GENERALIZATIONS OF OPIAL'S INEQUALITIES FOR RIEMANN-STIELTJES INTEGRAL WITH APPLICATIONS

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Abstract. In this paper we establish some generalizations of Opial's inequalities for Riemann-Stieltjes integral and for two functions. Applications related to trapezoid and Grüss' type inequalities are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that* $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ *is an absolutely continuous function on the interval* $[a,b]$ *and such that* $u'\in L_2[a,b]$.

(i) *If* $u(a)=u(b)=0$, *then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le\frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b,\end{cases}$$

where c *is an arbitrary constant.*

(ii) *If* $u(a)=0$, *then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ *for some constant* c.

The inequality (1.1) was obtained by Olech in [15] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [16].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [2], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [11]-[14] and [18].

In the recent paper [7] we obtained the following generalization of Opial's inequalities for two functions:

Theorem 2. *Assume that* $f,\ g:[a,b]\to\mathbb{C}$ *are absolutely continuous on* $[a,b]$ *with* $f',\ g'\in L_2[a,b]$.

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(i) *If* $g(a)=0$*, then*

$$\int_a^b |f'(t)g(t)|\,dt \le \left(\int_a^b (t-a)|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\int_a^b \left[(t-a)|f'(t)|^2+(b-t)|g'(t)|^2\right]dt. \tag{1.3}$$

(ii) *If* $g(b)=0$*, then*

$$\int_a^b |f'(t)g(t)|\,dt \le \left(\int_a^b (b-t)|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (t-a)|g'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}\int_a^b \left[(b-t)|f'(t)|^2+(t-a)|g'(t)|^2\right]dt. \tag{1.4}$$

(iii) *If* $g(a)=g(b)=0$*, then*

$$\begin{aligned}\int_a^b |f'(t)g(t)|\,dt &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2} \\ &\quad\times\left(\int_a^b\left|\frac{a+b}{2}-t\right||g'(t)|^2\,dt\right)^{1/2} \\ &\le \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,dt+\frac{1}{2}\int_a^b\left|\frac{a+b}{2}-t\right|\left(|g'(t)|^2-|f'(t)|^2\right)dt.\end{aligned} \tag{1.5}$$

By taking $g=f$ we obtain the following refinement of Opial's inequalities from Theorem 1:

Corollary 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous on* $[a,b]$ *with* $f'\in L_2[a,b]$.

(i) *If either* $f(a)=0$ *or* $f(b)=0$*, then*

$$\int_a^b |f'(t)f(t)|\,dt \le \left(\int_a^b (t-a)|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)|f'(t)|^2\,dt\right)^{1/2} \le \frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt. \tag{1.6}$$

(ii) *If* $f(a)=f(b)=0$*, then*

$$\begin{aligned}\int_a^b |f'(t)f(t)|\,dt &\le \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt-\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,dt\right)^{1/2} \\ &\quad\times\left(\int_a^b\left|\frac{a+b}{2}-t\right||f'(t)|^2\,dt\right)^{1/2} \le \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,dt.\end{aligned} \tag{1.7}$$

Motivated by the above results, in this paper we establish some generalizations of Opial's inequalities for Riemann-Stieltjes integral and for two functions. Applications related to trapezoid and Grüss' type inequalities are also given.

2. The Main Results

We have:

Theorem 3. *Assume that $f : [a,b] \to \mathbb{C}$ is in $C^1[a,b]$, namely is continuous on $[a,b]$, differentiable on (a,b) with the derivative f' continuous on (a,b), g is absolutely continuous on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b |f'(t)\, g(t)|\, du(t)$ exists and*

(i) *If $g(a) = 0$, then*

$$\begin{aligned}\int_a^b |f'(t)\, g(t)|\, du(t) &\le \left(\int_a^b (t-a)\, |f'(t)|^2\, du(t)\right)^{1/2} \left(\int_a^b [u(b)-u(t)]\, |g'(t)|^2\, dt\right)^{1/2} \\ &\le \frac{1}{2}\left\{\int_a^b (t-a)\, |f'(t)|^2\, du(t) + \int_a^b [u(b)-u(t)]\, |g'(t)|^2\, dt\right\}. \end{aligned} \tag{2.1}$$

(ii) *If $g(b) = 0$, then*

$$\begin{aligned}\int_a^b |f'(t)\, g(t)|\, du(t) &\le \left(\int_a^b (b-t)\, |f'(t)|^2\, du(t)\right)^{1/2} \left(\int_a^b [u(t)-u(a)]\, |g'(t)|^2\, dt\right)^{1/2} \\ &\le \frac{1}{2}\left\{\int_a^b (b-t)\, |f'(t)|^2\, du(t) + \int_a^b [u(t)-u(a)]\, |g'(t)|^2\, dt\right\}. \end{aligned} \tag{2.2}$$

Proof. (i) Since $g(a) = 0$, then $g(t) = \int_a^t g'(s)\, ds$ and

$$\begin{aligned}\int_a^b |f'(t)\, g(t)|\, du(t) &= \int_a^b |f'(t)| \left|\int_a^t g'(s)\, ds\right| du(t) \\ &= \int_a^b (t-a)^{1/2}\, |f'(t)|\, \frac{1}{(t-a)^{1/2}} \left|\int_a^t g'(s)\, ds\right| du(t) =: I. \end{aligned}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\frac{1}{(t-a)^{1/2}} \left|\int_a^t g'(s)\, ds\right| \le \left(\int_a^t |g'(s)|^2\, ds\right)^{1/2},\ t \in (a,b]$$

and then

$$I \le \int_a^b (t-a)^{1/2}\, |f'(t)| \left(\int_a^t |g'(s)|^2\, ds\right)^{1/2} du(t). \tag{2.3}$$

We use the following Cauchy-Bunyakovsky-Schwarz integral inequality for the Riemann-Stieltjes integral for monotonic nondecreasing integrators $u:[a,b]\to\mathbb{R}$ and continuous integrands $h,\ z:[a,b]\to\mathbb{C}$

$$\left|\int_a^b h(t)\,z(t)\,du(t)\right|^2 \le \int_a^b |h(t)|^2\,du(t)\int_a^b |z(t)|^2\,du(t). \tag{2.4}$$

Therefore,

$$\begin{aligned}&\int_a^b (t-a)^{1/2}\,|f'(t)|\left(\int_a^t |g'(s)|^2\,ds\right)^{1/2} du(t) \\ &\le \left(\int_a^b (t-a)\,|f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b\left(\int_a^t |g'(s)|^2\,ds\right)du(t)\right)^{1/2}.\end{aligned} \tag{2.5}$$

Using the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\begin{aligned}\int_a^b\left(\int_a^t |g'(s)|^2\,ds\right)du(t) &= \left(\int_a^b |g'(s)|^2\,ds\right)u(b) - \int_a^b u(t)\,|g'(t)|^2\,dt \\ &= \int_a^b [u(b)-u(t)]\,|g'(t)|^2\,dt.\end{aligned}$$

By (2.3) and (2.5) we then get (2.1).

The second part follows by the elementary inequality

$$\sqrt{\alpha\beta} \le \frac{1}{2}(\alpha+\beta)\ \text{ for } \alpha,\ \beta\ge 0. \tag{2.6}$$

(ii) Since $g(a)=0$, then $g(t) = -\int_t^b g'(s)\,ds$ and

$$\begin{aligned}\int_a^b |f'(t)\,g(t)|\,du(t) &= \int_a^b |f'(t)|\left|\int_t^b g'(s)\,ds\right| du(t) \\ &= \int_a^b (b-t)^{1/2}\,|f'(t)|\,\frac{1}{(b-t)^{1/2}}\left|\int_t^b g'(s)\,ds\right| du(t) =: J.\end{aligned}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\frac{1}{(b-t)^{1/2}}\left|\int_t^b g'(s)\,ds\right| \le \left(\int_t^b |g'(s)|^2\,ds\right)^{1/2},\ t\in(a,b]$$

and then

$$J \le \int_a^b (b-t)^{1/2}\,|f'(t)|\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2} du(t). \tag{2.7}$$

By (2.4) we then get

$$\begin{aligned}&\int_a^b (b-t)^{1/2}\,|f'(t)|\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2} du(t) \\ &\le \left(\int_a^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b\left(\int_t^b |g'(s)|^2\,ds\right)du(t)\right).\end{aligned} \tag{2.8}$$

Using the integration by parts formula for the Riemann-Stieltjes integral, we have

$$\int_a^b \left(\int_t^b |g'(s)|^2 ds \right) du(t) = -u(a) \left(\int_a^b |g'(s)|^2 ds \right) + \int_a^b u(t) |g'(t)|^2 dt$$
$$= \int_a^b [u(t) - u(a)] |g'(t)|^2 dt$$

and by (2.7) and (2.8) we get (2.2). □

Remark 1. *Since* u *is monotonic nondecreasing on* $[a,b]$, *then from (2.1) we get*

$$\begin{aligned} &\int_a^b |f'(t) g(t)| du(t) \\ &\leq [u(b) - u(a)]^{1/2} \left(\int_a^b (t-a) |f'(t)|^2 du(t) \right)^{1/2} \left(\int_a^b |g'(t)|^2 dt \right)^{1/2} \\ &\leq \frac{1}{2} [u(b) - u(a)]^{1/2} \left\{ \int_a^b (t-a) |f'(t)|^2 du(t) + \int_a^b |g'(t)|^2 dt \right\}, \end{aligned} \tag{2.9}$$

provided $g(a) = 0$ *and from (2.2) we get*

$$\begin{aligned} &\int_a^b |f'(t) g(t)| du(t) \\ &\leq [u(b) - u(a)]^{1/2} \left(\int_a^b (b-t) |f'(t)|^2 du(t) \right)^{1/2} \left(\int_a^b |g'(t)|^2 dt \right)^{1/2} \\ &\leq \frac{1}{2} [u(b) - u(a)]^{1/2} \left\{ \int_a^b (b-t) |f'(t)|^2 du(t) + \int_a^b |g'(t)|^2 dt \right\}, \end{aligned} \tag{2.10}$$

provided $g(b) = 0$.

Corollary 2. *Assume that* $f : [a,b] \to \mathbb{C}$ *is in* $C^1[a,b]$ *and* u *is monotonic nondecreasing on* $[a,b]$. *Then the Riemann-Stieltjes integral* $\int_a^b |f'(t) f(t)| du(t)$ *exists and*

(i) *If* $f(a) = 0$, *then*

$$\begin{aligned} &\int_a^b |f'(t) f(t)| du(t) \\ &\leq \left(\int_a^b (t-a) |f'(t)|^2 du(t) \right)^{1/2} \left(\int_a^b [u(b) - u(t)] |f'(t)|^2 dt \right)^{1/2} \\ &\leq \frac{1}{2} \left\{ \int_a^b (t-a) |f'(t)|^2 du(t) + \int_a^b [u(b) - u(t)] |f'(t)|^2 dt \right\}. \end{aligned} \tag{2.11}$$

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(ii) *If* $f(b)=0$, *then*

$$\begin{aligned}\int_a^b |f'(t) f(t)|\, du(t) &\le \left(\int_a^b (b-t)|f'(t)|^2 du(t)\right)^{1/2} \left(\int_a^b [u(t)-u(a)]|f'(t)|^2 dt\right)^{1/2} \\ &\le \frac{1}{2}\left\{\int_a^b (b-t)|f'(t)|^2 du(t) + \int_a^b [u(t)-u(a)]|f'(t)|^2 dt\right\}.\end{aligned} \tag{2.12}$$

Remark 2. *If* $w:[a,b]\to[0,\infty)$ *is continuous and we take* $u(t)=\int_a^t w(s)\,ds$ *in (2.11) and (2.12), then we get the weighted integral inequalities*

$$\begin{aligned}\int_a^b |f'(t) f(t)|\, w(t)\,dt &\le \left(\int_a^b (t-a)|f'(t)|^2 w(t)\,dt\right)^{1/2} \left(\int_a^b |f'(t)|^2 \left(\int_t^b w(s)\,ds\right) dt\right)^{1/2} \\ &\le \frac{1}{2}\left\{\int_a^b (t-a)|f'(t)|^2 w(t)\,dt + \int_a^b |f'(t)|^2 \left(\int_t^b w(s)\,ds\right) dt\right\}\end{aligned} \tag{2.13}$$

provided $f(a)=0$, *and*

$$\begin{aligned}\int_a^b |f'(t) f(t)|\, w(t)\,dt &\le \left(\int_a^b (b-t)|f'(t)|^2 w(t)\,dt\right)^{1/2} \left(\int_a^b |f'(t)|^2 \left(\int_a^t w(s)\,ds\right) dt\right)^{1/2} \\ &\le \frac{1}{2}\left\{\int_a^b (b-t)|f'(t)|^2 w(t)\,dt + \int_a^b |f'(t)|^2 \left(\int_a^t w(s)\,ds\right) dt\right\}\end{aligned} \tag{2.14}$$

provided $f(b)=0$.

If w *is nonincreasing on* $[a,b]$ *and* $f(a)=0$, *then by (2.13) we get*

$$\begin{aligned}\int_a^b |f'(t) f(t)|\, w(t)\,dt &\le \left(\int_a^b (t-a)|f'(t)|^2 w(t)\,dt\right)^{1/2} \left(\int_a^b (b-t)|f'(t)|^2 w(t)\,dt\right)^{1/2} \\ &\le \frac{1}{2}(b-a)\int_a^b |f'(t)|^2 w(t)\,dt,\end{aligned} \tag{2.15}$$

while if w is nondecreasing on $[a,b]$ and $f(b)=0$, then by (2.14) we get

$$\begin{aligned}\int_a^b |f'(t) f(t)| w(t)\,dt &\leq \left(\int_a^b (b-t)|f'(t)|^2 w(t)\,dt\right)^{1/2} \left(\int_a^b (t-a)|f'(t)|^2 w(t)\,dt\right)^{1/2} \\ &\leq \frac{1}{2}(b-a)\int_a^b |f'(t)|^2 w(t)\,dt.\end{aligned} \tag{2.16}$$

Furthermore, if $w \equiv 1$, then by (2.13) and (2.14) we get the inequalities from the first part of Theorem 2.

Corollary 3. *Assume that $f:[a,b]\to\mathbb{C}$ is in $C^1[a,b]$, u is monotonic nondecreasing on $[a,b]$ and Lipschitzian with the constant $L>0$, namely $|u(s)-u(t)|\leq L|t-s|$ for any t, $s\in[a,b]$ and g is absolutely continuous on $[a,b]$.*

(i) *If $g(a)=0$, then*

$$\begin{aligned}\int_a^b |f'(t) g(t)|\,du(t) &\leq L\left(\int_a^b (t-a)|f'(t)|^2 dt\right)^{1/2}\left(\int_a^b (b-t)|g'(t)|^2 dt\right)^{1/2} \\ &\leq \frac{1}{2}L\int_a^b \left[(t-a)|f'(t)|^2 + (b-t)|g'(t)|^2\right] dt.\end{aligned} \tag{2.17}$$

(ii) *If $g(b)=0$, then*

$$\begin{aligned}\int_a^b |f'(t) g(t)|\,du(t) &\leq L\left(\int_a^b (b-t)|f'(t)|^2 dt\right)^{1/2}\left(\int_a^b (t-a)|g'(t)|^2 dt\right)^{1/2} \\ &\leq \frac{1}{2}L\int_a^b \left[(b-t)|f'(t)|^2 + (t-a)|g'(t)|^2\right] dt.\end{aligned} \tag{2.18}$$

Proof. Using Riemann-Stieltjes integral inequality properties, we have

$$\int_a^b (t-a)|f'(t)|^2 du(t) \leq L\int_a^b (t-a)|f'(t)|^2 dt.$$

Also

$$\int_a^b [u(b)-u(t)]|g'(t)|^2 dt \leq L\int_a^b (b-t)|g'(t)|^2 dt.$$

By using the inequality (2.1) we obtain (2.17).

The inequality (2.18) follows in a similar way from (2.2). □

Remark 3. *Assume that $f:[a,b]\to\mathbb{C}$ is in $C^1[a,b]$ and g is absolutely continuous on $[a,b]$. If $w:[a,b]\to[0,\infty)$ is continuous, then the function $u(t)=\int_a^t w(s)\,ds$*

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is Lipschitzian with the constant $\|w\|_{[a,b],\infty} = \max_{t\in[a,b]} w(t)$ *and by (2.17) we get*

$$\int_a^b |f'(t) g(t)| w(t) dt \tag{2.19}$$
$$\leq \|w\|_{[a,b],\infty} \left(\int_a^b (t-a) |f'(t)|^2 dt \right)^{1/2} \left(\int_a^b (b-t) |g'(t)|^2 dt \right)^{1/2}$$
$$\leq \frac{1}{2} \|w\|_{[a,b],\infty} \int_a^b \left[(t-a) |f'(t)|^2 + (b-t) |g'(t)|^2 \right] dt,$$

provided $g(a) = 0$, *while from (2.18) we get*

$$\int_a^b |f'(t) g(t)| w(t) dt \tag{2.20}$$
$$\leq \|w\|_{[a,b],\infty} \left(\int_a^b (b-t) |f'(t)|^2 dt \right)^{1/2} \left(\int_a^b (t-a) |g'(t)|^2 dt \right)^{1/2}$$
$$\leq \frac{1}{2} \|w\|_{[a,b],\infty} \int_a^b \left[(b-t) |f'(t)|^2 + (t-a) |g'(t)|^2 \right] dt,$$

provided $g(b) = 0$.

In particular, if either $f(a) = 0$ *or* $f(b) = 0$, *then we have*

$$\int_a^b |f'(t) f(t)| w(t) dt \tag{2.21}$$
$$\leq \|w\|_{[a,b],\infty} \left(\int_a^b (t-a) |f'(t)|^2 dt \right)^{1/2} \left(\int_a^b (b-t) |f'(t)|^2 dt \right)^{1/2}$$
$$\leq \frac{1}{2} (b-a) \|w\|_{[a,b],\infty} \int_a^b |f'(t)|^2 dt.$$

3. Related Results

We have the following inequalities as well:

Theorem 4. *Assume that* $f : [a,b] \to \mathbb{C}$ *is in* $C^1[a,b]$, g *is absolutely continuous on* $[a,b]$ *with* $g(a) = g(b) = 0$ *and* u *is monotonic nondecreasing on* $[a,b]$. *Then*

$$\int_a^b |f'(t) g(t)| du(t) \tag{3.1}$$
$$\leq \frac{1}{2} (b-a)^{1/2} [u(b) - u(a)]^{1/2} \left(\int_a^b |f'(t)|^2 du(t) \right)^{1/2} \left(\int_a^b |g'(t)|^2 dt \right)^{1/2}$$
$$\leq \frac{1}{4} (b-a)^{1/2} [u(b) - u(a)]^{1/2} \left[\int_a^b |f'(t)|^2 du(t) + \int_a^b |g'(t)|^2 dt \right]$$

and

$$\begin{aligned}(3.2)\quad & \int_a^b |f'(t)\,g(t)|\,du(t) \\ & \leq \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,du(t) - \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\,du(t)\right)^{1/2} \\ & \times \left(\int_a^b \left|u(t)-u\left(\frac{a+b}{2}\right)\right| |g'(t)|^2\,dt\right)^{1/2} \\ & \leq \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,du(t) - \frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\,du(t) \\ & + \frac{1}{2}\int_a^b \left|u(t)-u\left(\frac{a+b}{2}\right)\right| |g'(t)|^2\,dt.\end{aligned}$$

Proof. If $g(a) = g(b) = 0$, then by adding the first parts of the inequalities (2.1) and (2.2), we get

$$\begin{aligned}(3.3)\quad & 2\int_a^b |f'(t)\,g(t)|\,du(t) \\ & \leq \left(\int_a^b (t-a)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_a^b [u(b)-u(t)]\,|g'(t)|^2\,dt\right)^{1/2} \\ & + \left(\int_a^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_a^b [u(t)-u(a)]\,|g'(t)|^2\,dt\right)^{1/2}.\end{aligned}$$

If we use the elementary (CBS) inequality

$$(3.4)\quad \alpha\beta+\gamma\delta \leq \left(\alpha^2+\gamma^2\right)^{1/2}\left(\beta^2+\delta^2\right)^{1/2},\ \alpha,\ \beta,\ \gamma,\ \delta \geq 0,$$

then we get

$$\begin{aligned}(3.5)\quad & \left(\int_a^b (t-a)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_a^b [u(b)-u(t)]\,|g'(t)|^2\,dt\right)^{1/2} \\ & + \left(\int_a^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_a^b [u(t)-u(a)]\,|g'(t)|^2\,dt\right)^{1/2} \\ & \leq \left(\int_a^b (t-a)\,|f'(t)|^2\,du(t) + \int_a^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2} \\ & \times \left(\int_a^b [u(b)-u(t)]\,|g'(t)|^2\,dt + \int_a^b [u(t)-u(a)]\,|g'(t)|^2\,dt\right)^{1/2} \\ & = (b-a)^{1/2}\,[u(b)-u(a)]^{1/2} \left(\int_a^b |f'(t)|^2\,du(t)\right)^{1/2} \left(\int_a^b |g'(t)|^2\,dt\right)^{1/2}.\end{aligned}$$

By making use of (3.3) and (3.5) we get the first part of (3.1). The second part follows by the inequality (2.6).

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Further on, if we write the first part of the inequality (2.1) on the interval $\left[a,\frac{a+b}{2}\right]$, we get

$$\int_a^{\frac{a+b}{2}} |f'(t)\,g(t)|\,du(t) \tag{3.6}$$
$$\le \left(\int_a^{\frac{a+b}{2}} (t-a)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right)-u(t)\right] |g'(t)|^2\,dt\right)^{1/2},$$

while from the first part of (2.2) written on $\left[\frac{a+b}{2},b\right]$ we have

$$\int_{\frac{a+b}{2}}^b |f'(t)\,g(t)|\,du(t) \tag{3.7}$$
$$\le \left(\int_{\frac{a+b}{2}}^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left[u(t)-u\left(\frac{a+b}{2}\right)\right] |g'(t)|^2\,dt\right)^{1/2}.$$

If we add (3.6) with (3.7), then we get

$$\int_a^b |f'(t)\,g(t)|\,du(t) \tag{3.8}$$
$$\le \left(\int_a^{\frac{a+b}{2}} (t-a)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right)-u(t)\right] |g'(t)|^2\,dt\right)^{1/2}$$
$$+\left(\int_{\frac{a+b}{2}}^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left[u(t)-u\left(\frac{a+b}{2}\right)\right] |g'(t)|^2\,dt\right)^{1/2}.$$

Moreover, if we use the elementary (CBS) inequality (3.4), then we have

(3.9)
$$\left(\int_a^{\frac{a+b}{2}} (t-a)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right)-u(t)\right] |g'(t)|^2\,dt\right)^{1/2}$$
$$+\left(\int_{\frac{a+b}{2}}^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left[u(t)-u\left(\frac{a+b}{2}\right)\right] |g'(t)|^2\,dt\right)^{1/2}$$
$$\le \left(\int_a^{\frac{a+b}{2}} (t-a)\,|f'(t)|^2\,du(t)+\int_{\frac{a+b}{2}}^b (b-t)\,|f'(t)|^2\,du(t)\right)^{1/2}$$
$$\times\left(\int_a^{\frac{a+b}{2}} \left[u\left(\frac{a+b}{2}\right)-u(t)\right] |g'(t)|^2\,dt+\int_{\frac{a+b}{2}}^b \left[u(t)-u\left(\frac{a+b}{2}\right)\right] |g'(t)|^2\,dt\right)^{1/2}$$
$$=\left(\int_a^b K(t)\,|f'(t)|^2\,du(t)\right)^{1/2} \left(\int_a^b \left|u(t)-u\left(\frac{a+b}{2}\right)\right| |g'(t)|^2\,dt\right)^{1/2},$$

where

$$K(t):=\begin{cases} t-a \text{ if } t\in\left[a,\frac{a+b}{2}\right], \\ b-t \text{ if } t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Since

$$K(t)=\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|,\ t\in[a,b],$$

hence by (3.8) and (3.9) we get the first inequality in (3.2). The second part follows by (2.6). □

Remark 4. *Assume that $f:[a,b]\to\mathbb{C}$ is in $C^1[a,b]$ with $f(a)=f(b)=0$ and u is monotonic nondecreasing on $[a,b]$. Then*

$$\begin{aligned}(3.10)\quad &\int_a^b |f'(t)f(t)|\,du(t)\\ &\leq \frac{1}{2}(b-a)^{1/2}[u(b)-u(a)]^{1/2}\left(\int_a^b |f'(t)|^2\,du(t)\right)^{1/2}\left(\int_a^b |f'(t)|^2\,dt\right)^{1/2}\\ &\leq \frac{1}{4}(b-a)^{1/2}[u(b)-u(a)]^{1/2}\left[\int_a^b |f'(t)|^2\,du(t)+\int_a^b |f'(t)|^2\,dt\right]\end{aligned}$$

and

$$\begin{aligned}(3.11)\quad &\int_a^b |f'(t)f(t)|\,du(t)\\ &\leq \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,du(t)-\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,du(t)\right)^{1/2}\\ &\quad\times\left(\int_a^b\left|u(t)-u\left(\frac{a+b}{2}\right)\right||f'(t)|^2\,dt\right)^{1/2}\\ &\leq \frac{1}{4}(b-a)\int_a^b |f'(t)|^2\,du(t)-\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right||f'(t)|^2\,du(t)\\ &\quad+\frac{1}{2}\int_a^b\left|u(t)-u\left(\frac{a+b}{2}\right)\right||f'(t)|^2\,dt.\end{aligned}$$

Corollary 4. *Assume that $f:[a,b]\to\mathbb{C}$ is in $C^1[a,b]$, g is absolutely continuous on $[a,b]$ with $g(a)=g(b)=0$ and u is monotonic nondecreasing and L-Lipschitzian on $[a,b]$. Then*

$$\begin{aligned}(3.12)\quad &\int_a^b |f'(t)g(t)|\,du(t)\\ &\leq \frac{1}{2}(b-a)L\left(\int_a^b |f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2}\\ &\leq \frac{1}{4}(b-a)L\int_a^b\left(|f'(t)|^2+|g'(t)|^2\right)dt\end{aligned}$$

and

$$\int_a^b |f'(t)\, g(t)|\, du(t) \tag{3.13}$$
$$\leq L\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\, dt - \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\, dt\right)^{1/2}$$
$$\times\left(\int_a^b \left|t-\frac{a+b}{2}\right| |g'(t)|^2\, dt\right)^{1/2}$$
$$\leq \frac{1}{2}L\left[\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\, dt - \int_a^b \left|t-\frac{a+b}{2}\right| \left[|f'(t)|^2 - |g'(t)|^2\right] dt\right].$$

In particular, if $f(a)=f(b)=0$, *then*

$$\int_a^b |f'(t)\, f(t)|\, du(t) \leq \frac{1}{2}(b-a)\, L\int_a^b |f'(t)|^2\, dt \tag{3.14}$$

and

$$\int_a^b |f'(t)\, f(t)|\, du(t) \tag{3.15}$$
$$\leq L\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\, dt - \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\, dt\right)^{1/2}$$
$$\times\left(\int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\, dt\right)^{1/2}$$
$$\leq \frac{1}{4}(b-a)\, L\int_a^b |f'(t)|^2\, dt.$$

Remark 5. *Assume that* $f:[a,b]\to\mathbb{C}$ *is in* $C^1[a,b]$ *and* g *is absolutely continuous on* $[a,b]$ *with* $g(a)=g(b)=0$. *If* $w:[a,b]\to[0,\infty)$ *is continuous and we take* $u(t)=\int_a^t w(s)\, ds$ *in (3.1) and (3.2), then we get*

$$\int_a^b |f'(t)\, g(t)|\, w(t)\, dt \tag{3.16}$$
$$\leq \frac{1}{2}(b-a)^{1/2}\left(\int_a^b w(s)\, ds\right)^{1/2}\left(\int_a^b |f'(t)|^2 w(t)\, dt\right)^{1/2}\left(\int_a^b |g'(t)|^2\, dt\right)^{1/2}$$
$$\leq \frac{1}{4}(b-a)^{1/2}\left(\int_a^b w(s)\, ds\right)^{1/2}\left[\int_a^b |f'(t)|^2 w(t)\, dt + \int_a^b |g'(t)|^2\, dt\right]$$

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and

$$
\begin{aligned}
(3.17)\quad & \int_a^b |f'(t)\, g(t)|\, w(t)\, dt \\
& \leq \left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2 w(t)\, dt - \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2 w(t)\, dt\right)^{1/2} \\
& \qquad \times \left(\int_a^b \left|\int_{\frac{a+b}{2}}^t w(s)\, ds\right| |g'(t)|^2\, dt\right)^{1/2} \\
& \leq \frac{1}{4}(b-a)\int_a^b |f'(t)|^2 w(t)\, dt - \frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2 w(t)\, dt \\
& \qquad + \frac{1}{2}\int_a^b \left|\int_{\frac{a+b}{2}}^t w(s)\, ds\right| |g'(t)|^2\, dt.
\end{aligned}
$$

We also have, by taking $L = \|w\|_{[a,b],\infty}$, *in (3.12) and (3.13), that*

$$
\begin{aligned}
(3.18)\quad & \int_a^b |f'(t)\, g(t)|\, w(t)\, dt \\
& \leq \frac{1}{2}(b-a)\|w\|_{[a,b],\infty}\left(\int_a^b |f'(t)|^2\, dt\right)^{1/2}\left(\int_a^b |g'(t)|^2\, dt\right)^{1/2} \\
& \leq \frac{1}{4}(b-a)\|w\|_{[a,b],\infty}\int_a^b \left(|f'(t)|^2 + |g'(t)|^2\right) dt
\end{aligned}
$$

and

$$
\begin{aligned}
(3.19)\quad & \int_a^b |f'(t)\, g(t)|\, w(t)\, dt \\
& \leq \|w\|_{[a,b],\infty}\left(\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\, dt - \int_a^b \left|t-\frac{a+b}{2}\right| |f'(t)|^2\, dt\right)^{1/2} \\
& \qquad \times \left(\int_a^b \left|t-\frac{a+b}{2}\right| |g'(t)|^2\, dt\right)^{1/2} \\
& \leq \frac{1}{2}\|w\|_{[a,b],\infty}\left[\frac{1}{2}(b-a)\int_a^b |f'(t)|^2\, dt - \int_a^b \left|t-\frac{a+b}{2}\right| \left[|f'(t)|^2 - |g'(t)|^2\right] dt\right].
\end{aligned}
$$

By taking $w \equiv 1$ *in (3.19), we get the inequality (1.5) from Introduction.*

4. Some Inequalities for Functions of Bounded Variation

The following lemma was obtained by the author in 2007, [6] and is of interest in itself as well (see also [5]):

Lemma 1. *If* $p:[a,b]\to\mathbb{C}$ *is continuous on* $[a,b]$ *and* $v:[a,b]\to\mathbb{C}$ *is of bounded variation on* $[a,b]$, *then*

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dV(t) \le \left(\int_a^b |p(t)|^p\,dV(t)\right)^{1/p}\left(\bigvee_a^b(v)\right)^{1/q} \le \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{4.1}$$

where $V(t):=\bigvee_a^t(v)$ *is the total variation of* v *on* $[a,t]$ *with* $t\in[a,b]$.

The function V is nondecreasing on $[a,b]$ with $V(a)=0$ and $V(b)=\bigvee_a^b(v)$. If we put $\overline{V}(t):=\bigvee_t^b(v)=\bigvee_a^b(v)-V(t)$, then $\overline{V}$is nonincreasing with $\overline{V}(a)=\bigvee_a^b(v)$ and $\overline{V}(b)=0$.

We have:

Proposition 1. *Assume that* $h:[a,b]\to\mathbb{C}$ *is continuous,* g *is absolutely continuous on* $[a,b]$ *and* v *is of bounded variation on* $[a,b]$, *then the Riemann-Stieltjes integral* $\int_a^b h(t)g(t)\,dv(t)$ *exists and:*

(i) *If* $g(a)=0$, *then*

$$\begin{aligned}\left|\int_a^b h(t)g(t)\,dv(t)\right| &\le \left(\int_a^b (t-a)|h(t)|^2\,dV(t)\right)^{1/2}\left(\int_a^b \overline{V}(t)|g'(t)|^2\,dt\right)^{1/2}\\ &\le \frac{1}{2}\left\{\int_a^b (t-a)|h(t)|^2\,dV(t)+\int_a^b \overline{V}(t)|g'(t)|^2\,dt\right\}.\end{aligned} \tag{4.2}$$

(ii) *If* $g(b)=0$, *then*

$$\begin{aligned}\left|\int_a^b h(t)g(t)\,dv(t)\right| &\le \left(\int_a^b (b-t)|h(t)|^2\,dV(t)\right)^{1/2}\left(\int_a^b V(t)|g'(t)|^2\,dt\right)^{1/2}\\ &\le \frac{1}{2}\left\{\int_a^b (b-t)|h(t)|^2\,dV(t)+\int_a^b V(t)|g'(t)|^2\,dt\right\}.\end{aligned} \tag{4.3}$$

Proof. Using the first inequality in (4.1), we get

$$\left|\int_a^b h(t)g(t)\,dv(t)\right| \le \int_a^b |h(t)g(t)|\,dV(t). \tag{4.4}$$

Using now Theorem 3 for $f=\int_a h$ and $u=V$, we get, for $g(a)=0$, that

$$\begin{aligned}(4.5)\qquad & \int_a^b |h(t)\,g(t)|\,dV(t) \\ & \leq \left(\int_a^b (t-a)\,|h(t)|^2\,dV(t)\right)^{1/2}\left(\int_a^b [V(b)-V(t)]\,|g'(t)|^2\,dt\right)^{1/2} \\ & \leq \frac{1}{2}\left\{\int_a^b (t-a)\,|f'(t)|^2\,dV(t)+\int_a^b [V(b)-V(t)]\,|g'(t)|^2\,dt\right\}.\end{aligned}$$

If $g(b)=0$, then

$$\begin{aligned}(4.6)\qquad & \int_a^b |h(t)\,g(t)|\,dV(t) \\ & \leq \left(\int_a^b (b-t)\,|f'(t)|^2\,dV(t)\right)^{1/2}\left(\int_a^b V(t)\,|g'(t)|^2\,dt\right)^{1/2} \\ & \leq \frac{1}{2}\left\{\int_a^b (b-t)\,|f'(t)|^2\,dV(t)+\int_a^b V(t)\,|g'(t)|^2\,dt\right\}.\end{aligned}$$

By utilising (4.4)-(4.6) we get the desired results (4.2) and (4.3). □

The case $h\equiv 1$ is of interest since in this case

$$\begin{aligned}\int_a^b (t-a)\,dV(t) &= (b-a)\,V(b)-\int_a^b V(t)\,dt \\ &= \int_a^b (V(b)-V(t))\,dt=\int_a^b \overline{V}(t)\,dt\end{aligned}$$

and

$$\int_a^b (b-t)\,dV(t)=\int_a^b V(t)\,dt.$$

We then can state:

Corollary 5. *Assume that g is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then the Riemann-Stieltjes integral $\int_a^b g(t)\,dv(t)$ exists and:*

(i) *If $g(a)=0$, then*

$$\begin{aligned}(4.7)\qquad \left|\int_a^b g(t)\,dv(t)\right| &\leq \left(\int_a^b \overline{V}(t)\,dt\right)^{1/2}\left(\int_a^b \overline{V}(t)\,|g'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{1}{2}\int_a^b \overline{V}(t)\left(1+|g'(t)|^2\right)dt.\end{aligned}$$

(ii) *If $g(b)=0$, then*

$$\begin{aligned}(4.8)\qquad \left|\int_a^b g(t)\,dv(t)\right| &\leq \left(\int_a^b V(t)\,dt\right)^{1/2}\left(\int_a^b V(t)\,|g'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{1}{2}\int_a^b V(t)\left(1+|g'(t)|^2\right)dt.\end{aligned}$$

We also have:

Proposition 2. *Assume that* $h:[a,b]\to\mathbb{C}$ *is continuous,* g *is absolutely continuous on* $[a,b]$ *with* $g(a)=g(b)=0$ *and* v *is of bounded variation on* $[a,b]$, *then*

$$\begin{aligned}(4.9)\quad &\left|\int_a^b h(t)\,g(t)\,dv(t)\right| \\ &\leq \frac{1}{2}(b-a)^{1/2}\left[\bigvee_a^b(v)\right]^{1/2}\left(\int_a^b |h(t)|^2\,dV(t)\right)^{1/2}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)^{1/2}\left[\bigvee_a^b(v)\right]^{1/2}\left[\int_a^b |h(t)|^2\,dV(t)+\int_a^b |g'(t)|^2\,dt\right]\end{aligned}$$

and

$$\begin{aligned}(4.10)\quad &\left|\int_a^b h(t)\,g(t)\,dv(t)\right| \\ &\leq \left(\frac{1}{2}(b-a)\int_a^b |h(t)|^2\,dV(t)-\int_a^b\left|t-\frac{a+b}{2}\right||h(t)|^2\,dV(t)\right)^{1/2} \\ &\qquad\times\left(\int_a^b\left|V(t)-V\left(\frac{a+b}{2}\right)\right||g'(t)|^2\,dt\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)\int_a^b |h(t)|^2\,dV(t)-\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right||h(t)|^2\,dV(t) \\ &\qquad+\frac{1}{2}\int_a^b\left|V(t)-V\left(\frac{a+b}{2}\right)\right||g'(t)|^2\,dt.\end{aligned}$$

The proof follows by Theorem 4 and Lemma 1.

Corollary 6. *Assume that* g *is absolutely continuous on* $[a,b]$ *with* $g(a)=g(b)=0$ *and* v *is of bounded variation on* $[a,b]$, *then*

$$(4.11)\quad \left|\int_a^b g(t)\,dv(t)\right|\leq\frac{1}{2}(b-a)^{1/2}\left(\int_a^b |g'(t)|^2\,dt\right)^{1/2}\bigvee_a^b(v)$$

and

$$\begin{aligned}(4.12)\quad &\left|\int_a^b g(t)\,dv(t)\right| \\ &\leq\left(\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right)^{1/2}\left(\int_a^b\left|V(t)-V\left(\frac{a+b}{2}\right)\right||g'(t)|^2\,dt\right)^{1/2}.\end{aligned}$$

Proof. The inequality (4.11) follows by (4.9) for $h\equiv 1$.

The same choice produces in (4.10)

$$\left|\int_a^b g(t)\,dv(t)\right| \le \left(\frac{1}{2}(b-a)\int_a^b dV(t) - \int_a^b \left|t-\frac{a+b}{2}\right| dV(t)\right)^{1/2} \times \left(\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right| |g'(t)|^2\,dt\right)^{1/2}. \tag{4.13}$$

Since

$$K(t) := \begin{cases} t-a \text{ if } t\in\left[a,\frac{a+b}{2}\right], \\ b-t \text{ if } t\in\left(\frac{a+b}{2},b\right], \end{cases}$$

hence, integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\frac{1}{2}(b-a)\int_a^b dV(t) - \int_a^b \left|t-\frac{a+b}{2}\right| dV(t) \\
&= \int_a^{\frac{a+b}{2}} (t-a)\,dV(t) + \int_{\frac{a+b}{2}}^b (b-t)\,dV(t) \\
&= \frac{1}{2}(b-a)V\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} V(t)\,dt - \frac{1}{2}(b-a)V\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b V(t)\,dt \\
&= \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) V(t)\,dt
\end{aligned}$$

and by (4.13) we get (4.12). □

5. Applications for Trapezoid Inequality

We have the following equalities:

Lemma 2. *Let f, $v:[a,b]\to\mathbb{C}$ be such that one is continuous and the other is of bounded variation. Then*

$$\begin{aligned}
T(f,v;[a,b]) &:= \int_a^b f(t)\,dv(t) \\
&- f(b)\left[v(b) - \frac{1}{b-a}\int_a^b v(t)\,dt\right] - f(a)\left[\frac{1}{b-a}\int_a^b v(t)\,dt - v(a)\right] \\
&= \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt - \int_a^b v(t)\,df(t) \\
&= \int_a^b \left[f(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right] dv(t).
\end{aligned} \tag{5.1}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^b \left[f(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right] dv(t) \\
&= \int_a^b f(t)\,dv(t) - \int_a^b \frac{f(a)(b-t)+f(b)(t-a)}{b-a} dv(t)
\end{aligned}$$

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$$
\begin{aligned}
&= \int_a^b f(t)\,dv(t) - \left.\frac{f(a)(b-t)+f(b)(t-a)}{b-a} v(t)\right|_a^b \\
&+ \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt \\
&= \int_a^b f(t)\,dv(t) - f(b)v(b) + f(a)v(a) + \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt \\
&= \int_a^b f(t)\,dv(t) \\
&- f(b)\left[v(b) - \frac{1}{b-a}\int_a^b v(t)\,dt\right] - f(a)\left[\frac{1}{b-a}\int_a^b v(t)\,dt - v(a)\right].
\end{aligned}
$$

Integrating by parts again, we also have

$$
\begin{aligned}
&\int_a^b f(t)\,dv(t) - f(b)v(b) + f(a)v(a) + \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt \\
&= \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt - \int_a^b v(t)\,df(t).
\end{aligned}
$$

These prove the required identities. □

We have the following trapezoid type inequality:

Proposition 3. *Assume that f is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then*

$$
|T(f,v;[a,b])| \le \frac{1}{2}(b-a)^{1/2}\left(\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^2 dt\right)^{1/2} \bigvee_a^b (v) \tag{5.2}
$$

and

$$
\begin{aligned}
|T(f,v;[a,b])| &\le \left(\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) V(t)\,dt\right)^{1/2} \\
&\times \left(\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right| \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^2 dt\right)^{1/2},
\end{aligned} \tag{5.3}
$$

where $V(t) := \bigvee_a^t (v)$ is the total variation of v on $[a,t]$ with $t \in [a,b]$.

Proof. Consider the function $g : [a,b] \to \mathbb{C}$ defined by

$$
g(t) := f(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a}, \quad t \in [a,b].
$$

Then g is absolutely continuous on $[a,b]$, $g(a) = g(b) = 0$ and

$$
g'(t) = f'(t) - \frac{f(b)-f(a)}{b-a} \text{ for a.e. } t \in [a,b].
$$

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By using (4.11), we then have

$$\left|\int_a^b \left[f(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right] dv(t)\right|$$
$$\leq \frac{1}{2}(b-a)^{1/2}\left(\int_a^b \left|f'(t) - \frac{f(b)-f(a)}{b-a}\right|^2 dt\right)^{1/2} \bigvee_a^b (v),$$

which by Lemma 2 produces the inequality (5.2).

The inequality (5.3) follows by (4.12). □

For a function $h:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform* $\breve{h}$ defined by

$$\widehat{h}(t) := \frac{1}{2}[h(t)+h(a+b-t)],\ t\in[a,b]$$

and the *antisymmetrical transform* $\widetilde{h}$ defined by

$$\widetilde{h}(t) := \frac{1}{2}[h(t)-h(a+b-t)],\ t\in[a,b].$$

Proposition 4. *Assume that f is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then*

$$B(f,v;[a,b]) := \int_a^b \widehat{f}(t)\,dv(t) - \frac{f(a)+f(b)}{2}[v(b)-v(a)] = \int_a^b f(t)\,d\widetilde{v}(t) - \frac{f(a)+f(b)}{2}[v(b)-v(a)] \tag{5.4}$$

and we have the inequalities

$$|B(f,v;[a,b])| \leq \frac{1}{2}(b-a)^{1/2}\left(\int_a^b \left|\widetilde{f'}(t)\right|^2 dt\right)^{1/2}\bigvee_a^b (v) \tag{5.5}$$

and

$$|B(f,v;[a,b])| \leq \left(\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right)^{1/2}\left(\int_a^b \left|V(t)-V\left(\frac{a+b}{2}\right)\right|\left|\widetilde{f'}(t)\right|^2 dt\right)^{1/2}. \tag{5.6}$$

Proof. Consider the function $g:[a,b]\to\mathbb{C}$ defined by

$$g(t) := \widehat{f}(t) - \frac{f(a)+f(b)}{2},\ t\in[a,b].$$

Then g is absolutely continuous on $[a,b]$, $g(a)=g(b)=0$,

$$g'(t) = \frac{f'(t)-f'(a+b-t)}{2} = \widetilde{f'}(t) \text{ for a.e. } t\in[a,b]$$

and

$$\int_a^b g(t)\,dv(t) = \int_a^b \left(\widehat{f}(t) - \frac{f(a)+f(b)}{2}\right)dv(t) = \int_a^b \widehat{f}(t)\,dv(t) - \frac{f(a)+f(b)}{2}[v(b)-v(a)].$$

Using the change of variable formula for the Riemann-Stieltjes integral, see for instance [1, p. 144], we have

$$\begin{aligned}\int_a^b \widehat{f}(t)\,dv(t) &= \frac{1}{2}\int_a^b [f(t)+f(a+b-t)]\,dv(t)\\ &= \frac{1}{2}\left[\int_a^b f(t)\,dv(t)+\int_a^b f(a+b-t)\,dv(t)\right]\\ &= \frac{1}{2}\left[\int_a^b f(t)\,dv(t)+\int_b^a f(u)\,dv(a+b-u)\right]\\ &= \frac{1}{2}\left[\int_a^b f(t)\,dv(t)-\int_a^b f(u)\,dv(a+b-u)\right]=\int_a^b f(t)\,d\widetilde{v}(t),\end{aligned}$$

which proves the equality (5.4).

The inequalities (5.5) and (5.6) follows by Corollary 6. □

6. Some Grüss' Type Inequalities

For two *Lebesgue integrable* functions $f,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt. \tag{6.1}$$

In 1935, Grüss [10] showed that

$$|C(f,g)|\le\frac{1}{4}(M-m)(N-n), \tag{6.2}$$

provided that there exists the real numbers $m,\ M,\ n,\ N$ such that

$$m\le f(t)\le M \quad\text{and}\quad n\le g(t)\le N \quad\text{for a.e. } t\in[a,b]. \tag{6.3}$$

The constant $\frac{1}{4}$ is best possible in (4.1) in the sense that it cannot be replaced by a smaller quantity.

Proposition 5. *Assume that* $h:[a,b]\to\mathbb{C}$ *is integrable on* $[a,b]$ *and* $v:[a,b]\to\mathbb{C}$ *of bounded variation on* $[a,b]$. *Then*

$$|C(h,v)|\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b |h(t)|^2-\left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right]^{1/2}\bigvee_a^b(v) \tag{6.4}$$

and

$$|C(h,v)| \leq \left(\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right)^{1/2} \tag{6.5}$$
$$\times\left(\frac{1}{b-a}\int_a^b \left|V(t)-V\left(\frac{a+b}{2}\right)\right|\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 dt\right)^{1/2}$$
$$\leq \max_{t\in[a,b]}\left|V(t)-V\left(\frac{a+b}{2}\right)\right|\left(\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right)^{1/2}$$
$$\times\left[\frac{1}{b-a}\int_a^b |h(t)|^2-\left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right]^{1/2}.$$

Proof. Using the integration by parts for the Riemann-Stieltjes integral, we have

$$\int_a^b\left(\int_a^t h(s)\,ds-\frac{t-a}{b-a}\int_a^b h(s)\,ds\right)dv(t)$$
$$=\left(\int_a^t h(s)\,ds-\frac{t-a}{b-a}\int_a^b h(s)\,ds\right)v(t)\Big|_a^b$$
$$-\int_a^b v(t)\,d\left(\int_a^t h(s)\,ds-\frac{t-a}{b-a}\int_a^b h(s)\,ds\right)$$
$$=-\int_a^b v(t)\,h(t)\,dt+\frac{1}{b-a}\int_a^b h(s)\,ds\int_a^b v(t)\,dt,$$

which gives that

$$C(h,v)=\frac{1}{b-a}\int_a^b\left(\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right)dv(t). \tag{6.6}$$

Consider

$$g(t):=\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds,\ t\in[a,b],$$

then g is absolutely continuous, $g(a)=g(b)=0$ and by (4.11) we get

$$\left|\int_a^b\left(\frac{t-a}{b-a}\int_a^b h(s)\,ds-\int_a^t h(s)\,ds\right)dv(t)\right| \tag{6.7}$$
$$\leq\frac{1}{2}(b-a)^{1/2}\left(\int_a^b\left|h(t)-\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 dt\right)^{1/2}\bigvee_a^b(v)$$
$$=\frac{1}{2}(b-a)\left[\frac{1}{b-a}\int_a^b |h(t)|^2-\left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right]^{1/2}\bigvee_a^b(v)$$

which by the equality (6.6) is equivalent to (6.4).

Using (4.12) we get

$$\left|\int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds\right) dv(t)\right|$$
$$\leq \left(\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) V(t)\,dt\right)^{1/2}$$
$$\times \left(\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right| \left|h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 dt\right)^{1/2},$$

namely

$$\left|\frac{1}{b-a}\int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds\right) dv(t)\right|$$
$$\leq \left(\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) V(t)\,dt\right)^{1/2}$$
$$\times \left(\frac{1}{b-a}\int_a^b \left|V(t) - V\left(\frac{a+b}{2}\right)\right| \left|h(t) - \frac{1}{b-a}\int_a^b h(s)\,ds\right|^2 dt\right)^{1/2},$$

and the first inequality in (6.5) is obtained.

The second part is obvious. □

Consider now the *weighted Čebyšev functional*

$$C_w(f,g) := \frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t) f(t) g(t)\,dt$$
$$- \frac{1}{\left(\int_a^b w(t)\,dt\right)^2}\int_a^b w(t) f(t)\,dt \int_a^b w(t) g(t)\,dt,$$

where f, g, $w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt > 0$.

Proposition 6. *Assume that* $h : [a,b] \to \mathbb{C}$ *is integrable on* $[a,b]$, $w : [a,b] \to \mathbb{R}$, $w(t) \geq 0$ *for a.e.* $t \in [a,b]$ *is integrable with* $\int_a^b w(t)\,dt > 0$ *and* $v : [a,b] \to \mathbb{C}$ *is of*

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bounded variation on $[a,b]$. *Then*

$$
\begin{aligned}
(6.8)\quad & \left|C_w(h,v)\right| \\
& \leq \frac{1}{2}\frac{(b-a)^{1/2}}{\int_a^b w(s)\,ds}\left(\int_a^b\left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds-h(t)\right|^2 w^2(t)\,dt\right)^{1/2}\bigvee_a^b(v) \\
& \leq \frac{1}{2}\left(\frac{(b-a)\|w\|_{[a,b],\infty}}{\int_a^b w(s)\,ds}\right)^{1/2} \\
& \times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\left|h(t)\right|^2 dt-\left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds\right|^2\right)^{1/2}\bigvee_a^b(v)
\end{aligned}
$$

and

$$
\begin{aligned}
(6.9)\quad & \left|C_w(h,v)\right| \\
& \leq \frac{1}{\int_a^b w(s)\,ds}\left(\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right)^{1/2} \\
& \times\left(\int_a^b\left|V(t)-V\left(\frac{a+b}{2}\right)\right|\left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds-h(t)\right|^2 w^2(t)\,dt\right)^{1/2} \\
& \leq\left(\frac{(b-a)\sup_{t\in[a,b]}\left\{\left|V(t)-V\left(\frac{a+b}{2}\right)\right|w(t)\right\}}{\int_a^b w(s)\,ds}\right)^{1/2} \\
& \times\left(\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right)^{1/2} \\
& \times\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\left|h(t)\right|^2 dt-\left|\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds\right|^2\right)^{1/2}.
\end{aligned}
$$

Proof. Consider

$$
g(t):=\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds-\int_a^t h(s)w(s)\,ds,\ t\in[a,b],
$$

then g is absolutely continuous, $g(a)=g(b)=0$,

$$
g'(t):=\left(\frac{1}{\int_a^b w(s)\,ds}\int_a^b h(s)w(s)\,ds-h(t)\right)w(t)\ \text{ for a.e. } t\in[a,b]
$$

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and, by using the integration by parts for the Riemann-Stieltjes integral,

$$\begin{aligned}
&\int_a^b \left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds - \int_a^t h(s) w(s)\,ds \right) dv(t) \\
&= -\int_a^b v(t)\, d\left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds - \int_a^t h(s) w(s)\,ds \right) \\
&= \int_a^b v(t) h(t) w(t)\,dt - \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds \int_a^b v(t) w(t)\,dt,
\end{aligned}$$

which gives the identity

$$\begin{aligned}
&C_w(h,v) \\
&= \frac{1}{\int_a^b w(s)\,ds} \int_a^b \left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds - \int_a^t h(s) w(s)\,ds \right) dv(t).
\end{aligned}$$

By using the inequality (4.11) we have

$$\begin{aligned}
&\left| \int_a^b \left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds - \int_a^t h(s) w(s)\,ds \right) dv(t) \right| \\
&\leq \frac{1}{2}(b-a)^{1/2} \left(\int_a^b \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds - h(t) \right|^2 w^2(t)\,dt \right)^{1/2} \bigvee_a^b (v),
\end{aligned}$$

which by division with $\int_a^b w(s)\,ds > 0$ gives the first inequality in (6.8).

We also have

$$\begin{aligned}
&\frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds - h(t) \right|^2 w^2(t)\,dt \\
&\leq \|w\|_{[a,b],\infty} \frac{1}{\int_a^b w(s)\,ds} \int_a^b \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds - h(t) \right|^2 w(t)\,dt \\
&= \|w\|_{[a,b],\infty} \left[\frac{1}{\int_a^b w(s)\,ds} \int_a^b w(t) |h(t)|^2\,dt - \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds \right|^2 \right],
\end{aligned}$$

which proves the second part of (6.8).

Using (4.12) we also have

$$\begin{aligned}
&\left| \int_a^b \left(\frac{\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds - \int_a^t h(s) w(s)\,ds \right) dv(t) \right| \\
&\qquad\qquad \leq \left(\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) V(t)\,dt \right)^{1/2} \\
&\times \left(\int_a^b \left| V(t) - V\left(\frac{a+b}{2}\right) \right| \left| \frac{1}{\int_a^b w(s)\,ds} \int_a^b h(s) w(s)\,ds - h(t) \right|^2 w^2(t)\,dt \right)^{1/2},
\end{aligned}$$

which proves the first inequality in (6.9).

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The second part is obvious. □

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