda = 0$ or $\lambda = 1$ in the first two inequalities in (2.2), then we get (1.6).

If we take $\lambda = \frac{1}{2}$ and use the h -convexity of $f \circ g^{-1}$, then we get from (2.2) that

$$\begin{aligned}
 (2.7) \quad & \frac{1}{4h^2\left(\frac{1}{2}\right)} f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) \\
 & \leq \frac{1}{4h\left(\frac{1}{2}\right)} \left\{ f \circ g^{-1} \left(\frac{g(x) + 3g(y)}{4} \right) + f \circ g^{-1} \left(\frac{3g(x) + g(y)}{4} \right) \right\} \\
 & \leq \frac{1}{g(y) - g(x)} \int_x^y f(t) g'(t) dt \\
 & \leq \left[f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) + \frac{f(x) + f(y)}{2} \right] \int_0^1 h(t) dt \\
 & \leq \left[h\left(\frac{1}{2}\right) + \frac{1}{2} \right] [f(x) + f(y)] \int_0^1 h(t) dt,
 \end{aligned}$$

where $y, x \in I$ with $y \neq x$.

Remark 2. In general, if $h(\lambda) > 0$ for $\lambda \in (0, 1)$, then for $y, x \in I$ with $y \neq x$

$$\begin{aligned}
 & (1 - \lambda) f \circ g^{-1} \left[\frac{(1 - \lambda) g(x) + (\lambda + 1) g(y)}{2} \right] \\
 & \quad + \lambda f \circ g^{-1} \left[\frac{(2 - \lambda) g(x) + \lambda g(y)}{2} \right] \\
 & = \frac{1 - \lambda}{h(1 - \lambda)} h(1 - \lambda) f \circ g^{-1} \left[\frac{(1 - \lambda) g(x) + (\lambda + 1) g(y)}{2} \right] \\
 & \quad + \frac{\lambda}{h(\lambda)} h(\lambda) f \circ g^{-1} \left[\frac{(2 - \lambda) g(x) + \lambda g(y)}{2} \right] \\
 & \geq \min \left\{ \frac{1 - \lambda}{h(1 - \lambda)}, \frac{\lambda}{h(\lambda)} \right\} \\
 & \quad \times \left\{ h(1 - \lambda) f \circ g^{-1} \left[\frac{(1 - \lambda) g(x) + (\lambda + 1) g(y)}{2} \right] \right. \\
 & \quad \left. + h(\lambda) f \circ g^{-1} \left[\frac{(2 - \lambda) g(x) + \lambda g(y)}{2} \right] \right\} \\
 & \geq \min \left\{ \frac{1 - \lambda}{h(1 - \lambda)}, \frac{\lambda}{h(\lambda)} \right\} \\
 & \quad \times f \circ g^{-1} \left[(1 - \lambda) \frac{(1 - \lambda) g(x) + (\lambda + 1) g(y)}{2} + \lambda \frac{(2 - \lambda) g(x) + \lambda g(y)}{2} \right] \\
 & = \min \left\{ \frac{1 - \lambda}{h(1 - \lambda)}, \frac{\lambda}{h(\lambda)} \right\} f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right)
 \end{aligned}$$

and from (2.2) we get the sequence of inequalities

$$\begin{aligned}
 (2.8) \quad & \frac{1}{2h\left(\frac{1}{2}\right)} \min \left\{ \frac{1 - \lambda}{h(1 - \lambda)}, \frac{\lambda}{h(\lambda)} \right\} f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) \\
 & \leq \frac{1}{2h\left(\frac{1}{2}\right)} \left\{ (1 - \lambda) f \circ g^{-1} \left[\frac{(1 - \lambda) g(x) + (\lambda + 1) g(y)}{2} \right] \right. \\
 & \quad \left. + \lambda f \circ g^{-1} \left[\frac{(2 - \lambda) g(x) + \lambda g(y)}{2} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{g(y) - g(x)} \int_x^y f(t) g'(t) dt \\
&\leq [f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y)) + (1-\lambda)f(y) + \lambda f(x)] \int_0^1 h(t) dt \\
&\leq \{[h(1-\lambda) + \lambda]f(x) + [h(\lambda) + 1-\lambda]f(y)\} \int_0^1 h(t) dt,
\end{aligned}$$

for $y, x \in I$ with $y \neq x$.

In particular, we have

$$\begin{aligned}
(2.9) \quad &\frac{1}{4h^2\left(\frac{1}{2}\right)} f \circ g^{-1}\left(\frac{g(x) + g(y)}{2}\right) \\
&\leq \frac{1}{4h\left(\frac{1}{2}\right)} \left\{ f \circ g^{-1}\left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2}\right] \right. \\
&\quad \left. + f \circ g^{-1}\left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2}\right] \right\} \\
&\leq \frac{1}{g(y) - g(x)} \int_x^y f(t) g'(t) dt \\
&\leq \left[f \circ g^{-1}\left(\frac{g(x) + g(y)}{2}\right) + \frac{f(y) + f(x)}{2} \right] \int_0^1 h(t) dt \\
&\leq \left[h\left(\frac{1}{2}\right) + \frac{1}{2} \right] [f(x) + f(y)] \int_0^1 h(t) dt.
\end{aligned}$$

In a similar way, if $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ is a composite- g^{-1} h -concave function, then

$$\begin{aligned}
(2.10) \quad &\frac{1}{2h\left(\frac{1}{2}\right)} \max\left\{\frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)}\right\} f \circ g^{-1}\left(\frac{g(x) + g(y)}{2}\right) \\
&\geq \frac{1}{2h\left(\frac{1}{2}\right)} \left\{ (1-\lambda) f \circ g^{-1}\left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2}\right] \right. \\
&\quad \left. + \lambda f \circ g^{-1}\left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2}\right] \right\} \\
&\geq \frac{1}{g(y) - g(x)} \int_x^y f(t) g'(t) dt \\
&\geq [f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y)) + (1-\lambda)f(y) + \lambda f(x)] \int_0^1 h(t) dt \\
&\geq \{[h(1-\lambda) + \lambda]f(x) + [h(\lambda) + 1-\lambda]f(y)\} \int_0^1 h(t) dt,
\end{aligned}$$

for $y, x \in I$ with $y \neq x$.

In particular,

$$\begin{aligned}
 (2.11) \quad & \frac{1}{4h^2\left(\frac{1}{2}\right)} f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) \\
 & \geq \frac{1}{4h\left(\frac{1}{2}\right)} \left\{ f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \right. \\
 & \quad \left. + f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right] \right\} \\
 & \geq \frac{1}{g(y) - g(x)} \int_x^y f(t) g'(t) dt \\
 & \geq \left[f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) + \frac{f(y) + f(x)}{2} \right] \int_0^1 h(t) dt \\
 & \geq \left[h\left(\frac{1}{2}\right) + \frac{1}{2} \right] [f(x) + f(y)] \int_0^1 h(t) dt.
 \end{aligned}$$

Corollary 1. Let $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ be a composite- g^{-1} convex function on the interval I in $\mathbb{R}$. Then for any $y, x \in I$ with $y \neq x$ and for any $\lambda \in [0, 1]$ we have the inequalities

$$\begin{aligned}
 (2.12) \quad & f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) \leq (1-\lambda) f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \\
 & \quad + \lambda f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right] \\
 & \leq \frac{1}{g(y) - g(x)} \int_x^y f(t) g'(t) dt \\
 & \leq \frac{1}{2} [f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y)) + (1-\lambda)f(y) + \lambda f(x)] \\
 & \leq \frac{f(y) + f(x)}{2}.
 \end{aligned}$$

We have:

Corollary 2. Let $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ be a composite- g^{-1} Breckner s -convex function on the interval I with $s \in (0, 1]$. Then for any $y, x \in I$ with $y \neq x$ and for any $\lambda \in [0, 1]$ we have the inequalities

$$\begin{aligned}
 (2.13) \quad & \frac{1}{2^{1-s}} \left(\frac{1}{2} - \left| \lambda - \frac{1}{2} \right| \right)^{1-s} f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) \\
 & \leq \frac{1}{2^{1-s}} \left\{ (1-\lambda) f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \right. \\
 & \quad \left. + \lambda f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 (2.14) \quad & \leq \frac{1}{g(y) - g(x)} \int_x^y f(t) g'(t) dt \\
 & \leq \frac{1}{s+1} [f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y)) + (1-\lambda)f(y) + \lambda f(x)] \\
 & \leq \frac{1}{s+1} \{[(1-\lambda)^s + \lambda]f(x) + (\lambda^s + 1 - \lambda)f(y)\}.
 \end{aligned}$$

We also have:

Corollary 3. Let $f : I \subseteq \mathbb{R} \rightarrow [0, \infty)$ be a composite- g^{-1} of s -Godunova-Levin type on the interval I with $s \in (0, 1)$. Then for any $y, x \in I$ with $y \neq x$ and for any $\lambda \in (0, 1)$ we have the inequalities

$$\begin{aligned}
 (2.15) \quad & \frac{1}{2^{1+s}} \left(\frac{1}{2} - \left| \lambda - \frac{1}{2} \right| \right)^{1+s} f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) \\
 & \leq \frac{1}{2^{1+s}} \left\{ (1-\lambda) f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \right. \\
 & \quad \left. + \lambda f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right] \right\} \\
 & \leq \frac{1}{g(y) - g(x)} \int_x^y f(t) g'(t) dt \\
 & \leq \frac{1}{1-s} [f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y)) + (1-\lambda)f(y) + \lambda f(x)] \\
 & \leq \frac{1}{1-s} \left\{ [(1-\lambda)^{-s} + \lambda]f(x) + (\lambda^{-s} + 1 - \lambda)f(y) \right\}.
 \end{aligned}$$

More generally, we have:

Corollary 4. Assume that $g : [a, b] \rightarrow [g(a), g(b)]$ is a continuous strictly increasing function that is differentiable on (a, b) , $f : [a, b] \rightarrow \mathcal{J}$, $\mathcal{J}$ an interval of real numbers and $k : \mathcal{J} \rightarrow \mathbb{R}$ is a continuous function on $\mathcal{J}$ that is strictly increasing. If the function $f : [a, b] \rightarrow \mathcal{J}$ is k -composite- g^{-1} h -convex on $[a, b]$, then

$$\begin{aligned}
 (2.16) \quad & \frac{1}{2h(\frac{1}{2})} \min \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\} k \circ f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) \\
 & \leq \frac{1}{2h(\frac{1}{2})} \left\{ (1-\lambda) k \circ f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \right. \\
 & \quad \left. + \lambda k \circ f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right] \right\} \\
 & \leq \frac{1}{g(y) - g(x)} \int_x^y (k \circ f)(t) g'(t) dt \\
 & \leq [k \circ f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y)) + (1-\lambda)(k \circ f)(y) + \lambda(k \circ f)(x)] \\
 & \quad \times \int_0^1 h(t) dt \\
 & \leq \{[h(1-\lambda) + \lambda](k \circ f)(x) + [h(\lambda) + 1 - \lambda](k \circ f)(y)\} \int_0^1 h(t) dt,
 \end{aligned}$$

for $y, x \in [a, b]$ with $y \neq x$.

If the function $f : [a, b] \rightarrow \mathcal{J}$ is k -composite- g^{-1} h -concave on $[a, b]$, then

$$\begin{aligned}
 (2.17) \quad & \frac{1}{2h\left(\frac{1}{2}\right)} \max \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\} k \circ f \circ g^{-1} \left(\frac{g(x) + g(y)}{2} \right) \\
 & \geq \frac{1}{2h\left(\frac{1}{2}\right)} \left\{ (1-\lambda) k \circ f \circ g^{-1} \left[\frac{(1-\lambda)g(x) + (\lambda+1)g(y)}{2} \right] \right. \\
 & \quad \left. + \lambda k \circ f \circ g^{-1} \left[\frac{(2-\lambda)g(x) + \lambda g(y)}{2} \right] \right\} \\
 & \geq \frac{1}{g(y) - g(x)} \int_x^y (k \circ f)(t) g'(t) dt \\
 & \geq [k \circ f \circ g^{-1}((1-\lambda)g(x) + \lambda g(y)) + (1-\lambda)(k \circ f)(y) + \lambda(k \circ f)(x)] \\
 & \quad \times \int_0^1 h(t) dt \\
 & \geq \{[h(1-\lambda) + \lambda](k \circ f)(x) + [h(\lambda) + 1 - \lambda](k \circ f)(y)\} \int_0^1 h(t) dt,
 \end{aligned}$$

for $y, x \in [a, b]$ with $y \neq x$.

The proof follows by the inequalities (2.8) and (2.10) and we omit the details.

In 1906, Fejér [51], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 2 (Fejér's Inequality). Consider the integral $\int_a^b h(x) w(x) dx$, where h is a convex function in the interval (a, b) and w is a positive function in the same interval such that

$$w(x) = w(a + b - x), \text{ for any } x \in [a, b]$$

i.e., $y = w(x)$ is a symmetric curve with respect to the straight line which contains the point $(\frac{1}{2}(a+b), 0)$ and is normal to the x -axis. Under those conditions the following inequalities are valid:

$$(2.18) \quad h\left(\frac{a+b}{2}\right) \int_a^b w(x) dx \leq \int_a^b h(x) w(x) dx \leq \frac{h(a) + h(b)}{2} \int_a^b w(x) dx.$$

If h is concave on (a, b) , then the inequalities reverse in (2.18).

If $w : [a, b] \rightarrow \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \rightarrow [0, \infty)$, $W(x) := \int_a^x w(s) ds$ is strictly increasing and differentiable on (a, b) and the inverse $W^{-1} : [a, \int_a^b w(s) ds] \rightarrow [a, b]$ exists.

Remark 3. Assume that $w : [a, b] \rightarrow \mathbb{R}$ is continuous and positive on the interval $[a, b]$, $f : [a, b] \rightarrow \mathcal{J}$, $\mathcal{J}$ an interval of real numbers and $k : \mathcal{J} \rightarrow \mathbb{R}$ is a continuous

function on $\mathcal{J}$ that is strictly increasing. If the function $f : [a, b] \rightarrow \mathcal{J}$ is k -composite- W^{-1} h -convex on $[a, b]$, then we have the weighted inequality

$$\begin{aligned}
 (2.19) \quad & \frac{1}{2h\left(\frac{1}{2}\right)} \min \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\} k \circ f \circ W^{-1} \left(\frac{\int_a^x w(s) ds + \int_a^y w(s) ds}{2} \right) \\
 & \leq \frac{1}{2h\left(\frac{1}{2}\right)} \left\{ (1-\lambda) k \circ f \circ W^{-1} \left[\frac{(1-\lambda) \int_a^x w(s) ds + (\lambda+1) \int_a^y w(s) ds}{2} \right] \right. \\
 & \quad \left. + \lambda k \circ f \circ W^{-1} \left[\frac{(2-\lambda) \int_a^x w(s) ds + \lambda \int_a^y w(s) ds}{2} \right] \right\} \\
 & \leq \frac{1}{\int_x^y w(s) ds} \int_x^y (k \circ f)(t) w(t) dt \\
 & \leq \left[k \circ f \circ W^{-1} \left((1-\lambda) \int_a^x w(s) ds + \lambda \int_a^y w(s) ds \right) \right. \\
 & \quad \left. + (1-\lambda) (k \circ f)(y) + \lambda (k \circ f)(x) \right] \int_0^1 h(t) dt \\
 & \leq \{ [h(1-\lambda) + \lambda] (k \circ f)(x) + [h(\lambda) + 1 - \lambda] (k \circ f)(y) \} \int_0^1 h(t) dt,
 \end{aligned}$$

for any $\lambda \in [0, 1]$ and for $y, x \in [a, b]$ with $y \neq x$.

3. APPLICATIONS FOR AG AND AH - h -CONVEX FUNCTIONS

The function $f : [a, b] \rightarrow (0, \infty)$ is AG - h -convex means that f is k -composite h -convex on $[a, b]$ with $k(t) = \ln t$, $t > 0$. By making use of Corollary 4 for $g(t) = t$, we get

$$\begin{aligned}
 (3.1) \quad & \left[f \left(\frac{x+y}{2} \right) \right]^{\frac{1}{2h\left(\frac{1}{2}\right)} \min \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\}} \\
 & \leq \left\{ f^{(1-\lambda)} \left[\frac{(1-\lambda)x + (\lambda+1)y}{2} \right] f^\lambda \left[\frac{(2-\lambda)x + \lambda y}{2} \right] \right\}^{\frac{1}{2h\left(\frac{1}{2}\right)}} \\
 & \leq \exp \left(\frac{1}{y-x} \int_x^y \ln f(t) dt \right) \\
 & \leq \left[f((1-\lambda)x + \lambda y) f^{(1-\lambda)}(y) f^\lambda(x) \right]^{\int_0^1 h(t) dt} \\
 & \leq \left\{ f^{[h(1-\lambda)+\lambda]}(x) f^{[h(\lambda)+1-\lambda]}(y) \right\}^{\int_0^1 h(t) dt},
 \end{aligned}$$

for any $\lambda \in [0, 1]$ and $x, y \in [a, b]$ with $y \neq x$.

The function $f : [a, b] \rightarrow (0, \infty)$ is AH - h -convex on $[a, b]$ means that f is k -composite h -concave on $[a, b]$ with $k : (0, \infty) \rightarrow (0, \infty)$, $k(t) = \frac{1}{t}$. By making use

of Corollary 4 for $g(t) = t$, we get

$$\begin{aligned}
 (3.2) \quad & \frac{1}{2h\left(\frac{1}{2}\right)} \max \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\} f^{-1} \left(\frac{x+y}{2} \right) \\
 & \geq \frac{1}{2h\left(\frac{1}{2}\right)} \left\{ (1-\lambda) f^{-1} \left[\frac{(1-\lambda)x + (\lambda+1)y}{2} \right] \right. \\
 & \quad \left. + \lambda f^{-1} \left[\frac{(2-\lambda)x + \lambda y}{2} \right] \right\} \\
 & \geq \frac{1}{y-x} \int_x^y f^{-1}(t) dt \\
 & \geq [f^{-1}((1-\lambda)x + \lambda y) + (1-\lambda)f^{-1}(y) + \lambda f^{-1}(x)] \int_0^1 h(t) dt \\
 & \geq \{[h(1-\lambda) + \lambda] f^{-1}(x) + [h(\lambda) + 1 - \lambda] f^{-1}(y)\} \int_0^1 h(t) dt,
 \end{aligned}$$

for any $\lambda \in [0, 1]$ and $x, y \in [a, b]$ with $y \neq x$.

4. APPLICATIONS FOR GA , GG AND GH - h -CONVEX FUNCTIONS

If we take $g(t) = \ln t$, $t \in [a, b] \subset (0, \infty)$, then $f : [a, b] \rightarrow \mathbb{R}$ is GA - h -convex on $[a, b]$ means that that $f : [a, b] \rightarrow \mathbb{R}$ composite- g^{-1} h -convex on $[a, b]$. By making use of Corollary 4 for $k(t) = t$, we get

$$\begin{aligned}
 (4.1) \quad & \frac{1}{2h\left(\frac{1}{2}\right)} \min \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\} f(\sqrt{xy}) \\
 & \leq \frac{1}{2h\left(\frac{1}{2}\right)} \left\{ (1-\lambda) f \left(x^{\frac{1-\lambda}{2}} y^{\frac{\lambda+1}{2}} \right) + \lambda f \left(x^{\frac{2-\lambda}{2}} y^{\frac{\lambda}{2}} \right) \right\} \\
 & \leq \frac{1}{\ln\left(\frac{y}{x}\right)} \int_x^y \frac{f(t)}{t} dt \\
 & \leq [f(x^{1-\lambda}y^\lambda) + (1-\lambda)f(y) + \lambda f(x)] \int_0^1 h(t) dt \\
 & \leq \{[h(1-\lambda) + \lambda] f(x) + [h(\lambda) + 1 - \lambda] f(y)\} \int_0^1 h(t) dt,
 \end{aligned}$$

for any $\lambda \in [0, 1]$ and for $y, x \in [a, b]$ with $y \neq x$.

The function $f : I \subset (0, \infty) \rightarrow (0, \infty)$ is GG - h -convex means that f is k -composite- g^{-1} h -convex on $[a, b]$ with $k : (0, \infty) \rightarrow \mathbb{R}$, $k(t) = \ln t$ and $g(t) = \ln t$, $t \in [a, b]$. By making use of Corollary 4 we get

$$\begin{aligned}
 (4.2) \quad & [f(\sqrt{xy})]^{\frac{1}{2h\left(\frac{1}{2}\right)} \min \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\}} \\
 & \leq \left\{ f^{(1-\lambda)} \left(x^{\frac{1-\lambda}{2}} y^{\frac{\lambda+1}{2}} \right) f^\lambda \left(x^{\frac{2-\lambda}{2}} y^{\frac{\lambda}{2}} \right) \right\}^{\frac{1}{2h\left(\frac{1}{2}\right)}}
 \end{aligned}$$

$$\begin{aligned}
&\leq \exp \left(\frac{1}{\ln \left(\frac{y}{x} \right)} \int_x^y \frac{\ln f(t)}{t} dt \right) \\
&\leq [f(x^{1-\lambda}y^\lambda) f^\lambda(x) f^{1-\lambda}(y)] \int_0^1 h(t) dt \\
&\leq \left\{ f^{[h(1-\lambda)+\lambda]}(x) f^{[h(\lambda)+1-\lambda]}(y) \right\} \int_0^1 h(t) dt,
\end{aligned}$$

for any $\lambda \in [0, 1]$ and for $y, x \in [a, b]$ with $y \neq x$.

We also have that $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ is GH - h -convex on $[a, b]$ is equivalent to the fact that f is k -composite- g^{-1} h -concave on $[a, b]$ with $k : (0, \infty) \rightarrow (0, \infty)$, $k(t) = \frac{1}{t}$ and $g(t) = \ln t$, $t \in I$. By making use of Corollary 4 we get

$$\begin{aligned}
(4.3) \quad &\frac{1}{2h\left(\frac{1}{2}\right)} \max \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\} f^{-1}(\sqrt{xy}) \\
&\geq \frac{1}{2h\left(\frac{1}{2}\right)} \left\{ \lambda f^{-1}\left(x^{\frac{2-\lambda}{2}} y^{\frac{\lambda}{2}}\right) + (1-\lambda) f^{-1}\left(x^{\frac{1-\lambda}{2}} y^{\frac{\lambda+1}{2}}\right) \right\} \\
&\geq \frac{1}{\ln\left(\frac{y}{x}\right)} \int_x^y \frac{f^{-1}(t)}{t} dt \\
&\geq [f^{-1}(x^{1-\lambda}y^\lambda) + \lambda f^{-1}(x) + (1-\lambda) f^{-1}(y)] \int_0^1 h(t) dt \\
&\geq \{[h(1-\lambda) + \lambda] f^{-1}(x) + [h(\lambda) + 1 - \lambda] f^{-1}(y)\} \int_0^1 h(t) dt,
\end{aligned}$$

for any $\lambda \in [0, 1]$ and for $y, x \in [a, b]$ with $y \neq x$.

5. APPLICATIONS FOR HA , HG AND HH - h -CONVEX FUNCTIONS

Let $f : [a, b] \subset (0, \infty) \rightarrow \mathbb{R}$ be an HA - h -convex function on the interval $[a, b]$. This is equivalent to the fact that f is composite- g^{-1} h -convex on $[a, b]$ with the increasing function $g(t) = -\frac{1}{t}$. Then by applying Corollary 4 for $k(t) = t$, we have the inequalities

$$\begin{aligned}
(5.1) \quad &\frac{1}{2h\left(\frac{1}{2}\right)} \min \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\} f\left(\frac{2xy}{x+y}\right) \\
&\leq \frac{1}{2h\left(\frac{1}{2}\right)} \left\{ (1-\lambda) f\left(\frac{2xy}{(1-\lambda)x + (\lambda+1)y}\right) + \lambda f\left(\frac{2xy}{(2-\lambda)x + \lambda y}\right) \right\} \\
&\leq \frac{xy}{y-x} \int_x^y \frac{f(t)}{t^2} dt \\
&\leq \frac{1}{2} \left[f\left(\frac{xy}{(1-\lambda)x + \lambda y}\right) + (1-\lambda) f(x) + \lambda f(y) \right] \int_0^1 h(t) dt \\
&\leq \{[h(1-\lambda) + \lambda] f(x) + [h(\lambda) + 1 - \lambda] f(y)\} \int_0^1 h(t) dt
\end{aligned}$$

for any $\lambda \in [0, 1]$ and for $y, x \in [a, b]$ with $y \neq x$.

Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ be an HG - h -convex function on the interval $[a, b]$. This is equivalent to the fact that f is k -composite- g^{-1} h -convex on $[a, b]$

with $k : (0, \infty) \rightarrow \mathbb{R}$, $k(t) = \ln t$ and $g(t) = -\frac{1}{t}$, $t \in [a, b]$. Then by applying Corollary 4, we have the inequalities

$$\begin{aligned}
 (5.2) \quad & \left[f \left(\frac{2xy}{x+y} \right) \right]^{\frac{1}{2h(\frac{1}{2})} \min \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\}} \\
 & \leq \left[f^{1-\lambda} \left(\frac{2xy}{(1-\lambda)x + (\lambda+1)y} \right) f^{\lambda} \left(\frac{2xy}{(2-\lambda)x + \lambda y} \right) \right]^{\frac{1}{2h(\frac{1}{2})}} \\
 & \leq \exp \left(\frac{xy}{y-x} \int_x^y \frac{\ln f(t)}{t^2} dt \right) \\
 & \leq \left[f \left(\frac{xy}{(1-\lambda)x + \lambda y} \right) [f(x)]^{1-\lambda} [f(y)]^{\lambda} \right]^{\int_0^1 h(t) dt} \\
 & \leq \left\{ f^{[h(1-\lambda)+\lambda]}(x) f^{[h(\lambda)+1-\lambda]}(y) \right\}^{\int_0^1 h(t) dt}
 \end{aligned}$$

for any $\lambda \in [0, 1]$ and for $y, x \in [a, b]$ with $y \neq x$.

Let $f : [a, b] \subset (0, \infty) \rightarrow (0, \infty)$ be an HH - h -convex function on the interval $[a, b]$. This is equivalent to the fact that f is k -composite- g^{-1} h -concave on $[a, b]$ with $k : (0, \infty) \rightarrow (0, \infty)$, $k(t) = \frac{1}{t}$ and $g(t) = -\frac{1}{t}$, $t \in [a, b]$. Then by applying Corollary 4, we have the inequalities

$$\begin{aligned}
 (5.3) \quad & \frac{1}{2h(\frac{1}{2})} \max \left\{ \frac{1-\lambda}{h(1-\lambda)}, \frac{\lambda}{h(\lambda)} \right\} f^{-1} \left(\frac{2xy}{x+y} \right) \\
 & \geq \frac{1}{2h(\frac{1}{2})} \left\{ \lambda f^{-1} \left(\frac{2xy}{(2-\lambda)x + \lambda y} \right) + (1-\lambda) f^{-1} \left(\frac{2xy}{(1-\lambda)x + (\lambda+1)y} \right) \right\} \\
 & \geq \frac{xy}{y-x} \int_x^y \frac{f^{-1}(t)}{t^2} dt \\
 & \geq \left[f^{-1} \left(\frac{xy}{(1-\lambda)x + \lambda y} \right) + \lambda f^{-1}(x) + (1-\lambda) f^{-1}(y) \right] \int_0^1 h(t) dt \\
 & \geq \{ [h(1-\lambda) + \lambda] f^{-1}(x) + [h(\lambda) + 1 - \lambda] f^{-1}(y) \} \int_0^1 h(t) dt,
 \end{aligned}$$

for $y, x \in [a, b]$ with $y \neq x$.

Applications for p , r -convex and $LogExp$ convex functions can also be provided. However the details are not presented here.

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