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WEIGHTED INTEGRAL INEQUALITIES OF OSTROWSKI, ČEBYŠEV AND LUPAŞ TYPE WITH APPLICATIONS

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Abstract. In this paper we establish some weighted integral inequalities of Ostrowski, Čebyšev and Lupaş type. Applications for continuous probability density functions supported on infinite intervals with two examples are also given.

1. Introduction

For two *Lebesgue integrable* functions $f,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{1.1}$$

In 1935, Grüss [17] showed that

$$|C(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that there exists the real numbers $m,\ M,\ n,\ N$ such that

$$m\le f(t)\le M \quad\text{and}\quad n\le g(t)\le N \quad\text{for a.e. } t\in[a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882, [4], states that

$$|C(f,g)|\le\frac{1}{12}\|f'\|_\infty\|g'\|_\infty(b-a)^2, \tag{1.4}$$

provided that $f',\ g'$ exist and are continuous on $[a,b]$ and $\|f'\|_\infty=\sup_{t\in[a,b]}|f'(t)|.$ The constant $\frac{1}{12}$ cannot be improved in the general case.

The Čebyšev inequality (1.4) also holds if $f,\ g:[a,b]\to\mathbb{R}$ are assumed to be *absolutely continuous* and $f',\ g'\in L_\infty[a,b]$ while $\|f'\|_\infty=\operatorname{essup}_{t\in[a,b]}|f'(t)|.$

A mixture between Grüss' result (1.2) and Čebyšev's one (1.4) is the following inequality obtained by Ostrowski in 1970, [24]:

$$|C(f,g)|\le\frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{1.5}$$

provided that f is *Lebesgue integrable* and satisfies (1.3) while g is absolutely continuous and $g'\in L_\infty[a,b].$ The constant $\frac{1}{8}$ is best possible in (1.5).

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The case of *euclidean norms* of the derivative was considered by A. Lupaş in [21] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a), \tag{1.6}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Consider now the *weighted Čebyšev functional*

$$\begin{aligned} C_w(f,g) := {} & \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t) g(t)\,dt \\ & - \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) f(t)\,dt \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) g(t)\,dt \end{aligned} \tag{1.7}$$

where $f, g, w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt > 0$.

In [6], Cerone and Dragomir obtained, among others, the following inequalities:

$$\begin{aligned} & |C_w(f,g)| \\ & \leq \frac{1}{2}(M-m) \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right| dt \\ & \leq \frac{1}{2}(M-m) \left[\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right|^p dt \right]^{\frac{1}{p}} \\ & \leq \frac{1}{2}(M-m) \operatorname*{essup}_{t\in[a,b]} \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right| \end{aligned} \tag{1.8}$$

for $p > 1$, provided $-\infty < m \leq f(t) \leq M < \infty$ for a.e. $t \in [a,b]$ and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (1.8) in the sense that it cannot be replaced by a smaller constant.

In addition, if $-\infty < n \leq g(t) \leq N < \infty$ for a.e. $t \in [a,b]$, then the following refinement of the celebrated Grüss inequality is obtained:

$$\begin{aligned} & |C_w(f,g)| \\ & \leq \frac{1}{2}(M-m) \frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right| dt \\ & \leq \frac{1}{2}(M-m) \left[\frac{1}{\int_a^b w(t)\,dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s)\,ds} \int_a^b w(s) g(s)\,ds \right|^2 dt \right]^{\frac{1}{2}} \\ & \leq \frac{1}{4}(M-m)(N-n). \end{aligned} \tag{1.9}$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

For other inequality of Grüss' type see [1]-[5], [7]-[16], [18]-[23] and [25]-[28].

Motivated by the above results, in this paper we establish some weighted integral inequalities of Ostrowski, Čebyšev and Lupaş type. Applications for continuous

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probability density functions supported on infinite intervals with two examples are also given.

2. Weighted Inequalities

We can define, as above

$$C_{h'}(f,g) := \frac{1}{h(b)-h(a)}\int_a^b f(t)\,g(t)\,h'(t)\,dt - \frac{1}{h(b)-h(a)}\int_a^b f(t)\,h'(t)\,dt\,\frac{1}{h(b)-h(a)}\int_a^b g(t)\,h'(t)\,dt, \tag{2.1}$$

where h is absolutely continuous and f, g are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

The following weighted version of Ostrowski's inequality holds:

Theorem 1. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f is Lebesgue integrable and satisfies the condition $m\le f(t)\le M$ for $t\in[a,b]$ and $g:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and $\frac{g'}{h'}$ is essentially bounded, namely $\frac{g'}{h'}\in L_\infty[a,b]$, then we have*

$$|C_{h'}(f,g)| \le \frac{1}{8}[h(b)-h(a)](M-m)\left\|\frac{g'}{h'}\right\|_{[a,b],\infty}. \tag{2.2}$$

The constant $\frac{1}{8}$ is best possible.

Proof. Assume that $[c,d]\subset[a,b]$. If $g:[c,d]\to\mathbb{C}$ is absolutely continuous on $[c,d]$, then $g\circ h^{-1}:[h(c),h(d)]\to\mathbb{C}$ is absolutely continuous on $[h(c),h(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(g\circ h^{-1}\right)'(z) = \left(g'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z) = \frac{\left(g'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)} \tag{2.3}$$

for almost every (a.e.) $z\in[h(c),h(d)]$.

If $x\in[c,d]$, then by taking $z=h(x)$, we get

$$\left(g\circ h^{-1}\right)'(z) = \frac{\left(g'\circ h^{-1}\right)(h(x))}{\left(h'\circ h^{-1}\right)(h(x))} = \frac{g'(x)}{h'(x)}.$$

Therefore, since $\frac{g'}{h'}\in L_\infty[c,d]$, hence $\left(g\circ h^{-1}\right)'\in L_\infty[h(c),h(d)]$. Also

$$\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(c),h(d)],\infty} = \left\|\frac{g'}{h'}\right\|_{[c,d],\infty}.$$

Now, if we use the Ostrowski's inequality (1.5) for the functions $f\circ h^{-1}$ and $g\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\left|\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)} f\circ h^{-1}(u)g\circ h^{-1}(u)du - \frac{1}{[h(b)-h(a)]^2}\int_{h(a)}^{h(b)} f\circ h^{-1}(u)du\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du\right| \le \frac{1}{8}[h(b)-h(a)](M-m)\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty} \tag{2.4}$$

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since $m \le f \circ h^{-1}(u) \le M$ for all $u \in [h(a), h(b)]$.

Observe also that, by the change of variable $t = h^{-1}(u)$, $u \in [g(a), g(b)]$, we have $u = h(t)$ that gives $du = h'(t)\,dt$ and

$$\int_{h(a)}^{h(b)} \left(f \circ h^{-1}\right)(u)\,du = \int_a^b f(t)\,h'(t)\,dt,$$

$$\int_{h(a)}^{h(b)} g \circ h^{-1}(u)du = \int_a^b g(t)\,h'(t)\,dt,$$

$$\int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u)du = \int_a^b f(t)\,g(t)\,h'(t)\,dt$$

and

$$\left\|\left(g \circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty} = \left\|\frac{g'}{h'}\right\|_{[a,b],\infty}.$$

By making use of (2.4) we then get the desired result (2.2).

The best constant follows by Ostrowski's inequality (1.5). □

If $w : [a, b] \to \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \to [0, \infty)$, $W(x) := \int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a, b). We have $W'(x) = w(x)$ for any $x \in (a, b)$.

Corollary 1. *Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$, f is Lebesgue integrable and satisfies the condition $m \le f(t) \le M$ for $t \in [a, b]$ and $g : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ with $\frac{g'}{w}$ is essentially bounded, namely $\frac{g'}{w} \in L_\infty[a, b]$, then we have*

$$|C_w(f, g)| \le \frac{1}{8}(M - m)\left\|\frac{g'}{w}\right\|_{[a,b],\infty} \int_a^b w(s)\,ds. \tag{2.5}$$

The constant $\frac{1}{8}$ is best possible.

Remark 1. *Under the assumptions of Corollary 1 and if there exists a constant $K > 0$ such that $|g'(t)| \le Kw(t)$ for a.e. $t \in [a, b]$, then by (2.5) we get*

$$|C_w(f, g)| \le \frac{1}{8}(M - m)\,K \int_a^b w(s)\,ds. \tag{2.6}$$

We have the following weighted version of Čebyšev inequality:

Theorem 2. *Let $h : [a, b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a, b). If f, $g : [a, b] \to \mathbb{R}$ are absolutely continuous on $[a, b]$ and $\frac{f'}{h'}$, $\frac{g'}{h'} \in L_\infty[a, b]$, then we have*

$$|C_{h'}(f, g)| \le \frac{1}{12}[h(b) - h(a)]^2 \left\|\frac{f'}{h'}\right\|_{[a,b],\infty} \left\|\frac{g'}{h'}\right\|_{[a,b],\infty}. \tag{2.7}$$

The constant $\frac{1}{12}$ is best possible.

The proof follows by the use of Čebyšev inequality (1.4) for the functions $f \circ h^{-1}$ and $g \circ h^{-1}$ on the interval $[h(a), h(b)]$.

Corollary 2. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$. If $f,\ g:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{f'}{w},\ \frac{g'}{w}\in L_\infty[a,b]$, then we have*

$$|C_w(f,g)|\le\frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\left(\int_a^b w(s)\,ds\right)^2. \tag{2.8}$$

The constant $\frac{1}{12}$ is best possible.

Remark 2. *Under the assumptions of Corollary 2 and if there exists the constants $K,\ L>0$ such that $|f'(t)|\le Lw(t)$, $|g'(t)|\le Kw(t)$ for a.e. $t\in[a,b]$, then by (2.8) we get*

$$|C_w(f,g)|\le\frac{1}{12}LK\left(\int_a^b w(s)\,ds\right)^2. \tag{2.9}$$

We also have the following version of Lupaş inequality:

Theorem 3. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If $f,\ g:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{f'}{(h')^{1/2}},\ \frac{g'}{(h')^{1/2}}\in L_2[a,b]$, then we have*

$$|C_{h'}(f,g)|\le\frac{1}{\pi^2}\left\|\frac{f'}{(h')^{1/2}}\right\|_{[a,b],2}\left\|\frac{g'}{(h')^{1/2}}\right\|_{[a,b],2}[h(b)-h(a)]. \tag{2.10}$$

The constant $\frac{1}{\pi^2}$ is best possible.

Proof. Using the identity (2.3) above, we have

$$\int_{h(a)}^{h(b)}\left|\left(g\circ h^{-1}\right)'(u)\right|^2du=\int_{h(a)}^{h(b)}\left|\frac{\left(g'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}\right|^2du.$$

By the change of variable $t=h^{-1}(u)$, $u\in[h(a),h(b)]$, we have $u=h(t)$ that gives $du=h'(t)\,dt$. Therefore

$$\begin{aligned}\int_{h(a)}^{h(b)}\left|\frac{\left(g'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}\right|^2du &= \int_b^b\left|\frac{g'(t)}{h'(t)}\right|^2h'(t)\,dt\\ &= \int_b^b\left|\frac{g'(t)}{[h'(t)]^{1/2}}\right|^2dt=\left\|\frac{g'}{(h')^{1/2}}\right\|_{[a,b],2}^2.\end{aligned}$$

In a similar way, we also have

$$\int_{h(a)}^{h(b)}\left|\frac{\left(f'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}\right|^2du=\left\|\frac{f'}{(h')^{1/2}}\right\|_{[a,b],2}^2.$$

By making use of Lupaş inequality (1.6) for the functions $f\circ h^{-1}$ and $g\circ h^{-1}$ on the interval $[h(a),h(b)]$ we get

$$\begin{aligned}&\left|\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)}f\circ h^{-1}(u)g\circ h^{-1}(u)du\right.\\&\qquad\left.-\frac{1}{[h(b)-h(a)]^2}\int_{h(a)}^{h(b)}f\circ h^{-1}(u)du\int_{h(a)}^{h(b)}g\circ h^{-1}(u)du\right|\\&\qquad\le\frac{1}{\pi^2}\left\|\left(f\circ h^{-1}\right)'\right\|_{[h(a),h(b)],2}\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(a),h(b)],2}[h(b)-h(a)],\end{aligned}$$

which together with the above calculations produces the desired result (2.10). □

Corollary 3. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$. If $f,\ g:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{f'}{w^{1/2}},\ \frac{g'}{w^{1/2}}\in L_2[a,b]$, then we have*

$$|C_w(f,g)|\le\frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,b],2}\int_a^b w(s)\,ds. \tag{2.11}$$

The constant $\frac{1}{\pi^2}$ is best possible.

We can give some examples of interest for several function $h:[a,b]\to[h(a),h(b)]$ that are *continuous strictly increasing functions* and *differentiable* on (a,b).

a). If we take $h:[a,b]\subset(0,\infty)\to\mathbb{R}$, $h(t)=\ln t$, in (2.2), then we get for $\ell(t):=t$, that

$$|C_{\ell^{-1}}(f,g)|\le\frac{1}{8}(M-m)\|\ell g'\|_{[a,b],\infty}\ln\left(\frac{b}{a}\right) \tag{2.12}$$

where

$$C_{\ell^{-1}}(f,g):=\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{f(t)g(t)}{t}dt-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{f(t)}{t}dt\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{g(t)}{t}dt, \tag{2.13}$$

and provided that f is *Lebesgue integrable* and satisfies the condition $m\le f(t)\le M$ for $t\in[a,b]$ and $g:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and $\ell g'\in L_\infty[a,b]$.

If $f,\ g:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $\ell f',\ \ell g'\in L_\infty[a,b]$, then by (2.7) we have

$$|C_{\ell^{-1}}(f,g)|\le\frac{1}{12}\|\ell f'\|_{[a,b],\infty}\|\ell g'\|_{[a,b],\infty}\left[\ln\left(\frac{b}{a}\right)\right]^2. \tag{2.14}$$

Also, if $f,\ g:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $\ell^{1/2}f',\ \ell^{1/2}g'\in L_2[a,b]$, then we have by (2.10)

$$|C_{\ell^{-1}}(f,g)|\le\frac{1}{\pi^2}\left\|\ell^{1/2}f'\right\|_{[a,b],2}\left\|\ell^{1/2}g'\right\|_{[a,b],2}\ln\left(\frac{b}{a}\right). \tag{2.15}$$

b). If we take $h:[a,b]\subset\mathbb{R}\to(0,\infty)$, $h(t)=\exp t$, in (2.2), then we get

$$|C_{\exp}(f,g)|\le\frac{1}{8}(M-m)\left\|\frac{g'}{\exp}\right\|_{[a,b],\infty}(\exp b-\exp a), \tag{2.16}$$

where

$$\begin{aligned} (2.17) \quad C_{\exp}(f,g) &:= \frac{1}{\exp b - \exp a}\int_a^b f(t)\,g(t)\exp t dt \\ &\quad - \frac{1}{\exp b - \exp a}\int_a^b f(t)\exp t dt \frac{1}{\exp b - \exp a}\int_a^b g(t)\exp t dt, \end{aligned}$$

and provided that f is *Lebesgue integrable* and satisfies the condition $m \le f(t) \le M$ for $t \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $\frac{g'}{\exp} \in L_\infty [a,b]$.

If f, $g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{f'}{\exp}$, $\frac{g'}{\exp} \in L_\infty [a,b]$, then by (2.7) we have

$$(2.18) \qquad |C_{\exp}(f,g)| \le \frac{1}{12}\left\|\frac{f'}{\exp}\right\|_{[a,b],\infty}\left\|\frac{g'}{\exp}\right\|_{[a,b],\infty}(\exp b - \exp a)^2.$$

Also, if f, $g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{f'}{\exp^{1/2}}$, $\frac{g'}{\exp^{1/2}} \in L_2 [a,b]$, then we have by (2.10) that

$$(2.19) \qquad |C_{\exp}(f,g)| \le \frac{1}{\pi^2}\left\|\frac{f'}{\exp^{1/2}}\right\|_{[a,b],2}\left\|\frac{g'}{\exp^{1/2}}\right\|_{[a,b],2}(\exp b - \exp a).$$

c). If we take $h : [a,b] \subset (0,\infty) \to \mathbb{R}$, $h(t) = t^r$, $r > 0$ in (2.2), then we get

$$(2.20) \qquad |C_{r\ell^{r-1}}(f,g)| \le \frac{1}{8r}(b^r - a^r)(M - m)\left\|\ell^{1-r}g'\right\|_{[a,b],\infty},$$

where

$$\begin{aligned} (2.21) \quad C_{r\ell^{r-1}}(f,g) &:= \frac{r}{b^r - a^r}\int_a^b f(t)\,g(t)\,t^{r-1}dt \\ &\quad - \frac{r}{b^r - a^r}\int_a^b f(t)\,t^{r-1}dt \frac{r}{b^r - a^r}\int_a^b g(t)\,t^{r-1}dt, \end{aligned}$$

and provided that f is *Lebesgue integrable* and satisfies the condition $m \le f(t) \le M$ for $t \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ and $\ell^{1-r}g' \in L_\infty [a,b]$.

If f, $g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$ and $\ell^{1-r}f'$, $\ell^{1-r}g' \in L_\infty [a,b]$, then by (2.7) we have

$$(2.22) \qquad |C_{r\ell^{r-1}}(f,g)| \le \frac{1}{12}\left\|\ell^{1-r}f'\right\|_{[a,b],\infty}\left\|\ell^{1-r}g'\right\|_{[a,b],\infty}(b^r - a^r)^2.$$

Also, if f, $g : [a,b] \to \mathbb{R}$ are absolutely continuous on $[a,b]$ and $\ell^{\frac{1-r}{2}}f'$, $\ell^{\frac{1-r}{2}}g' \in L_2 [a,b]$, then we have by (2.10) that

$$(2.23) \qquad |C_{r\ell^{r-1}}(f,g)| \le \frac{1}{\pi^2}\left\|\ell^{\frac{1-r}{2}}f'\right\|_{[a,b],2}\left\|\ell^{\frac{1-r}{2}}g'\right\|_{[a,b],2}(b^r - a^r).$$

3. Applications for Probability Density Functions

The above result can be extended for infinite intervals I by assuming that the function $f : I \to \mathbb{C}$ is locally absolutely continuous on I.

For instance, if $I = [a,\infty)$, $w(s) > 0$ for $s \in [a,\infty)$ with $\int_a^\infty w(s)\,ds = 1$, namely w is a probability density function on $[a,\infty)$, f is *Lebesgue measurable* and satisfies the condition $m \leq f(t) \leq M$ for $t \in [a,\infty)$ and $g:[a,\infty) \to \mathbb{R}$ is locally absolutely continuous on $[a,\infty)$ with $\frac{g'}{w} \in L_\infty [a,\infty)$, then by considering the functional

$$C_w(f,g) := \int_a^\infty w(t) f(t) g(t)\, dt - \int_a^\infty w(t) f(t)\, dt \int_a^\infty w(t) g(t)\, dt$$

we have from (2.2) that

$$|C_w(f,g)| \leq \frac{1}{8}(M-m) \left\| \frac{g'}{w} \right\|_{[a,\infty),\infty}. \tag{3.1}$$

Moreover, if $\frac{f'}{w} \in L_\infty [a,\infty)$ then also by (2.7)

$$|C_w(f,g)| \leq \frac{1}{12} \left\| \frac{f'}{w} \right\|_{[a,\infty),\infty} \left\| \frac{g'}{w} \right\|_{[a,\infty),\infty}. \tag{3.2}$$

If $\frac{f'}{w^{1/2}}, \frac{g'}{w^{1/2}} \in L_2[a,\infty)$, then we have by (2.10)

$$|C_w(f,g)| \leq \frac{1}{\pi^2} \left\| \frac{f'}{w^{1/2}} \right\|_{[a,\infty),2} \left\| \frac{g'}{w^{1/2}} \right\|_{[a,\infty),2}. \tag{3.3}$$

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x > 0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

where B is *Beta function*

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}, \ \alpha,\beta > 0.$$

The cumulative distribution function is

$$W_{\alpha,\beta}(x) = I_{\frac{x}{1+x}}(\alpha,\beta),$$

where I is the *regularized incomplete beta function* defined by

$$I_z(\alpha,\beta) := \frac{B(z;\alpha,\beta)}{B(\alpha,\beta)}.$$

Here $B(\cdot;\alpha,\beta)$ is the *incomplete beta function* defined by

$$B(z;\alpha,\beta) := \int_0^z t^{\alpha-1}(1-t)^{\beta-1}, \ \alpha, \ \beta, \ z > 0.$$

Consider the functional

$$C_{B,\alpha,\beta}(f,g) := B(\alpha,\beta) \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t) g(t)\, dt$$
$$- \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\, dt \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} g(t)\, dt$$

where $\alpha, \ \beta > 0$.

Therefore, by (3.1)-(3.3) we have for $\ell(t) = t$, that

$$|C_{B,\alpha,\beta}(f,g)| \leq \frac{1}{8}(M-m) B^3(\alpha,\beta) \left\| g' \ell^{1-\alpha}(1+\ell)^{\alpha+\beta} \right\|_{[0,\infty),\infty}; \tag{3.4}$$

provided $m \leq f(t) \leq M$ for $t \in [0,\infty)$ and $g'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta} \in L_\infty[0,\infty)$,

$$|C_{B,\alpha,\beta}(f,g)| \leq \frac{1}{12}B^4(\alpha,\beta)\left\|f'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta}\right\|_{[0,\infty),\infty}\left\|g'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta}\right\|_{[0,\infty),\infty}; \tag{3.5}$$

provided $f'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta}$, $g'\ell^{1-\alpha}(1+\ell)^{\alpha+\beta} \in L_\infty[0,\infty)$ and

$$|C_{B,\alpha,\beta}(f,g)| \leq \frac{1}{\pi^2}B^3(\alpha,\beta)\left\|f'\ell^{\frac{1-\alpha}{2}}(1+\ell)^{\frac{\alpha+\beta}{2}}\right\|_{[0,\infty),2}\left\|g'\ell^{\frac{1-\alpha}{2}}(1+\ell)^{\frac{\alpha+\beta}{2}}\right\|_{[0,\infty),2}; \tag{3.6}$$

provided $f'\ell^{\frac{1-\alpha}{2}}(1+\ell)^{\frac{\alpha+\beta}{2}}$, $g'\ell^{\frac{1-\alpha}{2}}(1+\ell)^{\frac{\alpha+\beta}{2}} \in L_2[0,\infty)$.

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R} = (-\infty,\infty)$. Namely, if $I = (-\infty,\infty)$, $f : \mathbb{R} \to \mathbb{C}$ is locally absolutely continuous on $\mathbb{R}$ and $w(s) > 0$ for $s \in \mathbb{R}$ with $\int_{-\infty}^{\infty} w(s)\,ds = 1$, namely w is a probability density function on $(-\infty,\infty)$, f is *Lebesgue measurable* and satisfies the condition $m \leq f(t) \leq M$ for $t \in (-\infty,\infty)$ and $g : (-\infty,\infty) \to \mathbb{R}$ is locally absolutely continuous on $(-\infty,\infty)$ with $\frac{g'}{w} \in L_\infty(-\infty,\infty)$, then, by considering the functional

$$C_w(f,g) := \int_{-\infty}^{\infty} w(t)f(t)g(t)\,dt - \int_{-\infty}^{\infty} w(t)f(t)\,dt \int_{-\infty}^{\infty} w(t)g(t)\,dt,$$

we have

$$|C_w(f,g)| \leq \frac{1}{8}(M-m)\left\|\frac{g'}{w}\right\|_{(-\infty,\infty),\infty}. \tag{3.7}$$

Moreover, if $\frac{f'}{w} \in L_\infty(-\infty,\infty)$ then also

$$|C_w(f,g)| \leq \frac{1}{12}\left\|\frac{f'}{w}\right\|_{(-\infty,\infty),\infty}\left\|\frac{g'}{w}\right\|_{(-\infty,\infty),\infty}. \tag{3.8}$$

If $\frac{f'}{w^{1/2}}, \frac{g'}{w^{1/2}} \in L_2(-\infty,\infty)$, then we have

$$|C_w(f,g)| \leq \frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{(-\infty,\infty),2}\left\|\frac{g'}{w^{1/2}}\right\|_{(-\infty,\infty),2}. \tag{3.9}$$

In what follows we give an example.

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \ x \in \mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)dt.$$

Consider the functional

$$C_{N,\sigma,\mu}(f,g) := \sqrt{2\pi}\sigma \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t)\, g(t)\, dt - \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t)\, dt \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) g(t)\, dt$$

with the parameters μ and σ as above.

Therefore, by (3.7)-(3.9) we have

$$|C_{N,\sigma,\mu}(f,g)| \le \frac{1}{8}(M-m)\left(\sqrt{2\pi}\sigma\right)^3 \left\| g' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),\infty}; \tag{3.10}$$

provided $m \le f(t) \le M$ for $t \in (-\infty,\infty)$ and $g' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right) \in L_\infty(-\infty,\infty)$.

Moreover, if $f' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right) \in L_\infty(-\infty,\infty)$ then also

$$|C_{N,\sigma,\mu}(f,g)| \le \frac{1}{12}\left(\sqrt{2\pi}\sigma\right)^4 \left\| f' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),\infty} \left\| g' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),\infty}. \tag{3.11}$$

If $f' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)$, $g' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right) \in L_2(-\infty,\infty)$, then we have

$$|C_{N,\sigma,\mu}(f,g)| \le \frac{1}{\pi^2}\left(\sqrt{2\pi}\sigma\right)^3 \left\| f' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),2} \left\| g' \exp\left(\frac{(\ell-\mu)^2}{2\sigma^2}\right)\right\|_{(-\infty,\infty),2}. \tag{3.12}$$

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