Derivation of generalized Einstein's equations of gravitation based on a mechanical model of vacuum and a sink flow model of particles

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J. C. Maxwell, B. Riemann and H. Poincaré have proposed the idea that all microscopic particles are sink flows in a fluidic aether. Following this research program, a previous theory of gravitation based on a mechanical model of vacuum and a sink flow model of particles is generalized by methods of special relativistic continuum mechanics. In inertial reference frames, we construct a tensorial potential which satisfies the wave equation. Inspired by the equation of motion of a test particle, a definition of a metric tensor of a Riemannian spacetime is introduced. Applying Fock's theorem, generalized Einstein's equations in inertial systems are derived based on some assumptions. These equations reduce to Einstein's equations in case of weak field in harmonic reference frames. In some special non-inertial reference frames, generalized Einstein's equations are derived based on some assumptions. If the field is weak and the reference frame is quasi-inertial, these generalized Einstein's equations reduce to Einstein's equations. Thus, this theory may also explains all the experiments which support the theory of general relativity. There exists some differences between this theory and Einstein's theory of general relativity.

Keywords: Einstein's equations; gravitation; general relativity; sink; gravitational aether.

I. INTRODUCTION

The Einstein's equations of gravitational fields in the theory of general relativity can be written as [1, 2]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T^{m}_{\mu\nu}, \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor of a Riemannian spacetime, $R_{\mu\nu}$ is the Ricci tensor, $R \equiv g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, $g^{\mu\nu}$ is the contravariant metric tensor, $\kappa = 8\pi\gamma_N/c^4$, γ_N is Newton's gravitational constant, c is the speed of light in vacuum, $T^{m}_{\mu\nu}$ is the energy-momentum tensor of a matter system.

The Einstein's equations (1) is a fundamental assumption in the theory of general relativity [1, 2]. It is remarkable that Einstein's theory of general relativity, born in 1915, has held up under every experimental test, refers to, for instance, [3].

R. P. Feynman once said:" *What I cannot create, I do not understand.*" ([4], p. xxxii). New theories which can derive Einstein's field equations of gravitation and thus explain all known experiments of gravitational phenomena may be interesting. The reasons may be summarized as follows.

1. Many attempts to reconcile the theory of general relativity and quantum mechanics by using the techniques in quantum electrodynamics meet some mathematical difficulties ([5], p. 101). J. Maddox speculates that the failure of the familiar quantization procedures to cope with Einstein's equations may stem from two possible reasons. One possibility is that Einstein's equations are incomplete. The other possible reason may be that some underlying assumptions in Einstein's theory about the character of the space or time may be not suitable ([5], p. 101).

2. The value of the cosmological constant is a puzzle [6]. In 1917, A. Einstein thought that his equations should be revised to be ([2], p. 410)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\kappa T^{m}_{\mu\nu}, \tag{2}$$

where $g_{\mu\nu}$ is the metric tensor of a Riemannian spacetime, $R_{\mu\nu}$ is the Ricci tensor, $R \equiv g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, $g^{\mu\nu}$ is the contravariant metric tensor, $\kappa = 8\pi\gamma_N/c^4$, γ_N is Newton's gravitational constant, c is the speed of light in vacuum, $T^{m}_{\mu\nu}$ is the energy-momentum tensor of a matter system, Λ is the cosmological constant.

However, it seems that the cosmological constant Λ is unnecessary when Hubble discovered the expansion of the universe. Thus, Einstein abandoned the term $\Lambda g_{\mu\nu}$ in Eq. (2) and returned to his original equations ([2], p. 410). The value of the cosmological constant Λ is also related to the energy-momentum tensor of vacuum ([2], p. 411). Theoretical interpretation of the small value of Λ is still open [6].

3. The problem of the existence of black hole is still controversy [7]. Einstein believed that black hole can not exist in the real world [8]. Recently, the Event Horizon Telescope Collaboration (EHTC) reconstructed event-horizon-scale images of the supermassive black hole candidate in the center of the giant elliptical galaxy M87 [9]. EHTC reports that the observed image is consistent with predictions for the shadow of a Kerr black hole based on the theory of general relativity.

4. The existences and characters of dark matter and dark energy are still controversy, refers to, for instance, [10–14].

5. The existence and characters of gravitational aether are still not clear. Sir I. Newton pointed out that his inverse-square law of gravitation did not touch on the mechanism of gravitation ([15], p. 28;[16], p. 91). Newton warned ([17], p. 204):" *That Gravity should be innate, inherent and essential to Matter, so that one Body may*

act upon another at a Distance thro' a Vacuum, without the Mediation of any thing else, by and through which their Action and Force may be conveyed from one to another, is to me so great an Absurdity, that I believe no Man who has in philosophical Matters a competent Faculty of thinking, can ever fall into it. " He conjectured that gravitation may be explained based on the action of an aether pervading the space ([15], p. 28;[16], p. 92). In the years 1905-1916, Einstein abandoned the concepts of electromagnetic aether and gravitational aether in his theory relativity ([18], p. 27-61). However, H. A. Lorentz believed that general relativity could be reconciled with the concept of an ether at rest and wrote a letter to A. Einstein ([18], p. 65). Einstein changed his view later and introduced his new concept of ether ([18], p. 63-113). In 1920, Einstein said ([18], p. 98):"*According to the general theory of relativity, space is endowed with physical qualities; in this sense, therefore, there exists an ether. According to the general theory of relativity, space without ether is unthinkable;*". In 1954, Einstein said ([18], p. 149):"*There is no such thing as an empty space, i.e., a space without field. Space-time does not claim existence on its own, but only as a structural quality of the field.*" Unfortunately, Einstein did not tell us how to derive his equations theoretically based on his new concept of the gravitational aether.

6. Whether Newton's gravitational constant γ_N depends on time and space is still not clear. It is known that γ_N is a constant in Newton's and Einstein's theory of gravitation. P. A. M. Dirac speculates that γ_N may depend on time based on his large number hypothesis [19]. R. P. Feynman thought that if γ_N decreases on time, then the earth's temperature a billion years ago was about $48°C$ higher than the present temperature ([4], p. 9). D. R. Long reports that γ_N depends on the distance between matters [20].

Furthermore, there exists some other problems related to the theories of gravity, for instance, gravitational waves [21], the speed of light in vacuum [22–24], the definition of inertial system, origin of inertial force, the velocity of the propagation of gravity [25], the velocity of individual photons [23, 24], unified field theory, etc.

There is a long history of researches of derivations or interpretations of Einstein's theory of general relativity. For instance, C. Misner et al. introduce six derivations of the Einstein's equations (1) in their great book ([2], p. 417). S. Weinberg proposed two derivations ([1], p. 151).

However, these theories still face the aforementioned difficulties. The gravitational interaction seems to differ in character from other interactions. Thus, it seems that new ideas about the gravitational phenomena are needed. In 1949, Einstein wrote in a letter to Solovine [26]:"*I am not convinced of the certainty of a simple concept, and I am uncertain as to whether I was even on the right track.*" Following Einstein, it may be better for us to keep an open and critical mind to explore all possible theories about gravity.

The purpose of this manuscript is to propose a derivation of the Einstein's equation (1) in some special reference frames based on a mechanical model of vacuum and a sink flow model of particles [27].

II. INTRODUCTION OF A PREVIOUS THEORY OF GRAVITATION BASED ON A SINK FLOW MODEL OF PARTICLES BY METHODS OF CLASSICAL FLUID MECHANICS

The idea that all microscopic particles are sink flows in a fluidic substratum is not new. For instance, in order to compare fluid motions with electric fields, J. C. Maxwell introduced an analogy between source or sink flows and electric charges ([15], p. 243). B. Riemann speculates that:"*I make the hypothesis that space is filled with a substance which continually flows into ponderable atoms, and vanishes there from the world of phenomena, the corporeal world*"([28], p. 507). H. Poincaré also suggests that matters may be holes in fluidic aether ([29], p. 171). A. Einstein and L. Infeld said ([30], p. 256-257):"*Matter is where the concentration of energy is great, field where the concentration of energy is small. $\cdots$ What impresses our senses as matter is really a great concentration of energy into a comparatively small space. We could regard matter as the regions in space where the field is extremely strong.*"

Following these researchers, we suppose that all the microscopic particles were made up of a kind of elementary sinks of a fluidic medium filling the space [27]. Thus, Newton's law of gravitation is derived by methods of hydrodynamics based on the fluid model of vacuum and the sink flow model of particles [27].

We briefly introduce this theory of gravitation [27]. Suppose that there exists a fluidic medium filling the interplanetary vacuum. For convenience, we may call this medium as the $\Omega(0)$ substratum, or gravitational aether, or tao [27]. Suppose that the following conditions are valid: (1) the $\Omega(0)$ substratum is an ideal fluid; (2) the ideal fluid is irrotational and barotropic; (3) the density of the $\Omega(0)$ substratum is homogeneous; (4) there are no external body forces exerted on the fluid; (5) the fluid is unbounded and the velocity of the fluid at the infinity is approaching to zero.

An illustration of the velocity field of a sink flow can be found in Figure 1.

If a point source is moving with a velocity $\mathbf{v}_s$, then there is a force [27]

$$\mathbf{F}_Q = -\rho_0 Q(\mathbf{u} - \mathbf{v}_s) \tag{3}$$

is exerted on the source by the fluid, where ρ_0 is the density of the fluid, Q is the strength of the source, $\mathbf{u}$ is the velocity of the fluid at the location of the source induced by all means other than the source itself.

We suppose that all the elementary sinks were created simultaneously [27]. For convenience, we may call these elementary sinks as monads. The initial masses and the strengths of the monads are the same. Suppose that (1)

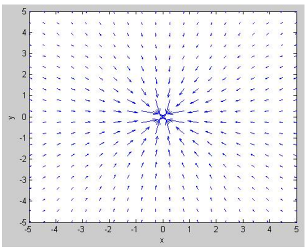

FIG. 1: an illustration of the velocity field of a sink flow.

$\mathbf{v}_i \ll \mathbf{u}_i, i = 1, 2$, where $\mathbf{v}_i$ is the velocity of the particle with mass m_i, $\mathbf{u}_i$ is the velocity of the $\Omega(0)$ substratum at the location of the particle with mass m_i induced by the other particle; (2) there are no other forces exerted on the particles, then the force $\mathbf{F}_{21}(t)$ exerted on the particle with mass $m_2(t)$ by the velocity field of $\Omega(0)$ substratum induced by the particle with mass $m_1(t)$ is [27]

$$\mathbf{F}_{21}(t) = -\gamma_N(t) \frac{m_1(t) m_2(t)}{r^2} \hat{\mathbf{r}}_{21}, \tag{4}$$

where $\hat{\mathbf{r}_{21}}$ denotes the unit vector directed outward along the line from the particle with mass $m_1(t)$ to the particle with mass $m_2(t)$, r is the distance between the two particles, $m_0(t)$ is the mass of monad at time t, $-q_0(q_0 > 0)$ is the strength of a monad, and

$$\gamma_N(t) = \frac{\rho_0 q_0^2}{4\pi m_0^2(t)}. \tag{5}$$

For continuously distributed matter, we have

$$\frac{\partial \rho_0}{\partial t} + \nabla \cdot (\rho_0 \mathbf{u}) = -\rho_0 \rho_s, \tag{6}$$

where $\mathbf{u}$ is the velocity of the $\Omega(0)$ substratum, $\nabla = \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y + \mathbf{k}\partial/\partial z$ is the nabla operator introduced by Hamilton, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are basis vectors, $-\rho_s(\rho_s > 0)$ is the density of continuously distributed sinks, i.e.,

$$-\rho_s = \lim_{\triangle V \to 0} \frac{\triangle Q}{\triangle V}, \tag{7}$$

where $\triangle Q$ is the source strength of the continuously distributed matter in the volume $\triangle V$ of the $\Omega(0)$ substratum.

Since the $\Omega(0)$ substratum is homogeneous, i.e., $\partial\rho_0/\partial t = \partial\rho_0/\partial x = \partial\rho_0/\partial y = \partial\rho_0/\partial z = 0$, and irrotational, i.e., $\nabla \times \mathbf{u} = 0$, Eq. (6) can be written as [31]

$$\nabla^2 \varphi = -\rho_s, \tag{8}$$

where φ is a velocity potential such that $\mathbf{u} = \nabla\varphi$, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator.

We introduce the following definitions

$$\Phi = -\frac{\rho_0 q_0}{m_0}\varphi, \quad \rho_m = \frac{m_0 \rho_s}{q_0}, \tag{9}$$

where ρ_m denotes the mass density of continuously distributed particles.

Using Eq. (9) and Eq. (5), Eq. (8) can be written as

$$\nabla^2 \Phi = 4\pi\gamma_N \rho_m. \tag{10}$$

III. A MECHANICAL MODEL OF VACUUM

According to our previous paper [32] we suppose that vacuum is filled with a kind of continuously distributed material which may be called $\Omega(1)$ substratum or electromagnetic aether. Maxwell's equations in vacuum are derived by methods of continuum mechanics based on this mechanical model of vacuum and a source and sink flow model of electric charges [32]. We speculate that the electromagnetic aether may also generate gravity. Thus, we introduce the following assumption.

Assumption 1 *The particles that constitute the $\Omega(1)$ substratum, or the electromagnetic aether, are sinks in the $\Omega(0)$ substratum.*

Then, according to the previous theory of gravitation [27], these $\Omega(1)$ particles gravitate with each other and also attract with matters. Thus, vacuum is composed of at least two kinds of interacting substratums, i.e., the gravitational aether $\Omega(0)$ and the electromagnetic aether $\Omega(1)$.

From Eq. (3), we see that there exists a following universal damping force $\mathbf{F}_d = -\rho_0 q_0 m \mathbf{v}_p / m_0$ exerted on each particle by the $\Omega(0)$ substratum [27], where $\mathbf{v}_p$ is the velocity of the particle. Based on this universal damping force $\mathbf{F}_d$ and some assumptions, we derive a generalized Schrödinger equation for microscopic particles [33]. For convenience, we may call these theories [27, 32, 33] as the theory of vacuum mechanics.

IV. CONSTRUCTION OF A LAGRANGIAN FOR FREE FIELDS OF THE $\Omega(0)$ SUBSTRATUM BASED ON A TENSORIAL POTENTIAL IN THE GALILEAN COORDINATES

There exists some approaches ([4], page vii;[2], p. 424), which regards Einstein's general relativity as a special relativistic field theory in an unobservable flat spacetime, to derive the Einstein's equations (1). However, these theories can not provide a physical definition of the tensorial potential of gravitational fields, refers to, for instance, [2, 34, 35]. Thus, similar to the theory of general relativity, these theories may be regarded as phenomenological theories of gravitation.

Inspired by these special relativistic field theories of gravitation, we explore the possibility of establishing a similar theory based on the theory of vacuum mechanics [27, 32, 33]. Thus, first of all, we need to construct a Lagrangian for free fields of the $\Omega(0)$ substratum based on a tensorial potential in the Galilean coordinates. In this section, we will regard the $\Omega(0)$ substratum in the previous theory of gravitation [27] as a special relativistic fluid. Then, we will study the $\Omega(0)$ substratum by methods of special relativistic continuum mechanics [36].

In this article, we adopt the mathematical framework of the theory of special relativity [1]. However, the physical interpretation of the mathematics of the theory of special relativity may be different from Einstein's theory. It is known that Maxwell's equations are valid in the frames of reference that attached to the $\Omega(1)$ substratum [32]. We introduce a Cartesian coordinate system $\{o, x, y, z\}$ for a three-dimensional Euclidean space that attached to the $\Omega(1)$ substratum. Let $\{0, t\}$ be a one-dimensional time coordinate. We denote this reference frame as $S_{\Omega(1)}$.

Based on the Maxwell's equations, the law of propagation of an electromagnetic wave front in this reference frame $S_{\Omega(1)}$ can be derived and can be written as ([37],p. 13)

$$\frac{1}{c^2}\left(\frac{\partial\omega}{\partial t}\right)^2 - \left(\frac{\partial\omega}{\partial x}\right)^2 - \left(\frac{\partial\omega}{\partial y}\right)^2 - \left(\frac{\partial\omega}{\partial z}\right)^2 = 0, \quad (11)$$

where $\omega(t, x, y, z)$ is an electromagnetic wave front, c is the velocity of light in the reference frame $S_{\Omega(1)}$.

An electromagnetic wave front is a characteristics. According to Fock's theorem of characteristics ([37], p. 432), we obtain the following metric tensor $\eta_{\alpha\beta} = \mathrm{diag}[c^2, -1, -1, -1]$ of a Minkowski spacetime for vacuum ([38], p. 57).

For convenience, we introduce the following Galilean coordinate system

$$x^0 \equiv ct, \qquad x^1 \equiv x, \qquad x^2 \equiv y, \qquad x^3 \equiv z. \quad (12)$$

We will use Greek indices α, β, μ, ν, etc., denote the range $\{0, 1, 2, 3\}$ and use Latin indices i, j, k, etc., denote the range $\{1, 2, 3\}$. We will use Einstein's summation convention, that is, any repeated Greek superscript or subscript appearing in a term of an equation is to be summed from 0 to 3. We introduce the following definition of spacetime interval

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \quad (13)$$

where $\eta_{\mu\nu}$ is the metric tensor of the Minkowski spacetime defined by $\eta_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$.

Suppose that the $\Omega(0)$ substratum is an incompressible viscous fluid. Then, there is no elastic deformations in the fluid and the internal stress states depend on the instantaneous velocity field. Thus, we can choose the reference frame $S_{\Omega(1)}$ as the co-moving coordinate system. The internal energy U is the sum of the internal elastic energy U_e and the dissipative energy U_d, i.e., $U = U_e + U_d$. Since there is no elastic deformations in the fluid, we have $U_e = 0$. We introduce the following definition of deviatoric tensor of strain rate $\dot{\gamma}^i_j$ ([39],p. 331)

$$\dot{\gamma}^i_j = \dot{S}^i_j - \dot{S}^k_k \delta^i_j, \quad (14)$$

where $\dot{S}^i_j$ is the tensor of strain rate, $\dot{S}^k_k$ is the rate of volume change, δ^i_j is the Kronecker delta.

Suppose that the rate of dissipative energy $\dot{U}_d$ is the Rayleigh type, then, we have ([39],p. 332)

$$\dot{U}_d = \mu_0 \dot{\gamma}^i_j \dot{\gamma}^j_i, \quad (15)$$

where μ_0 is the coefficient of viscosity.

Since the $\Omega(0)$ substratum is incompressible, we have $\dot{S}^k_k = 0$. Thus, from Eqs. (15) and Eqs. (14), we have

$$\dot{U}_d = \mu_0 \dot{S}^i_j \dot{S}^j_i. \quad (16)$$

In the low velocity limit, i.e., $u/c \ll 1$, where $u = |\mathbf{u}|$, the Lagrangian $L_{\Omega(0)}$ for free fields of the $\Omega(0)$ substratum can be written as ([39],p. 332)

$$L_{\Omega(0)} = \frac{1}{2}\rho_0 u^2 + \int_{t_0}^{t} \dot{U}_d(\dot{S}^i_j) dt, \quad (17)$$

where $u = |\mathbf{u}|$, t_0 is an initial time.

Suppose that the $\Omega(0)$ substratum is a Newtonian fluid and the stress tensor σ^i_j is symmetric, then we have ([40],p. 46)

$$\sigma^i_j = -p\delta^i_j + 2\mu_0 \dot{S}^i_j, \quad (18)$$

where p is the pressure of the $\Omega(0)$ substratum.

Using Eqs. (18) and Eqs. (16), Eqs. (17) can be written as

$$L_{\Omega(0)} = \frac{1}{2}\rho_0 u^2 + \int_{t_0}^{t} (\sigma^i_j + p\delta^i_j)\frac{\dot{S}^j_i}{2} dt, \quad (19)$$

For a macroscopic observer, the relaxation time t_ε of the $\Omega(0)$ substratum is so small that the tensor of strain rate $\dot{S}^i_j$ may be regarded as a slow varying function of time, i.e., $\partial \dot{S}^i_j/\partial t \ll 1$. Thus, in a small time interval $[t_0, t]$, we have $\dot{S}^i_j \geq 0$, or, $\dot{S}^i_j \leq 0$. Then, it is possible to choose a value $\bar{\sigma}^i_j + \bar{p}\delta^i_j$ of $\sigma^i_j + p\delta^i_j$ in the time interval $[t_0, t]$ such that Eqs. (19) can be written as

$$L_{\Omega(0)} = \frac{1}{2}\rho_0 u^2 + (\bar{\sigma}^i_j + \bar{p}\delta^i_j) \int_{t_0}^{t} \frac{\dot{S}^j_i}{2} dt. \quad (20)$$

We introduce the following definition

$$\psi_{ij} \triangleq \int_{t_0}^{t} \frac{\dot{S}_{ij}}{2f_0} dt, \quad (21)$$

where f_0 is a parameter to be determined.

Using Eqs. (21), Eqs. (20) can be written as

$$L_{\Omega(0)} = \frac{1}{2}\rho_0 u^2 + f_0 \psi^j_i (\bar{\sigma}^i_j + \bar{p}\delta^i_j). \quad (22)$$

Since the coefficient of viscosity μ_0 of the $\Omega(0)$ substratum may be very small, we introduce the following assumption.

Assumption 2 *In the low velocity limit, i.e., $u/c \ll 1$, where $u = |\boldsymbol{u}|$, $\boldsymbol{u}$ is the velocity of the $\Omega(0)$ substratum, we suppose that $\mu_0 \approx 0$ and we have the following conditions*

$$\psi_{ij} \approx 0, \quad \partial_\mu \psi_{ij} \approx 0, \quad \partial_\mu \partial_\nu \psi_{ij} \approx 0, \tag{23}$$

where

$$\partial_\mu \equiv \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right). \tag{24}$$

According to the Stokes-Helmholtz resolution theorem, refers to, for instance, [41], every sufficiently smooth vector field can be decomposed into irrotational and solenoidal parts. Thus, there exists a scalar function φ and a vector function $\mathbf{R}$ such that the velocity field $\mathbf{u}$ of the $\Omega(0)$ substratum can be represented by [41]

$$\mathbf{u} = \nabla\varphi + \nabla \times \mathbf{R}, \tag{25}$$

where $\nabla \times \varphi = 0$, $\nabla \cdot \mathbf{R} = 0$.

We introduce the following definition of a vector function $\vec{\xi}$

$$\frac{\partial \vec{\xi}}{\partial(ct)} = \nabla \times \mathbf{R}. \tag{26}$$

Putting Eq. (26) into Eq. (25), we have

$$\mathbf{u} = \nabla\varphi + \frac{\partial \vec{\xi}}{\partial(ct)}. \tag{27}$$

Based on Assumption 2 and using Eq. (9) and Eq. (27), Eq. (22) can be written as

$$\begin{aligned} L_{\Omega(0)} &= \frac{1}{2}\rho_0 u^2 = \frac{1}{2}\rho_0 \left(\nabla\varphi + \frac{\partial \vec{\xi}}{\partial(ct)}\right)^2 \\ &= \frac{1}{2}\rho_0 \left(-\frac{m_0}{\rho_0 q_0}\nabla\Phi + \frac{\partial \vec{\xi}}{\partial(ct)}\right)^2. \end{aligned} \tag{28}$$

We introduce the following definitions

$$\psi_{00} = a_{00}\Phi, \quad \psi_{0i} = \psi_{i0} = a_{0i}\xi_i, \tag{29}$$
$$\vec{\psi}_0 = \psi_{01}\mathbf{i} + \psi_{02}\mathbf{j} + \psi_{03}\mathbf{k}. \tag{30}$$

where $a_{00} > 0$ and $a_{0i} > 0$ are 4 parameters to be determined.

Eqs. (29) and Eqs. (21) have defined a rank 2 symmetric tensor $\psi_{\mu\nu}$. We require that for some special values of a_{00} and a_{0i}, Eq. (28) can be written as

$$\begin{aligned} L_{\Omega(0)} &= \left(\frac{m_0}{q_0\sqrt{2\rho_0}}\frac{1}{a_{00}}\nabla\psi_{00} - \sqrt{\frac{\rho_0}{2}}\frac{\partial(\psi_{0i}/a_{0i})}{\partial(ct)}\mathbf{e}^i\right)^2 \\ &\equiv \left(\nabla\psi_{00} - \frac{\partial\vec{\psi}_0}{\partial(ct)}\right)^2, \end{aligned} \tag{31}$$

where $\mathbf{e}^1 \equiv \mathbf{i}$, $\mathbf{e}^2 \equiv \mathbf{j}$, $\mathbf{e}^3 \equiv \mathbf{k}$.

Comparing the left- and right-hand parts of Eq. (31), we have

$$a_{00} = \sqrt{\frac{m_0^2}{2\rho_0 q_0^2}}, \quad a_{0i} = \sqrt{\frac{\rho_0}{2}}. \tag{32}$$

In order to construct the Lagrangian $L_{\Omega(0)}$ described in Eq. (31) based on the tensorial potential $\psi_{\mu\nu}$, we should consider all the possible products of derivatives of the tensor $\psi_{\mu\nu}$. If we require that the two tensor indices of $\psi_{\mu\nu}$ are different from each other and the two tensor indices of $\psi_{\mu\nu}$ are different from the derivative index, we have the following two possible products ([4], p. 43):

$$L_1 = \partial_\sigma\psi_{\mu\nu}\partial^\sigma\psi^{\mu\nu}, \quad L_2 = \partial_\sigma\psi_{\mu\nu}\partial^\mu\psi^{\sigma\nu}, \tag{33}$$

where $\psi^{\mu\nu} = \eta^{\mu\lambda}\eta^{\nu\sigma}\psi_{\lambda\sigma}$ is the corresponding contravariant tensor of $\psi_{\mu\nu}$.

If there are two indices of $\psi_{\mu\nu}$ which are equal, or one of the indices of $\psi_{\mu\nu}$ is the same as the derivative index, we may have the following three possible products ([4], p. 43):

$$L_3 = \partial_\nu\psi^{\mu\nu}\partial_\sigma\psi^\sigma{}_\mu, \quad L_4 = \partial^\mu\psi_{\mu\nu}\partial^\nu\psi, \tag{34}$$
$$L_5 = \partial_\lambda\psi\partial^\lambda\psi. \tag{35}$$

where ψ is the trace of $\psi_{\mu\nu}$, i.e., $\psi \equiv \psi^\lambda_\lambda = \eta_{\alpha\beta}\psi^{\alpha\beta}$,

$$\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial x^0}, -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}\right). \tag{36}$$

L_3 may be omitted because it can be converted to L_2 by integration by parts ([4], p. 43).

Proposition 3 *Suppose that we have the following conditions*

$$\frac{\partial\psi_{00}}{\partial(ct)} \approx 0, \quad \frac{\partial\psi_{0i}}{\partial x^j} \approx 0. \tag{37}$$

If we set

$$c_1 = \frac{1}{2}, \quad c_2 = -2, \quad c_4 = -6, \quad c_5 = -\frac{3}{2}, \tag{38}$$

then we have

$$\begin{aligned} c_1L_1 + c_2L_2 + c_4L_4 + c_5L_5 &\approx \left(\nabla\psi_{00} - \frac{\partial\vec{\psi}_0}{\partial(ct)}\right)^2 \\ &= \frac{1}{2}\rho_0 u^2. \end{aligned} \tag{39}$$

Proof of Proposition 3. Based on Eqs. (23) and Eqs. (37) and noticing $\psi^{00} = \psi_{00}, \psi^{0i} = -\psi_{0i}$, we have

$$L_1 \approx -(\nabla\psi_{00})^2 - 2\left(\frac{\partial\vec{\psi}_0}{\partial(ct)}\right)^2, \tag{40}$$

$$L_2 \approx -2(\nabla\psi_{00}) \cdot \frac{\partial\vec{\psi}_0}{\partial(ct)} - \left(\frac{\partial\vec{\psi}_0}{\partial(ct)}\right)^2, \tag{41}$$

$$L_3 \approx (\nabla\psi_{00}) \cdot \frac{\partial \vec{\psi}_0}{\partial(ct)}, \tag{42}$$

$$L_4 \approx -(\nabla\psi_{00})^2. \tag{43}$$

Using Eqs. (40-43) and Eqs. (38), we obtain Eq. (39). □

Inspired by W. Thirring [34] and R. P. Feynman ([4], p. 43), we introduce the following assumption.

Assumption 4 *The Lagrangian $L_{\Omega(0)}$ for free fields of the $\Omega(0)$ substratum can be written as*

$$L_{\Omega(0)} = c_1 L_1 + c_2 L_2 + c_4 L_4 + c_5 L_5 + L_{\text{more}}, \tag{44}$$

where $c_1 = 1/2$, $c_2 = -2$, $c_4 = -6$, $c_5 = -3/2$, L_{more} denotes those terms involving more than two derivatives of $\psi_{\mu\nu}$.

V. INTERACTION TERMS OF THE LAGRANGIAN OF A SYSTEM OF THE $\Omega(0)$ SUBSTRATUM, THE $\Omega(1)$ SUBSTRATUM AND MATTER

In order to derive the field equations, we should explore the possible interaction terms of the Lagrangian of a system of the $\Omega(0)$ substratum, the $\Omega(1)$ substratum and matter. According to Assumption 2, the coefficient of viscosity μ_0 of the $\Omega(0)$ substratum may be very small. Thus, we may regard the $\Omega(0)$ substratum as an ideal fluid approximately. Then from Eq. (25) we have $\mathbf{u} = \nabla\varphi$. Ignoring the damping force $\rho_0 Q \mathbf{v}_s$ in Eq. (3) and using $\mathbf{u} = \nabla\varphi$, Eq. (3) can be written as

$$\mathbf{F}_Q = -\rho_0 Q \nabla\varphi. \tag{45}$$

A particle is modelled as a point sink of the $\Omega(0)$ substratum [27, 32, 33]. Thus, the interaction term of the Lagrangian of a system of the $\Omega(0)$ substratum and a particle can be written as

$$L_{\text{int1}} = \rho_0 Q \varphi. \tag{46}$$

Therefore, the interaction term of the Lagrangian of a system of the $\Omega(0)$ substratum and continuously distributed particles can be written as

$$L_{\text{int}} = -\rho_0 \rho_s \varphi. \tag{47}$$

Putting Eq. (9) into Eq. (47), we have

$$L_{\text{int}} = \rho_m \Phi. \tag{48}$$

The 00 term of the energy-momentum tensor $T^m_{\mu\nu}$ of a matter system is $T^{00}_m = \rho_m c^2$. Thus, using Eqs. (29), Eq. (48) can be written as

$$L_{\text{int}} = f_0 \psi_{00} T^{00}_m, \tag{49}$$

where

$$f_0 = \frac{1}{a_{00} c^2}. \tag{50}$$

From Eq. (50), Eq. (32) and Eq. (5), we have

$$f_0 = \sqrt{\frac{2\rho_0 q_0^2}{m_0^2 c^4}} = \sqrt{\frac{8\pi\gamma_N}{c^4}}, \quad \frac{1}{a_{00}^2} = 8\pi\gamma_N. \tag{51}$$

Inspired by Eq. (49) and Eq. (22), we introduce the following assumption.

Assumption 5 *The interaction terms of the Lagrangian of a system of the $\Omega(0)$ substratum, the $\Omega(1)$ substratum and matter can be written in the following form:*

$$L_{\text{int}} = f_0 \psi_{\mu\nu} T^{\mu\nu}_m + f_0 \psi_{\mu\nu} T^{\mu\nu}_{\Omega(1)} + O[(f_0\psi_{\mu\nu})^2], \tag{52}$$

where $T^{\mu\nu}_m$ and $T^{\mu\nu}_{\Omega(1)}$ are the contravariant energy-momentum tensors of the system of the matter and the $\Omega(1)$ substratum respectively, $O[(f_0\psi_{\mu\nu})^2]$ denotes those terms which are small quantities of the order of $(f_0\psi_{\mu\nu})^2$.

VI. FIELD EQUATIONS IN INERTIAL REFERENCE FRAMES

Based on Assumptions 4 and 5, the total Lagrangian L_{tot} of a system of the $\Omega(0)$ substratum, the $\Omega(1)$ substratum and matter can be written as

$$\begin{aligned} L_{\text{tot}} &= \frac{1}{2}\partial_\lambda\psi_{\mu\nu}\partial^\lambda\psi^{\mu\nu} - 2\partial_\lambda\psi_{\mu\nu}\partial^\mu\psi^{\lambda\nu} - 6\partial^\mu\psi_{\mu\nu}\partial^\nu\psi \\ &\quad -\frac{3}{2}\partial_\lambda\psi\partial^\lambda\psi + L_{\text{more}} + f_0\psi_{\mu\nu}(T^{\mu\nu}_m + T^{\mu\nu}_{\Omega(1)}) \\ &\quad +O[(f_0\psi_{\mu\nu})^2]. \end{aligned} \tag{53}$$

Theorem 6 *If we ignore those terms which are small quantities of the order of $(f_0\psi_{\mu\nu})^2$ and those terms involving more than two derivatives of $\psi_{\mu\nu}$ in Eq. (53), i.e., $O[(f_0\psi_{\mu\nu})^2]$ and L_{more}, then the field equations for the total Lagrangian L_{tot} in Eq. (53) can be written as*

$$\begin{aligned} &\partial_\sigma\partial^\sigma\psi_{\alpha\beta} - 2(\partial^\sigma\partial_\alpha\psi_{\beta\sigma} + \partial^\sigma\partial_\beta\psi_{\alpha\sigma}) - 6(\eta_{\alpha\beta}\partial_\sigma\partial_\lambda\psi^{\sigma\lambda} \\ &+\partial_\alpha\partial_\beta\psi) - 3\eta_{\alpha\beta}\partial_\sigma\partial^\sigma\psi = f_0(T^m_{\alpha\beta} + T^{\Omega(1)}_{\alpha\beta}). \end{aligned} \tag{54}$$

Proof of Theorem 6. We have the following Euler-Lagrange equations [42]

$$\frac{\partial L_{\text{tot}}}{\partial\psi^{\alpha\beta}} - \frac{\partial}{\partial x^\sigma}\left(\frac{\partial L_{\text{tot}}}{\partial(\partial_\sigma\psi^{\alpha\beta})}\right) = 0. \tag{55}$$

We can verify the following results ([4], p. 43; [34])

$$\frac{\partial}{\partial x^\sigma}\left[\frac{\partial(\partial_\lambda\psi_{\mu\nu}\partial^\lambda\psi^{\mu\nu})}{\partial(\partial_\sigma\psi^{\alpha\beta})}\right] = 2\partial_\sigma\partial^\sigma\psi_{\alpha\beta}, \tag{56}$$

$$\frac{\partial}{\partial x^\sigma}\left[\frac{\partial(\partial_\lambda\psi_{\mu\nu}\partial^\mu\psi^{\lambda\nu})}{\partial(\partial_\sigma\psi^{\alpha\beta})}\right] = \partial^\sigma\partial_\alpha\psi_{\beta\sigma} + \partial^\sigma\partial_\beta\psi_{\alpha\sigma}, \tag{57}$$

$$\frac{\partial}{\partial x^\sigma}\left[\frac{\partial(\partial^\mu\psi_{\mu\nu}\partial^\nu\psi)}{\partial(\partial_\sigma\psi^{\alpha\beta})}\right] = \partial_\alpha\partial_\beta\psi + \eta_{\alpha\beta}\partial_\sigma\partial_\lambda\psi^{\sigma\lambda}, \tag{58}$$

$$\frac{\partial}{\partial x^\sigma}\left[\frac{\partial(\partial_\lambda\psi\partial^\lambda\psi)}{\partial(\partial_\sigma\psi^{\alpha\beta})}\right] = 2\eta_{\alpha\beta}\partial_\sigma\partial^\sigma\psi, \tag{59}$$

$$\frac{\partial L_{\text{tot}}}{\partial\psi^{\alpha\beta}} = f_0(T^m_{\alpha\beta} + T^{\Omega(1)}_{\alpha\beta}). \tag{60}$$

Putting Eq. (53) into Eqs. (55) and using Eqs. (56-60), we obtain Eqs. (54). □

For convenience, we introduce the following notation

$$\begin{aligned}\Psi^{\mu\nu} &= \partial_\lambda\partial^\lambda\psi^{\mu\nu} - 2\partial_\lambda\partial^\mu\psi^{\nu\lambda} - 2\partial_\lambda\partial^\nu\psi^{\mu\lambda} \\ &-6\eta^{\mu\nu}\partial_\sigma\partial_\lambda\psi^{\sigma\lambda} - 6\partial^\mu\partial^\nu\psi - 3\eta^{\mu\nu}\partial_\lambda\partial^\lambda\psi. \quad (61)\end{aligned}$$

Thus, the field equations (54) can be written as

$$\Psi^{\mu\nu} = f_0(T^{\mu\nu}_m + T^{\mu\nu}_{\Omega(1)}). \quad (62)$$

We introduce the following definition of the total energy-momentum tensor $T^{\mu\nu}$ of the system of the matter, the $\Omega(1)$ substratum and the $\Omega(0)$ substratum

$$T^{\mu\nu} = T^{\mu\nu}_m + T^{\mu\nu}_{\Omega(1)} + T^{\mu\nu}_{\Omega(0)}, \quad (63)$$

where $T^{\mu\nu}_{\Omega(0)}$ is the energy-momentum tensor of the $\Omega(0)$ substratum.

Adding the term $f_0T^{\mu\nu}_{\Omega(0)}$ on both sides of Eqs. (62) and using Eqs. (63), the field equations (62) can be written as

$$\Psi^{\mu\nu} + f_0T^{\mu\nu}_{\Omega(0)} = f_0T^{\mu\nu}. \quad (64)$$

For the total system of matter, the $\Omega(1)$ substratum and the $\Omega(0)$ substratum, the law of conservation of energy and momentum is ([36], p. 169; [38], p. 155)

$$\partial_\mu T^{\mu\nu} = 0. \quad (65)$$

Comparing Eqs. (65) and Eqs. (64), we have

$$\partial_\mu(\Psi^{\mu\nu} + f_0T^{\mu\nu}_{\Omega(0)}) = 0. \quad (66)$$

Noticing Eqs. (56-60), we introduce the following notation ([4], p. 43)

$$\begin{aligned}H^{\mu\nu} &= f_1\partial_\lambda\partial^\lambda\psi^{\mu\nu} + f_2(\partial_\lambda\partial^\mu\psi^{\nu\lambda} + \partial_\lambda\partial^\nu\psi^{\mu\lambda}) \\ &+f_3(\partial^\mu\partial^\nu\psi + \eta^{\mu\nu}\partial_\sigma\partial_\lambda\psi^{\sigma\lambda}) + f_4\eta^{\mu\nu}\partial_\lambda\partial^\lambda\psi, \quad (67)\end{aligned}$$

where $f_i, i = 1,2,3,4$ are 4 arbitrary parameters.

If we require that

$$\partial_\mu H^{\mu\nu} = 0, \quad (68)$$

then, we can verify the following relationships ([4], p. 44; [34])

$$f_1 + f_2 = 0, \quad f_2 + f_3 = 0, \quad f_3 + f_4 = 0. \quad (69)$$

We choose $f_1 = 1$, $f_2 = -1$, $f_3 = 1$, $f_4 = -1$ in Eqs. (67) and introduce the following notation

$$\begin{aligned}\Theta^{\mu\nu} &= \partial_\lambda\partial^\lambda\psi^{\mu\nu} - (\partial_\lambda\partial^\mu\psi^{\nu\lambda} + \partial_\lambda\partial^\nu\psi^{\mu\lambda}) \\ &+(\partial^\mu\partial^\nu\psi + \eta^{\mu\nu}\partial_\sigma\partial_\lambda\psi^{\sigma\lambda}) - \eta^{\mu\nu}\partial_\lambda\partial^\lambda\psi. \quad (70)\end{aligned}$$

We can verify the following result ([4], p. 44; [34])

$$\partial_\mu\Theta^{\mu\nu} = 0. \quad (71)$$

From Eqs. (71) and Eqs. (66), we have

$$\partial_\mu\left(\frac{1}{f_0}\Psi^{\mu\nu} - \frac{b_0}{f_0}\Theta^{\mu\nu} + T^{\mu\nu}_{\Omega(0)}\right) = 0. \quad (72)$$

where b_0 is an arbitrary parameter.

Noticing Eqs. (72), it is convenient for us to introduce the following definition of a tensor $T^{\mu\nu}_\omega$

$$T^{\mu\nu}_\omega = \frac{1}{f_0}\Psi^{\mu\nu} - \frac{b_0}{f_0}\Theta^{\mu\nu} + T^{\mu\nu}_{\Omega(0)}, \quad (73)$$

where b_0 is a parameter to be determined.

From Eqs. (72), we have $\partial_\mu T^{\mu\nu}_\omega = 0$. In the present stage, we have no idea about the physical meaning of the tensor $T^{\mu\nu}_\omega$. Later, once we have determined the value of the parameter b_0, we may explore the meaning of $T^{\mu\nu}_\omega$. Using Eqs. (73), the field equations (64) can be written as

$$\Theta^{\mu\nu} = \frac{f_0}{b_0}(T^{\mu\nu} - T^{\mu\nu}_\omega). \quad (74)$$

Now our task is to determine the parameter b_0 in the field equations (74). A natural idea is that the 00 component of Eqs. (74) reduces to the field equations (10) in the case that the velocity of the $\Omega(0)$ substratum is much smaller than c, i.e., in the low velocity limit. Thus, it is necessary for us to introduce an estimation of the value of $T^{\mu\nu} - T^{\mu\nu}_\omega$ on the right hand side of Eqs. (74) in the low velocity limit. To this end, we introduce the following speculation about the interaction between the $\Omega(0)$ substratum and the $\Omega(1)$ substratum.

Assumption 7 *In the low velocity limit, i.e., $u/c \ll 1$, where $u = |\boldsymbol{u}|$, $\boldsymbol{u}$ is the velocity of the $\Omega(0)$ substratum, the following relationship is valid*

$$\Psi^{\mu\nu} - b_0\Theta^{\mu\nu} \approx 0, \quad (75)$$

where b_0 is a parameter to be determined.

Using Eqs. (64), Eqs. (73) and Eqs. (75), we have the following estimation of $T^{\mu\nu} - T^{\mu\nu}_\omega$ in the low velocity limit

$$T^{\mu\nu} - T^{\mu\nu}_\omega \approx T^{\mu\nu}_m + T^{\mu\nu}_{\Omega(1)}. \quad (76)$$

Theorem 8 *Suppose that (1) Assumption 7 is valid; (2) $T^{\mu\nu}_{\Omega(1)} \approx 0$. Then, $b_0 = -1$ and the field equations (54) can be written as*

$$\begin{aligned}&\partial_\lambda\partial^\lambda\psi^{\mu\nu} - \partial_\lambda\partial^\mu\psi^{\nu\lambda} - \partial_\lambda\partial^\nu\psi^{\mu\lambda} + \partial^\mu\partial^\nu\psi \\ &+\eta^{\mu\nu}\partial_\sigma\partial_\lambda\psi^{\sigma\lambda} - \eta^{\mu\nu}\partial_\lambda\partial^\lambda\psi = -f_0(T^{\mu\nu} - T^{\mu\nu}_\omega) \quad (77)\end{aligned}$$

Proof of Theorem 8. Noticing Eqs. (70), the 00 component of the field equations (74) is

$$\begin{aligned}\partial_\lambda\partial^\lambda\psi^{00} &- 2\partial_\lambda\partial^0\psi^{0\lambda} + \partial^0\partial^0\psi \\ &+ \partial_\sigma\partial_\lambda\psi^{\sigma\lambda} - \partial_\lambda\partial^\lambda\psi = \frac{f_0}{b_0}(T^{00} - T^{00}_\omega) \quad (78)\end{aligned}$$

Take the trace of the field equations (74), we have

$$\partial_\sigma \partial_\lambda \psi^{\sigma\lambda} - \partial_\lambda \partial^\lambda \psi = \frac{f_0}{2b_0}(T - T_\omega), \tag{79}$$

where T and T_ω are the traces of $T^{\mu\nu}$ and $T_\omega^{\mu\nu}$ respectively, i.e., $T \equiv T_\lambda^\lambda = \eta_{\alpha\beta} T^{\alpha\beta}$, $T_\omega \equiv T_{\omega\lambda}^{\ \lambda} = \eta_{\alpha\beta} T_\omega^{\alpha\beta}$.

Subtracting Eq. (79) from Eq. (78), we have

$$\begin{aligned}\partial_\lambda \partial^\lambda \psi^{00} \ &- \ 2\partial_\lambda \partial^0 \psi^{0\lambda} + \partial^0 \partial^0 \psi \\ &= \ \frac{f_0}{b_0}\left(T^{00} - \frac{T}{2} - T_\omega^{00} + \frac{T_\omega}{2}\right). \end{aligned} \tag{80}$$

If the field is time-independent, then Eq. (80) reduces to

$$-\nabla^2 \psi^{00} = \frac{f_0}{b_0}\left(T^{00} - \frac{T}{2} - T_\omega^{00} + \frac{T_\omega}{2}\right). \tag{81}$$

According to Eqs. (76), we have the following estimations in the low velocity limit

$$T^{00} - T_\omega^{00} \approx T_m^{00} + T_{\Omega(1)}^{00} = \rho_m c^2, \tag{82}$$

$$T - T_\omega \approx T_m + T_{\Omega(1)} \approx \rho_m c^2, \tag{83}$$

where T_m is the trace of $T_m^{\mu\nu}$, i.e., $T_m \equiv \eta_{\alpha\beta} T_m^{\alpha\beta}$, $T_{\Omega(1)}$ is the trace of $T_{\Omega(1)}^{\mu\nu}$,.

Noticing $\psi^{00} = \psi_{00}$ and using Eqs. (29), Eq. (32), Eq. (51) Eqs. (82) and Eqs. (83), Eq. (81) can be written as

$$-\nabla^2 \Phi = \frac{1}{b_0} 4\pi \gamma_N \rho_m. \tag{84}$$

Comparing Eq. (84) and Eq. (10), we obtain $b_0 = -1$. Therefore, using Eqs. (70) and $b_0 = -1$, the field equations (74) can be written as Eqs. (77). □

Now we discuss the physical meaning of $T_\omega^{\mu\nu}$. Noticing Eqs. (75) and Eqs. (73), we have the following estimation in the low velocity limit

$$T_\omega^{\mu\nu} \approx T_{\Omega(0)}^{\mu\nu}. \tag{85}$$

From Eqs. (85), we see that the tensor $T_\omega^{\mu\nu}$ is an estimation of $T_{\Omega(0)}^{\mu\nu}$ when the velocity u of the $\Omega(0)$ substratum is small comparing to c.

We can verify that the field equations (77) are invariant under the following gauge transformation ([4], p. 45; [34])

$$\psi^{\mu\nu} \to \psi^{\mu\nu} + \partial^\mu \Lambda^\nu + \partial^\nu \Lambda^\mu, \tag{86}$$

where Λ^μ is an arbitrary vector field.

We introduce the following definition

$$\phi^{\mu\nu} = \psi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\psi. \tag{87}$$

Using Eqs. (87), the field equations (77) can be written as

$$\begin{aligned}\partial_\lambda \partial^\lambda \phi^{\mu\nu} \ &- \ \partial_\lambda \partial^\mu \phi^{\nu\lambda} - \partial_\lambda \partial^\nu \phi^{\mu\lambda} \\ &+ \ \eta^{\mu\nu} \partial_\sigma \partial_\lambda \phi^{\sigma\lambda} = -f_0 (T^{\mu\nu} - T_\omega^{\mu\nu}). \end{aligned} \tag{88}$$

We introduce the following Hilbert gauge condition [34]

$$\partial_\mu \left(\psi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\psi\right) = 0. \tag{89}$$

Using Eqs. (87), the Hilbert gauge condition Eqs. (89) simplifies to

$$\partial_\mu \phi^{\mu\nu} = 0. \tag{90}$$

Applying Eqs. (90) in Eqs. (88), we obtain the following proposition [34].

Proposition 9 *If we impose the Hilbert gauge condition Eqs. (89) on the fields, then, the field equations (77) simplifies to*

$$\partial_\lambda \partial^\lambda \left(\psi^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\psi\right) = -f_0 (T^{\mu\nu} - T_\omega^{\mu\nu}). \tag{91}$$

If the tensorial potential $\psi^{\mu\nu}$ does not satisfy the Hilbert gauge condition Eqs. (89), then we can always construct a new tensorial potential $\bar{\psi}^{\mu\nu}$ by the following gauge transformation [34]

$$\bar{\psi}^{\mu\nu} = \psi^{\mu\nu} + \partial^\mu \Lambda^\nu + \partial^\nu \Lambda^\mu, \tag{92}$$

such that the new tensorial potential $\bar{\psi}^{\mu\nu}$ does satisfy the Hilbert gauge condition Eqs. (89).

Using Eqs. (87), the field equations (91) can be written as

$$\partial_\lambda \partial^\lambda \phi^{\mu\nu} = -f_0 (T^{\mu\nu} - T_\omega^{\mu\nu}). \tag{93}$$

The field equations (93) can also be written as

$$\eta^{\alpha\beta} \frac{\partial^2 \phi^{\mu\nu}}{\partial x^\alpha \partial x^\beta} = -f_0 (T^{\mu\nu} - T_\omega^{\mu\nu}). \tag{94}$$

We noticed that the tensorial field equations (94) are similar to the wave equations of electromagnetic fields.

VII. CONSTRUCTION OF A TENSORIAL POTENTIAL IN INERTIAL REFERENCE FRAMES

The existence of the $\Omega(1)$ substratum allows us to introduce the following definition of inertial reference frames.

Definition 10 *If a coordinates system S is static or moving with a constant velocity relative to the reference frame $S_{\Omega(1)}$, then, we call such a coordinates system as an inertial reference frame.*

The field equations (88) and Eqs. (91) are valid in the reference frame $S_{\Omega(1)}$. We will explore the possibility of constructing a tensorial potential in an arbitrary inertial system S'. In an inertial reference frame

S, an arbitrary event is characterized by the four spacetime coordinates (t, x, y, z). In an inertial system S', this event is characterized by four other coordinates (t', x', y', z'). We assume that the origins of the Cartesian coordinates in the two inertial systems S and S' coincide at the time $t = t' = 0$. Then, the connections between these spacetime coordinates are given by a homogeneous linear transformation keeping the quantity $s^2 = c^2t^2 - x^2 - y^2 - z^2$ invariant, i.e., ([36], p. 90)

$$s^2 = c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = s'^2. \quad (95)$$

We introduce the following two coordinate systems

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z,$$
$$x'^0 = ct', \quad x'^1 = x', \quad x'^2 = y', \quad x'^3 = z'. \quad (96)$$

The homogeneous linear transformation keeping the quantity s^2 invariant, which is usually called the Lorentz transformation, can be written as ([43], p. 57; [36], p. 90)

$$x'^\mu = a^\mu{}_\nu x^\nu, \quad (97)$$

where $a^\mu{}_\nu$ are coefficients depend only on the angles between the spatial axes in the two inertial systems S and S' and on the relative velocity of S and S'.

Applying the standard methods in theory of special relativity [36], we have the following results.

Proposition 11 *Suppose that the field equations (93) is valid in the the reference frame $S_{\Omega(1)}$. Then, in an arbitrary inertial system S', there exists a symmetric tensor $\phi'_{\mu\nu}$ satisfies the following wave equation*

$$\partial'_\lambda \partial'^\lambda \phi'^{\mu\nu} = -f_0(T'^{\mu\nu} - T'^{\mu\nu}_\omega), \quad (98)$$

where $T'^{\mu\nu}$ and $T'^{\mu\nu}_\omega$ are corresponding tensors of $T^{\mu\nu}$ and $T^{\mu\nu}_\omega$ in the arbitrary inertial reference frame S' respectively.

Proposition 12 *Suppose that the field equations (88) is valid in the reference frame $S_{\Omega(1)}$. Then, in an arbitrary inertial system S', there exists a symmetric tensor $\phi'_{\mu\nu}$ satisfies the following field equation*

$$\partial'_\lambda \partial'^\lambda \phi'^{\mu\nu} - \partial'_\lambda \partial'^\mu \phi'^{\nu\lambda} - \partial'_\lambda \partial'^\nu \phi'^{\mu\lambda} + \eta^{\mu\nu} \partial'_\sigma \partial'_\lambda \phi'^{\sigma\lambda} = -f_0(T'^{\mu\nu} - T'^{\mu\nu}_\omega). \quad (99)$$

VIII. THE EQUATIONS OF MOTION OF A POINT PARTICLE IN A GRAVITATIONAL FIELD AND INTRODUCTION OF AN EFFECTIVE RIEMANNIAN SPACETIME

In this section, we study the equations of motion of a free point particle in a gravitational field. The Lagrangian of a free point particle can be written as ([4], p. 57;[34])

$$L_0 = \frac{1}{2} m \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} = \frac{1}{2} m u^\mu u_\mu = \frac{1}{2} m \eta_{\mu\nu} u^\mu u^\nu, \quad (100)$$

where m is the rest mass of the point particle, $d\tau \equiv \frac{1}{c}\sqrt{dx^\mu dx_\mu}$ is the infinitesimal proper time interval, $u^\mu \equiv dx^\mu / d\tau$.

Suppose that $T^{\mu\nu}_{\Omega(1)} \approx 0$. Ignoring those higher terms $O[(f_0 \psi_{\mu\nu})^2]$ in Eq. (52), the interaction term of the Lagrangian of a system of the $\Omega(0)$ substratum, the $\Omega(1)$ substratum and the point particle can be written in the following form ([4], p. 57;[34])

$$L_{\text{int}} = f_0 \psi_{\mu\nu} m u^\mu u^\nu. \quad (101)$$

Using Eq. (101) and Eq. (100), the total Lagrangian L_{p} of a system of the $\Omega(0)$ substratum, the $\Omega(1)$ substratum and the point particle can be written as ([4], p. 57)

$$L_{\text{p}} = L_0 + L_{\text{int}} = \frac{1}{2} m u^\mu u_\mu + f_0 \psi_{\mu\nu} m u^\mu u^\nu. \quad (102)$$

The Euler-Lagrange equations for the total Lagrangian L_{p} can be written as ([43],p. 111)

$$\frac{d}{d\tau}\left[(\eta_{\mu\nu} + 2f_0\psi_{\mu\nu})\frac{dx^\nu}{d\tau}\right] - f_0 \frac{\partial \psi_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \quad (103)$$

We notice that the equations of motion (103) of a point particle in gravitational field are similar to the equations of a geodesic line (105) in a Riemannian spacetime. Thus, it is natural for us to introduce the following definition of a metric tensor $g_{\mu\nu}$ of a Riemannian spacetime ([4], p. 57)

$$g_{\mu\nu} = \eta_{\mu\nu} + 2f_0 \psi_{\mu\nu}. \quad (104)$$

Then, the equations of motion (103) can be written as ([4], p. 58)

$$\frac{d}{d\tau_g}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau_g}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{d\tau_g}\frac{dx^\beta}{d\tau_g}, \quad (105)$$

where $d\tau_g$ is the infinitesimal proper time interval in the Riemannian spacetime with a metric tensor $g_{\mu\nu}$.

Eqs. (105) represent a geodesic line in a Riemannian spacetime with a metric tensor $g_{\mu\nu}$, which can also be written as ([44], p. 51)

$$\frac{d^2 x^\mu}{d\tau_g^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau_g}\frac{dx^\sigma}{d\tau_g} = 0, \quad (106)$$

where

$$\Gamma^\nu_{\alpha\beta} \triangleq \frac{1}{2} g^{\mu\nu}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\beta} + \frac{\partial g_{\mu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\mu}\right) \quad (107)$$

are the Christoffel symbols.

Thus, we find that the equations of motion (103) of a point particle in gravitational field represent a geodesic line described in Eqs. (106) in a Riemannian spacetime with a metric tensor $g_{\mu\nu}$.

According to Assumption 1, the particles that constitute the $\Omega(1)$ substratum are sinks in the $\Omega(0)$ substratum. Thus, the movements of the $\Omega(1)$ substratum in gravitational field will be different from the Maxwell's equations. We notice that the equations of motion of a point particle in gravitational field (105) are generalizations of the equations of motion of a point particle in vacuum free of gravitational field. The law of propagation of an electromagnetic wave front in vacuum free of gravitational field is Eqs. (11). Thus, the law of propagation of an electromagnetic wave front in gravitational field may be a kind of generalization of Eq. (11). Therefore, we introduce the following assumption.

Assumption 13 *To first order of $f_0\psi_{\mu\nu}$, the law of propagation of an electromagnetic wave front $\omega(x^0,x^1,x^2,x^3)$ in gravitational field is*

$$g_{\mu\nu}\frac{\partial\omega}{\partial x^\mu}\frac{\partial\omega}{\partial x^\nu}=0, \tag{108}$$

where $\omega(x^0,x^1,x^2,x^3)$ is the electromagnetic wave front, $g_{\alpha\beta}$ is the metric tensor defined in Eqs. (104).

The measurements of spacetime intervals are carried out using light rays and point particles, which are only subject to inertial force and gravitation. Thus, according to Eqs. (105) and Eq. (108), the physically observable metric of spacetime, to first order of $f_0\psi_{\mu\nu}$, is $g_{\mu\nu}$. Thus, the initial flat background spacetime with metric $\eta_{\mu\nu}$ is no longer physically observable [34].

If we can further derive the Einstein's equations (1) using the definition (104) of a metric tensor $g_{\mu\nu}$ of a Riemannian spacetime, then, we may provide a geometrical interpretation of Einstein's theory of gravitation based on the theory of vacuum mechanics [27, 32, 33]. This is the task of the next section.

IX. GENERALIZED EINSTEIN EQUATIONS IN INERTIAL REFERENCE FRAMES

Definition 14 *The Einstein tensor $G_{\mu\nu}$ is defined by*

$$G_{\mu\nu}\triangleq R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R, \tag{109}$$

where $g_{\mu\nu}$ is a metric tensor of a Riemannian spacetime, $R_{\mu\nu}$ is the Ricci tensor, $R\triangleq g^{\mu\nu}R_{\mu\nu}$, $g^{\mu\nu}$ is the corresponding contravariant tensor of $g_{\mu\nu}$ such that $g_{\mu\lambda}g^{\lambda\nu}=\delta^\nu_\mu=g^\nu_\mu$ ([44], p. 40).

According to the geometrical interpretation of some theories of gravitation in flat spacetime [34], the physically observable metric $g_{\mu\nu}$ of spacetime in Eqs. (104) can be written as

$$g^{\mu\nu}=\eta^{\mu\nu}-2f_0\psi^{\mu\nu}+O[(f_0\psi^{\mu\nu})^2]. \tag{110}$$

Following the clue showed in Eqs. (110) and noticing the methods of S. N. Gupta [45] and W. Thirring [34], we introduce the following definition of a metric tensor of a Riemannian spacetime.

Definition 15

$$\tilde{g}^{\mu\nu}\triangleq\sqrt{-g_0}g^{\mu\nu}\triangleq\eta^{\mu\nu}-2f_0\phi^{\mu\nu}, \tag{111}$$

where $g_0=\mathrm{Det}\ g_{\mu\nu}$.

We have the following expansion of the contravariant metric tensor $g^{\mu\nu}$ [45]

$$\begin{aligned}g^{\mu\nu} &= \eta^{\mu\nu}-2f_0\phi^{\mu\nu}+f_0\eta^{\mu\nu}\eta_{\alpha\beta}\phi^{\alpha\beta}\\ &\quad -2f_0^2\eta_{\alpha\beta}\phi^{\alpha\beta}\phi^{\mu\nu}+f_0^2\eta^{\mu\nu}\eta_{\alpha\sigma}\eta_{\beta\lambda}\phi^{\alpha\beta}\phi^{\lambda\sigma}\\ &\quad +\frac{1}{2}f_0^2\eta^{\mu\nu}\eta_{\alpha\beta}\eta_{\lambda\sigma}\phi^{\alpha\beta}\phi^{\lambda\sigma}+O[(f_0\phi^{\mu\nu})^3].\end{aligned} \tag{112}$$

Definition 16 *If $\phi^{\mu\nu}$ and their first and higher derivatives satisfy the following conditions*

$$|2f_0\phi^{\mu\nu}|\ll 1, \tag{113}$$

$$\left|\frac{\partial^n(2f_0\phi^{\mu\nu})}{\partial(x^\alpha)^n}\right|\ll 1, n=1,2,3,\cdots \tag{114}$$

then we call this field $\phi^{\mu\nu}$ weak.

For weak fields, $\psi\approx\phi\approx 0$. Thus, $\phi^{\mu\nu}=\psi^{\mu\nu}-1/2\cdot\eta^{\mu\nu}\psi\approx\psi^{\mu\nu}$. From Eqs. (112), we see that the definition (111) is compatible with Eqs. (110).

Theorem 17 *Suppose that Assumption 7 is valid. Then, in an arbitrary inertial reference frame S_i, we have the following field equations*

$$\begin{aligned}G^{\mu\nu} &- \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta}-\eta^{\alpha\beta}\right)\frac{\partial^2(\sqrt{-g_0}g^{\mu\nu})}{\partial x^\alpha\partial x^\beta}\\ &- \frac{\sqrt{-g_0}}{2g_0}(\partial_\lambda\partial^\mu g^{\nu\lambda}+\partial_\lambda\partial^\nu g^{\mu\lambda}-\eta^{\mu\nu}\partial_\sigma\partial_\lambda g^{\sigma\lambda})\\ &- \Pi^{\mu,\alpha\beta}\Pi^\nu_{\alpha\beta}+\frac{1}{2}y^\mu y^\nu-\frac{1}{2}g^{\mu\nu}(L+B)\\ &+ B^{\mu\nu}=\frac{f_0^2}{g_0}(T^{\mu\nu}-T^{\mu\nu}_\omega),\end{aligned} \tag{115}$$

where $T^{\mu\nu}$ is the contravariant total energy-momentum tensor of the system of the matter, the $\Omega(1)$ substratum and the $\Omega(0)$ substratum in the inertial reference frame S_i, $T^{\mu\nu}_\omega$ is the contravariant energy-momentum tensor of vacuum in the low velocity limit in S_i,

$$\Pi^{\mu,\alpha\beta}\triangleq\frac{1}{2g_0}\left(\tilde{g}^{\alpha\lambda}\frac{\partial\tilde{g}^{\mu\beta}}{\partial x^\lambda}+\tilde{g}^{\beta\lambda}\frac{\partial\tilde{g}^{\mu\alpha}}{\partial x^\lambda}-\tilde{g}^{\mu\lambda}\frac{\partial\tilde{g}^{\alpha\beta}}{\partial x^\lambda}\right), \tag{116}$$

$$\Pi^\nu_{\alpha\beta}\triangleq g_{\alpha\lambda}g_{\beta\sigma}\Pi^{\nu,\lambda\sigma}, \tag{117}$$

$$\Gamma^\alpha\triangleq g^{\sigma\lambda}\Gamma^\alpha_{\sigma\lambda}, \tag{118}$$

$$\Gamma^{\mu\nu}\triangleq\frac{1}{2}\left(g^{\mu\alpha}\frac{\partial\Gamma^\nu}{\partial x^\alpha}+g^{\nu\alpha}\frac{\partial\Gamma^\mu}{\partial x^\alpha}-\frac{\partial g^{\mu\nu}}{\partial x^\alpha}\Gamma^\alpha\right), \tag{119}$$

$$y_\beta \triangleq \frac{\partial(\lg\sqrt{-g_0})}{\partial x^\beta},\ \ y^\alpha \triangleq g^{\alpha\beta}y_\beta, \tag{120}$$

$$L \triangleq -\frac{1}{2}\Gamma^\nu_{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x^\nu} - \Gamma^\alpha\frac{\partial(\lg\sqrt{-g_0})}{\partial x^\alpha}, \tag{121}$$

$$B^{\mu\nu} \triangleq \Gamma^{\mu\nu} + \frac{1}{2}(y^\mu\Gamma^\nu + y^\nu\Gamma^\mu),\ B \triangleq g_{\mu\nu}B^{\mu\nu}. \tag{122}$$

Proof of Theorem 17. According to a theorem of V. Fock ([37], p. 429), the contravariant Einstein tensor $G^{\mu\nu}$ can be written as

$$\begin{aligned} G^{\mu\nu} &= \frac{1}{2g_0}\tilde{g}^{\alpha\beta}\frac{\partial^2\tilde{g}^{\mu\nu}}{\partial x^\alpha\partial x^\beta} + \Pi^{\mu,\alpha\beta}\Pi^\nu_{\alpha\beta} - \frac{1}{2}y^\mu y^\nu \\ &\quad + \frac{1}{2}g^{\mu\nu}(L+B) - B^{\mu\nu}. \end{aligned} \tag{123}$$

Applying Eqs. (111), Eqs. (123) can be written as

$$\begin{aligned} G^{\mu\nu} &= \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta} - \eta^{\alpha\beta}\right)\frac{\partial^2(-2f_0\phi^{\mu\nu})}{\partial x^\alpha\partial x^\beta} \\ &\quad -\frac{f_0}{g_0}\eta^{\alpha\beta}\frac{\partial^2\phi^{\mu\nu}}{\partial x^\alpha\partial x^\beta} + \Pi^{\mu,\alpha\beta}\Pi^\nu_{\alpha\beta} \\ &\quad -\frac{1}{2}y^\mu y^\nu + \frac{1}{2}g^{\mu\nu}(L+B) - B^{\mu\nu}. \end{aligned} \tag{124}$$

Noticing Eqs. (111), the field equations (99) can be written as

$$\begin{aligned} \eta^{\alpha\beta}\frac{\partial^2\phi^{\mu\nu}}{\partial x^\alpha\partial x^\beta} &= -\frac{\sqrt{-g_0}}{2f_0}\left(\partial_\lambda\partial^\mu g^{\nu\lambda} + \partial_\lambda\partial^\nu g^{\mu\lambda}\right. \\ &\quad \left. - \eta^{\mu\nu}\partial_\sigma\partial_\lambda g^{\sigma\lambda}\right) - f_0(T^{\mu\nu} - T^{\mu\nu}_\omega). \end{aligned} \tag{125}$$

Using Eqs. (111) and Eqs. (125), Eqs. (124) can be written as

$$\begin{aligned} G^{\mu\nu} &= \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta} - \eta^{\alpha\beta}\right)\frac{\partial^2(\sqrt{-g_0}g^{\mu\nu})}{\partial x^\alpha\partial x^\beta} \\ &\quad + \frac{\sqrt{-g_0}}{2g_0}(\partial_\lambda\partial^\mu g^{\nu\lambda} + \partial_\lambda\partial^\nu g^{\mu\lambda} - \eta^{\mu\nu}\partial_\sigma\partial_\lambda g^{\sigma\lambda}) \\ &\quad + \frac{f_0^2}{g_0}(T^{\mu\nu} - T^{\mu\nu}_\omega) + \Pi^{\mu,\alpha\beta}\Pi^\nu_{\alpha\beta} - \frac{1}{2}y^\mu y^\nu \\ &\quad + \frac{1}{2}g^{\mu\nu}(L+B) - B^{\mu\nu}. \end{aligned} \tag{126}$$

Eqs. (126) can be written as Eqs. (115). □

Eqs. (115) have the same form in all inertial reference frames. Eqs. (115) is one of the main results in this manuscript. We need to further study the relationship between Eqs. (115) and the Einstein field equations (1).

Theorem 18 *If we impose the Hilbert gauge Eqs. (89) on the fields, then in an arbitrary inertial reference frame S_i we have the following field equations*

$$\begin{aligned} G^{\mu\nu} &- \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta} - \eta^{\alpha\beta}\right)\frac{\partial^2(\sqrt{-g_0}g^{\mu\nu})}{\partial x^\alpha\partial x^\beta} \\ &- \Pi^{\mu,\alpha\beta}\Pi^\nu_{\alpha\beta} + \frac{1}{2}y^\mu y^\nu - \frac{1}{2}g^{\mu\nu}(L+B) \\ &+ B^{\mu\nu} = \frac{f_0^2}{g_0}(T^{\mu\nu} - T^{\mu\nu}_\omega). \end{aligned} \tag{127}$$

Proof of Theorem 18. Using Eqs. (111) and Eqs. (98), Eqs. (124) can be written as

$$\begin{aligned} G^{\mu\nu} &= \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta} - \eta^{\alpha\beta}\right)\frac{\partial^2(\sqrt{-g_0}g^{\mu\nu})}{\partial x^\alpha\partial x^\beta} \\ &\quad + \frac{f_0^2}{g_0}(T^{\mu\nu} - T^{\mu\nu}_\omega) + \Pi^{\mu,\alpha\beta}\Pi^\nu_{\alpha\beta} - \frac{1}{2}y^\mu y^\nu \\ &\quad + \frac{1}{2}g^{\mu\nu}(L+B) - B^{\mu\nu}. \end{aligned} \tag{128}$$

Eqs. (128) can be written as Eqs. (127). □

Definition 19 *If each of the coordinates x^α satisfies the following generalized wave equations*

$$\frac{1}{\sqrt{-g_0}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g_0}g^{\mu\nu}\frac{\partial x^\alpha}{\partial x^\nu}\right) = 0, \tag{129}$$

then, we call such a coordinates system harmonic.

In a harmonic coordinates system, we have ([37], p. 254)

$$\Gamma^\nu = \Gamma^{\mu\nu} = B^{\mu\nu} = B = 0. \tag{130}$$

Putting Eqs. (130) into Eqs. (127), we have the following corollary.

Corollary 20 *If we apply the Hilbert gauge Eqs. (89) and the coordinates system is harmonic, then the field equations (127) can be written as*

$$\begin{aligned} G^{\mu\nu} &- \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta} - \eta^{\alpha\beta}\right)\frac{\partial^2(\sqrt{-g_0}g^{\mu\nu})}{\partial x^\alpha\partial x^\beta} \\ &- \Pi^{\mu,\alpha\beta}\Pi^\nu_{\alpha\beta} + \frac{1}{2}y^\mu y^\nu - \frac{1}{2}g^{\mu\nu}L \\ &= \frac{f_0^2}{g_0}(T^{\mu\nu} - T^{\mu\nu}_\omega). \end{aligned} \tag{131}$$

We can verify that each of the Galilean coordinates is harmonic. Any constant and any linear function of harmonic coordinates satisfy Eqs. (129). Thus, from Eqs. (97) we see that an inertial reference frame is harmonic and Eqs. (131) are valid for every inertial system. In order to study the case of weak fields in inertial systems, we introduce the following assumption.

Assumption 21 *Suppose that the dimensionless parameter $\varpi = m_0c/2\rho_0q_0$ satisfies the following condition*

$$\varpi = \frac{m_0c}{2\rho_0q_0} \le 1. \tag{132}$$

Using the 00 component of Eqs. (114) for the case $n = 1$ and noticing Eqs. (87), Eqs. (29), Eqs. (51) and Eqs. (9), we have

$$\left|\frac{\partial(2f_0\phi^{00})}{\partial(x^\alpha)}\right| = \left|\frac{2\rho_0q_0}{m_0c^2}\frac{\partial\varphi}{\partial x^\alpha}\right| \ll 1. \tag{133}$$

Noticing Eq. (27) and using Eq. (133) and Eq. (132), we have $|u| \approx |\nabla\varphi| \ll m_0c^2/(2\rho_0q_0) \le c$. Therefore, according to Assumption 7, Eqs. (75) and Eqs. (76) are valid for weak fields.

Corollary 22 *Suppose that (1) the Hilbert gauge Eqs. (89) is applied on the fields; (2) the filed is weak; (3) Assumption 7 is valid. Then in an arbitrary inertial reference frame the field equations (131) reduce to*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{f_0^2}{g_0} \left(T^m_{\mu\nu} + T^{\Omega(1)}_{\mu\nu} \right). \tag{134}$$

Proof of Corollary 22. According to Definition 16, $f_0 \phi^{\mu\nu}$ and their first and higher derivatives are small quantities of order ε, where $|\varepsilon| \ll 1$ is a small quantity. Thus, using Eqs. (111) and Eqs. (112), we have the following estimation of the order of magnitude of the following quantities

$$\sqrt{-g_0} g^{\mu\nu} - \eta^{\mu\nu} \sim \varepsilon, \quad \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \sim \frac{\partial g^{\mu\nu}}{\partial x^\alpha} \sim \varepsilon. \tag{135}$$

From Eqs. (111), we have the following estimation of the order of magnitude of the quantity

$$\frac{\partial^2 (\sqrt{-g_0} g^{\alpha\beta})}{\partial x^\alpha \partial x^\beta} = \frac{\partial^2 (-2 f_0 \phi^{\mu\nu})}{\partial x^\alpha \partial x^\beta} \sim \varepsilon. \tag{136}$$

Thus, using Eqs. (135) and Eqs. (136), we have the following estimation of the order of magnitude of the quantity

$$\left(\sqrt{-g_0} g^{\alpha\beta} - \eta^{\alpha\beta} \right) \frac{\partial^2 (\sqrt{-g_0} g^{\alpha\beta})}{\partial x^\alpha \partial x^\beta} \sim \varepsilon^2. \tag{137}$$

From Eqs. (116) and Eqs. (117), we have the following estimation of the order of magnitude of the following quantities

$$\Pi^{\mu,\alpha\beta} \sim \Pi^\nu_{\alpha\beta} \sim \varepsilon. \tag{138}$$

Using Eqs. (120), we have the following relationship ([37], p. 143)

$$y_\beta = \Gamma^\nu_{\beta\nu}. \tag{139}$$

We also have ([37], p. 143)

$$\Gamma^\nu_{\beta\nu} = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\beta}. \tag{140}$$

From Eqs. (139), Eqs. (140) and Eqs. (135), we have the following estimation of the order of magnitude

$$y_\beta \sim \varepsilon. \tag{141}$$

Using Eqs. (120) and Eqs. (141), we have the following estimation of the order of magnitude

$$y^\alpha \sim \varepsilon. \tag{142}$$

From Eq. (121) and Eqs. (130), we have

$$L = -\frac{1}{2} \Gamma^\nu_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial x^\nu}. \tag{143}$$

Using Eq. (143), Eqs. (107) and Eqs. (135), we have the following estimation of the order of magnitude

$$L \sim \varepsilon^2. \tag{144}$$

From Eqs. (137), Eqs. (138), Eqs. (142) and Eq. (144), we see that the second to the fifth term on the right side of Eqs. (131) are all small quantities of order ε^2. Ignoring all these small quantities of order ε^2 in Eqs. (131) and using Eqs. (76), we obtain

$$G^{\mu\nu} \approx \frac{f_0^2}{g_0} \left(T^{\mu\nu}_m + T^{\mu\nu}_{\Omega(1)} \right). \tag{145}$$

Applying the rules of lowering or raising the indexes of tensors, i.e., $G^{\mu\nu} = g^{\mu\sigma} g^{\nu\lambda} G_{\sigma\lambda}$, $T^{\mu\nu}_m = g^{\mu\sigma} g^{\nu\lambda} T^m_{\sigma\lambda}$, $T^{\mu\nu}_{\Omega(1)} = g^{\mu\sigma} g^{\nu\lambda} T^{\Omega(1)}_{\sigma\lambda}$, Eqs. (145) can be written as

$$G_{\lambda\sigma} \approx \frac{f_0^2}{g_0} \left(T^m_{\sigma\lambda} + T^{\Omega(1)}_{\sigma\lambda} \right). \tag{146}$$

Putting Eqs. (109) into Eqs. (146), we obtain Eqs. (134). □

Corollary 23 *Suppose that the following conditions are valid: (1) the Hilbert gauge Eqs. (89) is applied on the fields; (2) the filed is weak; (3) $g_0 \approx -1$; (4) Assumption 7 is valid; (5) $T^{\Omega(1)}_{\mu\nu} \approx 0$. Then in an arbitrary inertial reference frame the field equation Eqs. (134) reduce to*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -f_0^2 T^m_{\mu\nu}. \tag{147}$$

If we introduce the following notation

$$\kappa = f_0^2 = \frac{8\pi\gamma_N}{c^4}, \tag{148}$$

then, Eqs. (147) coincide with Einstein's equations (1). Thus, we see that the field equations (115) are generalizations of the Einstein's equations (1) in inertial reference frames.

X. EQUIVALENCE BETWEEN THE INERTIAL MASS AND THE GRAVITATIONAL MASS

Proposition 24 *The inertial mass of a microscope particle equals it's gravitational mass.*

Proof of Proposition 24. Newton's law of gravitation can be written as ([43], p. 2)

$$\mathbf{F}_{21} = -G_N \frac{m_{g1} m_{g2}}{r^2} \hat{\mathbf{r}}_{21}, \tag{149}$$

where m_{g1} and m_{g2} are the gravitational masses of two particles, G_N is Newton's gravitational constant, $\hat{\mathbf{r}_{21}}$ denotes the unit vector directed outward along the line from the particle with mass m_{g1} to the particle with mass m_{g2}, r is the distance between the two particles,.

In 2008, we show that the force $\mathbf{F}_{21}(t)$ exerted on the particle with inertial mass $m_{i2}(t)$ by the velocity field of the $\Omega(0)$ substratum induced by the particle with inertial mass $m_{i1}(t)$ is [27]

$$\mathbf{F}_{21}(t) = -\gamma_N(t)\frac{m_{i1}(t)m_{i2}(t)}{r^2}\hat{\mathbf{r}}_{21}, \tag{150}$$

where

$$\gamma_N(t) = \frac{\rho_0 q_0^2}{4\pi m_0^2(t)}, \tag{151}$$

ρ_0 is the density of the $\Omega(0)$ substratum or we say the gravitational aether, $m_0(t)$ is the inertial mass of monad at time t, $-q_0(q_0 > 0)$ is the strength of a monad.

Suppose that $G_N = \gamma_N(t)$. Comparing Eq. (149) and Eq. (150), we have

$$m_{i1}m_{i2} = m_{g1}m_{g2}. \tag{152}$$

Now we study a gravitational system of two protons. According to Eq. (152), we have

$$m_{ip}^2 = m_{gp}^2, \tag{153}$$

where m_{ip} and m_{gp} are the inertial mass and gravitational mass of a proton respectively.

Noticing $m_{ip} > 0$ and $m_{gp} > 0$, Eq. (153) can be written as

$$m_{ip} = m_{gp}. \tag{154}$$

Eq. (154) shows the inertial mass m_{ip} of a proton equals it's gravitational mass m_{gp}. Similarly, we can demonstrate that the inertial mass of another type of microscope particle equals it's gravitational mass. □

This result is called the principle of equivalence in the theory of general relativity [1, 2, 36].

XI. THE DYNAMICAL GRAVITATIONAL POTENTIALS IN INERTIAL REFERENCE FRAMES

The purpose of this section is to review the mathematical forms of the dynamical gravitational potentials in inertial reference frames. These results may provide us some clues to explore possible mathematical models of inertial potential and inertial force Lagrangian in non-inertial reference frames, which are introduced in the next section.

The definition of the strength $\mathbf{g}$ of a gravitational field is ([44], p. 24)

$$\mathbf{g} = \frac{\mathbf{F}_g}{m_{test}}, \tag{155}$$

where m_{test} is the mass of a test point particle, $\mathbf{F}_g$ is the gravitational force exerted on the test point particle by a gravitational field.

According to Newton's second law, we have

$$\mathbf{F}_g = m_{test}\mathbf{a}, \tag{156}$$

where $\mathbf{a}$ is the acceleration of the test point particle.

Comparing Eq. (156) and Eq. (155), we have

$$\mathbf{g} = \mathbf{a}. \tag{157}$$

The definition of the acceleration $\mathbf{a}$ is ([44], p. 24)

$$a_i = \gamma_{ik}\frac{d^2x^k}{dt^2}, \tag{158}$$

where

$$\gamma_{ik} = -g_{ik} + \frac{g_{0i}g_{0k}}{g_{00}}. \tag{159}$$

Based on the time tracks of free particles described by geodesic curves in Minkowski spacetime, we have the following results ([36], p. 279;[44], p. 26)

$$a_i = -\frac{\partial \Pi}{\partial x^i} - c\sqrt{1 + \frac{2\Pi}{c^2}}\frac{\partial \gamma_i}{\partial t}, \tag{160}$$

where

$$\Pi = -\frac{1 - g_{00}}{2}c^2, \quad \gamma_i = -\frac{g_{i0}}{\sqrt{g_{00}}}. \tag{161}$$

Eqs. (160) can also be written as

$$a_i = -c^2\frac{\partial}{\partial x^i}\left(-\frac{1 - g_{00}}{2}\right) - c^2\frac{\partial(-g_{i0})}{\partial(ct)}. \tag{162}$$

XII. INERTIAL POTENTIAL AND INERTIAL FORCE LAGRANGIAN IN NON-INERTIAL REFERENCE FRAMES

According to the theory of general relativity [1, 2], the Einstein's equations are valid not only in inertial reference frames and but also in non-inertial reference frames. Thus, it is needed to explore the possibility to derive the Einstein's equations in non-inertial reference frames.

When solving the Einstein's equations for an isolated system of masses, V. Fock introduces harmonic reference frame and obtains an unambiguous solution ([37], p. 369). Furthermore, in the case of an isolated system of masses, he concludes that there exists a harmonic reference frame which is determined uniquely apart from a Lorentz transformation if suitable supplementary conditions are imposed ([37], p. 373). It is known that wave equations keep the same form under Lorentz transformations [36]. Thus, we speculate that Fock's special harmonic reference frames may have provided us a clue to derive the Einstein's equations in some special class of non-inertial reference frames.

We introduce an arbitrary coordinate system $(x'^0,\ x'^1,\ x'^2,\ x'^3)$ and denote it as S_n. It is known that a particle in a non-inertial reference frame will experiences an inertial force. Unfortunately, we have no knowledge about the origin of inertial forces.

The equivalence between inertial mass and gravitational mass implies that to some degree gravitational forces behave in the same way as inertial forces that result from non-inertial reference frames ([43], p. 17). Thus, we speculate that inertial forces may originate from the interactions between matter systems and vacuum. Therefore, we introduce the following assumption.

Assumption 25 *The inertial force exerted on a matter system in a non-inertial reference frame stems from the interactions between the matter system and vacuum.*

Based on Assumption 25, we introduce the following concepts for inertial forces, which are similar to those concepts for gravitational interactions.

Definition 26 *Inertial potential $\psi^{\rm iner}_{\mu\nu}$ is an interaction potential between a matter system and vacuum resulting from the inertial force $\mathbf{F}_{iner}$ exerted on the matter system by vacuum in a non-inertial reference frame S_n.*

Definition 27 *Inertial force Lagrangian $L_{\rm iner}$ is an interaction Lagrangian between a matter system and vacuum resulting from the inertial force $\mathbf{F}_{iner}$ exerted on the matter system by vacuum in a non-inertial reference frame S_n.*

Now our task is to explore possible expressions of inertial potential $\psi^{\rm iner}_{\mu\nu}$ and inertial force Lagrangian $L_{\rm iner}$. Similar to Eqs. (162), the inertial acceleration $\mathbf{a}$ of a test point particle in the non-inertial reference frame S_n can be written as

$$a_i = -c^2\frac{\partial}{\partial x'^i}\left(-\frac{1-\eta'_{00}}{2}\right) - c^2\frac{\partial(-\eta'_{i0})}{\partial(x'^0)}, \qquad (163)$$

where $\eta'_{\mu\nu}$ is the corresponding metric tensor of the non-inertial reference frame S_n.

If η'_{i0} are time-independent, the inertial acceleration $\mathbf{a}$ of the test point particle in Eqs. (163) simplifies to ([36], p. 280)

$$a_i = -c^2\frac{\partial}{\partial x'^i}\left(-\frac{1-\eta'_{00}}{2}\right). \qquad (164)$$

Using Eqs. (164), the inertial force $\mathbf{F}_{iner}$ exerted on the test point particle can be written as

$$\mathbf{F}_{iner} = m\mathbf{a} = -mc^2\nabla'\left(-\frac{1-\eta'_{00}}{2}\right), \qquad (165)$$

where m is the mass of the test point particle, $\nabla' = \mathbf{i}'\partial/\partial {x'}^1 + \mathbf{j}'\partial/\partial {x'}^2 + \mathbf{k}'\partial/\partial {x'}^3$ is the corresponding Hamilton operator in the non-inertial reference frame $(x'^0,\ x'^1,\ x'^2,\ x'^3)$.

From Eq. (165), the inertial force Lagrangian of a system of vacuum and the test point particle can be written as

$$L_{\rm iner1} = mc^2\left(-\frac{1-\eta'_{00}}{2}\right). \qquad (166)$$

Therefore, the inertial force Lagrangian of a system of vacuum and continuously distributed particles can be written as

$$L_{\rm iner} = \rho_m c^2\left(-\frac{1-\eta'_{00}}{2}\right). \qquad (167)$$

Noticing $T^{00}_m = \rho_m c^2$ and $\eta_{00} = 1$, the inertial force Lagrangian $L_{\rm iner}$ in Eq. (167) can be written as

$$L_{\rm iner} = f_0\psi^{\rm iner}_{00}T^{00}_m, \qquad (168)$$

where

$$\psi^{\rm iner}_{00} = -\frac{1}{2f_0}(\eta_{00} - \eta'_{00}). \qquad (169)$$

Following Ref. [46], the parameter f_0 is

$$f_0 = \sqrt{\frac{2\rho_0 q_0^2}{m_0^2c^4}} = \sqrt{\frac{8\pi\gamma_N}{c^4}}. \qquad (170)$$

Inspired by Eqs. (168) and Eq. (169), we introduce the following assumption.

Assumption 28 *Suppose that the inertial force Lagrangian $L_{\rm iner}$ of a system of a free point particle and vacuum in the non-inertial reference frame S_n can be written as*

$$L_{\rm iner} = f_0\psi^{\rm iner}_{\mu\nu}mu'^\mu u'^\nu, \qquad (171)$$

where m is the rest mass of the point particle, $u'^\mu \equiv d{x'}^\mu/d\tau_{\eta'}$, $\tau_{\eta'}$ is the proper time,

$$\psi^{\rm iner}_{\mu\nu} = -\frac{1}{2f_0}(\eta_{\mu\nu} - \eta'_{\mu\nu}). \qquad (172)$$

Following similar methods in [46], we obtain the following result.

Proposition 29 *Suppose that Assumption 28 is valid. Then, the equations of motion of a free point particle can be written as*

$$\frac{d^2{x'}^\mu}{d\tau^2_{\eta'}} + {C'}^\mu_{\nu\sigma}\frac{d{x'}^\nu}{d\tau_{\eta'}}\frac{d{x'}^\sigma}{d\tau_{\eta'}} = 0, \qquad (173)$$

where

$${C'}^\nu_{\alpha\beta} \triangleq \frac{1}{2}{\eta'}^{\mu\nu}\left(\frac{\partial\eta'_{\mu\alpha}}{\partial x'^\beta} + \frac{\partial\eta'_{\mu\beta}}{\partial x'^\alpha} - \frac{\partial\eta'_{\alpha\beta}}{\partial x'^\mu}\right) \qquad (174)$$

are the corresponding Christoffel symbols in the non-inertial reference frame S_n.

Proof of Proposition 29. The Lagrangian of a free point particle in S_n can be written as ([4], p. 57;[34])

$$L'_0 = \frac{1}{2}m\frac{d{x'}^\mu}{d\tau_{\eta'}}\frac{dx'_\mu}{d\tau_{\eta'}} = \frac{1}{2}m{u'}^\mu u'_\mu = \frac{1}{2}m\eta'_{\mu\nu}{u'}^\mu{u'}^\nu, \qquad (175)$$

where m is the rest mass of the point particle, $d\tau \equiv \frac{1}{c}\sqrt{dx'^{\mu}dx'_{\mu}}$ is the infinitesimal proper time interval, $u'^{\mu} \equiv dx'^{\mu}/d\tau_{\eta'}$.

Suppose that $T'^{\mu\nu}_{\Omega(1)} \approx 0$. Using Eq. (175) and Eq. (171), the total Lagrangian $L_{\rm p}$ of a system of the $\Omega(0)$ substratum, the $\Omega(1)$ substratum and the point particle can be written as

$$L_{\rm p} = L'_0 + L_{\rm iner} = \frac{1}{2}mu'^{\mu}u'_{\mu} + f_0\psi^{\rm iner}_{\mu\nu}mu'^{\mu}u'^{\nu}. \quad (176)$$

The Euler-Lagrange equations for the total Lagrangian $L_{\rm p}$ can be written as ([43], p. 111)

$$\frac{\partial L_{\rm p}}{\partial x'^{\mu}} - \frac{d}{d\tau_{\eta'}}\frac{\partial L_{\rm p}}{\partial u'^{\mu}} = 0. \quad (177)$$

Putting Eq. (176) into Eqs. (177), we have

$$\frac{d}{d\tau_{\eta'}}\left[\left(\eta_{\mu\nu} + 2f_0\psi^{\rm iner}_{\mu\nu}\right)\frac{dx'^{\nu}}{d\tau_{\eta'}}\right] - f_0\frac{\partial\psi^{\rm iner}_{\alpha\beta}}{\partial x'^{\mu}}\frac{dx'^{\alpha}}{d\tau_{\eta'}}\frac{dx'^{\beta}}{d\tau_{\eta'}} = 0. \quad (178)$$

Using Eq. (172), Eqs. (178) can be written as

$$\frac{d}{d\tau_{\eta'}}\left(\eta'_{\mu\nu}\frac{dx'^{\nu}}{d\tau_{\eta'}}\right) - \frac{1}{2}\frac{\partial\eta'_{\alpha\beta}}{\partial x'^{\mu}}\frac{dx'^{\alpha}}{d\tau_{\eta'}}\frac{dx'^{\beta}}{d\tau_{\eta'}} = 0. \quad (179)$$

Eqs. (179) represent a geodesic line in a Riemannian spacetime with a metric tensor $\eta'_{\mu\nu}$, which can also be written as Eqs. (173) ([44], p. 50). □

Eqs. (173) is a geodesic curve in a Minkowski spacetime. It is known that a geodesic curve is a straight line in a Minkowski spacetime ([47], p. 235). For instance, according to Newton's first law, a free particle moves along a straight line in the Galilean coordinates. Therefore, Assumption 28 may be supports by some experiments. Thus, inspired by the inertial force Lagrangian for a free point particle in Eq. (171), we introduce the following assumption for a matter system.

Assumption 30 *The inertial force Lagrangian $L_{\rm iner}$ of a matter system and vacuum in the non-inertial reference frame S_n can be written as*

$$L_{\rm iner} = f_0\psi^{\rm iner}_{\mu\nu}(T'^{\mu\nu}_m + T'^{\mu\nu}_{\Omega(1)}) + O[(f_0\psi^{\rm iner}_{\mu\nu})^2], \quad (180)$$

where $T'^{\mu\nu}_m$ and $T'^{\mu\nu}_{\Omega(1)}$ are the contravariant energy-momentum tensors of the system of the matter and the $\Omega(1)$ substratum respectively, $O[(f_0\psi^{\rm iner}_{\mu\nu})^2]$ denotes those terms which are small quantities of the order of $(f_0\psi^{\rm iner}_{\mu\nu})^2$.

XIII. FIELD EQUATIONS IN A SPECIAL CLASS OF NON-INERTIAL REFERENCE FRAMES

Suppose that the transformation equations between a non-inertial coordinate system $(x'^0,\ x'^1,\ x'^2,\ x'^3)$ and the Galilean coordinates (ct, x, y, z) are

$$x'^{\alpha} = f^{\alpha}(x^0,\ x^1,\ x^2,\ x^3). \quad (181)$$

Following V. Fock ([37], p. 370-373), we introduce the following definition of a special class of reference frames.

Definition 31 *Suppose that a coordinate system $(x'^0,\ x'^1,\ x'^2,\ x'^3)$ satisfies the following conditions: (1) every coordinates x'^{α} satisfies the d'Alembert's equation ([37], p. 369), i.e.,*

$$\Box_{\eta'}x'^{\alpha} \triangleq \frac{1}{\sqrt{-\eta'_0}}\frac{\partial}{\partial x'^{\mu}}\left(\sqrt{-\eta'_0}\eta'^{\mu\nu}\frac{\partial x'^{\alpha}}{\partial x'^{\nu}}\right) = 0, \quad (182)$$

where $\eta'_{\mu\nu}$ is the metric of the reference frame S_n, $\eta'_0 =$ Det $\eta'_{\mu\nu}$; (2) every coordinates x'^{α} converges to the Galilean coordinates (ct, x, y, z) at large enough distance, i.e.,

$$\lim_{r\to\infty} x'^{\alpha} = x^{\alpha}, \quad (183)$$

where $r = \sqrt{x^2+y^2+z^2}$; (3) $\eta'^{\mu\nu} - (\eta'^{\mu\nu})_{\infty}$ are outgoing waves, i.e., $\eta'^{\mu\nu} - (\eta'^{\mu\nu})_{\infty}$ satisfy the following condition of outward radiation: for $r\to\infty$, and all values of $t'_0 = t + r/c$ in an arbitrary fixed interval the following limiting conditions are satisfied ([37], p. 365)

$$\lim_{r\to\infty}\left[\frac{\partial[r(\eta'^{\mu\nu} - (\eta'^{\mu\nu})_{\infty})]}{\partial r} + \frac{1}{c}\frac{\partial[r(\eta'^{\mu\nu} - (\eta'^{\mu\nu})_{\infty})]}{\partial t}\right] = 0, \quad (184)$$

where $(\eta'^{\mu\nu})_{\infty}$ denotes the value of $\eta'^{\mu\nu}$ at infinity. Then, we call this coordinate system $(x'^0,\ x'^1,\ x'^2,\ x'^3)$ as a Fock coordinate system.

We use S_F to denote a Fock coordinate system. The Galilean coordinate system (ct, x, y, z) is a Fock coordinate system. V. Fock points out an advantage of Fock coordinate system:"When solving Einstein's equations for an isolated system of masses we used harmonic coordinates and in this way obtained a perfectly unambiguous solution." ([37], p. 369) Here the harmonic coordinates called by V. Fock are Fock coordinate systems.

According to a theorem of Fock about Fock coordinate systems ([37], p. 369-373), the transformation equations (181) from one Fock coordinate system to another can be written as a Lorentz transformation, i.e.,

$$x'^{\mu} = a^{\mu}{}_{\nu}x^{\nu}, \quad (185)$$

where $a^{\mu}{}_{\nu}$ are coefficients of a Lorentz transformation.

For convenience, we introduce the following noatons

$$\partial'_{\mu} \equiv \left(\frac{\partial}{\partial x'^0}, \frac{\partial}{\partial x'^1}, \frac{\partial}{\partial x'^2}, \frac{\partial}{\partial x'^3}\right), \quad \partial'^{\mu} \equiv \eta'^{\mu\nu}\partial'_{\nu}. \quad (186)$$

Proposition 32 *Suppose that the reference frame S_F is a Fock coordinate system and Assumptions 30 is valid, then the total Lagrangian $L'_{\rm tot}$ of a system of the $\Omega(0)$*

substratum, the $\Omega(1)$ substratum, vacuum and matter in S_F can be written as

$$
\begin{aligned}
L'_{\rm tot} &= \frac{1}{2}\partial'_\lambda\psi'_{\mu\nu}\partial'^\lambda\psi'^{\mu\nu} - 2\partial'_\lambda\psi'_{\mu\nu}\partial'^\mu\psi'^{\lambda\nu} - 6\partial'^\mu\psi'_{\mu\nu}\partial'^\nu\psi' \\
&-\frac{3}{2}\partial'_\lambda\psi'\partial'^\lambda\psi' + L'_{\rm more} + f_0\psi'_{\mu\nu}(T'^{\mu\nu}_m + T'^{\mu\nu}_{\Omega(1)}) \\
&+f_0\psi^{\rm iner}_{\mu\nu}(T'^{\mu\nu}_m + T'^{\mu\nu}_{\Omega(1)}) + O[(f_0\psi^{\rm iner}_{\mu\nu})^2] \\
&+O[(f_0\psi'_{\mu\nu})^2], \qquad (187)
\end{aligned}
$$

where $L'_{\rm more}$ denotes those terms involving more than two derivatives of $\psi'_{\mu\nu}$, $O[(f_0\psi'_{\mu\nu})^2]$ denotes those terms which are small quantities of the order of $(f_0\psi'_{\mu\nu})^2$.

Proof of Proposition 32. Based on some assumptions, the total Lagrangian $L_{\rm tot}$ of a system of the $\Omega(0)$ substratum, the $\Omega(1)$ substratum and matter can be written as [46]

$$
\begin{aligned}
L_{\rm tot} &= \frac{1}{2}\partial_\lambda\psi_{\mu\nu}\partial^\lambda\psi^{\mu\nu} - 2\partial_\lambda\psi_{\mu\nu}\partial^\mu\psi^{\lambda\nu} - 6\partial^\mu\psi_{\mu\nu}\partial^\nu\psi \\
&-\frac{3}{2}\partial_\lambda\psi\partial^\lambda\psi + L_{\rm more} + f_0\psi_{\mu\nu}(T^{\mu\nu}_m + T^{\mu\nu}_{\Omega(1)}) \\
&+O[(f_0\psi_{\mu\nu})^2]. \qquad (188)
\end{aligned}
$$

The total Lagrangian $L'_{\rm tot}$ can be written as

$$L'_{\rm tot} = L_{\rm tot} + L_{\rm iner}. \qquad (189)$$

Similar to the case of inertial reference frames ([43], p. 59-60, 63), we also have the following results in the Fock coordinate system S_F

$$\partial'_\lambda = a_\lambda{}^\sigma\partial_\sigma, \quad \partial'^\lambda = a^\lambda{}_\sigma\partial^\sigma, \qquad (190)$$

$$\psi'^{\mu\nu} = a^\mu{}_\alpha a^\nu{}_\beta\psi^{\alpha\beta}, \qquad (191)$$

$$\psi'_{\mu\nu} = a_\mu{}^\alpha a_\nu{}^\beta\psi_{\alpha\beta}. \qquad (192)$$

The first term on the right hand side of Eqs. (187) can be written as

$$
\begin{aligned}
\frac{1}{2}\partial'_\lambda\psi'_{\mu\nu}\partial'^\lambda\psi'^{\mu\nu} &= \frac{1}{2}(a_\lambda{}^\sigma\partial_\sigma)(a_\mu{}^\alpha a_\nu{}^\beta\psi_{\alpha\beta}) \\
&\cdot(a^\lambda{}_\sigma\partial^\sigma)(a^\mu{}_\alpha a^\nu{}_\beta\psi^{\alpha\beta}). \qquad (193)
\end{aligned}
$$

We have the following result ([43], p. 60)

$$a_\beta{}^\mu a^\beta{}_\nu = \delta^\mu_\nu, \qquad (194)$$

where δ^μ_ν is the Kronecker delta.

Using Eq. (194), Eqs. (193) can be written as

$$\frac{1}{2}\partial'_\lambda\psi'_{\mu\nu}\partial'^\lambda\psi'^{\mu\nu} = \frac{1}{2}\partial_\sigma\psi_{\alpha\beta}\partial^\sigma\psi^{\alpha\beta}. \qquad (195)$$

Similarly, we can verify the following results

$$
\begin{aligned}
-2\partial'_\lambda\psi'_{\mu\nu}\partial'^\mu\psi'^{\lambda\nu} &= -2\partial_\sigma\psi_{\alpha\beta}\partial^\alpha\psi^{\sigma\beta}, \qquad (196)\\
-6\partial'^\mu\psi'_{\mu\nu}\partial'^\nu\psi' &= -6\partial^\alpha\psi_{\alpha\beta}\partial^\beta\psi, \qquad (197)\\
-\frac{3}{2}\partial'_\lambda\psi'\partial'^\lambda\psi' &= -\frac{3}{2}\partial_\sigma\psi\partial^\sigma\psi, \qquad (198)\\
f_0\psi'_{\mu\nu}T'^{\mu\nu}_m &= f_0\psi_{\alpha\beta}T^{\alpha\beta}_m, \qquad (199)\\
L'_{\rm more} &= L_{\rm more}, \qquad (200)\\
O[(f_0\psi'_{\mu\nu})^2] &= O[(f_0\psi_{\mu\nu})^2]. \qquad (201)
\end{aligned}
$$

Putting Eq. (188) and Eq. (180) into Eq. (189) and using Eqs. (195-201), we obtain Eq. (187). □

Applying similar methods in Ref. [46], we have the following result.

Theorem 33 *If we ignore those terms which are small quantities of the order of $(f_0\psi'_{\mu\nu})^2$ and $(f_0\psi^{\rm iner}_{\mu\nu})^2$ and those terms involving more than two derivatives of $\psi'_{\mu\nu}$ in Eq. (187), i.e., $O[(f_0\psi'_{\mu\nu})^2]$, $O[(f_0\psi^{\rm iner}_{\mu\nu})^2]$ and $L'_{\rm more}$, then the field equations for the total Lagrangian $L'_{\rm tot}$ in Eq. (187) can be written as*

$$
\begin{aligned}
&\partial'_\sigma\partial'^\sigma\psi'_{\alpha\beta} - 2(\partial'^\sigma\partial'_\alpha\psi'_{\beta\sigma} + \partial'^\sigma\partial'_\beta\psi'_{\alpha\sigma}) \\
&-6(\eta'_{\alpha\beta}\partial'_\sigma\partial'_\lambda\psi'^{\sigma\lambda} + \partial'_\alpha\partial'_\beta\psi') - 3\eta'_{\alpha\beta}\partial'_\sigma\partial'^\sigma\psi' \\
&= f_0\left(T'^m_{\alpha\beta} + T'^{\Omega(1)}_{\alpha\beta}\right). \qquad (202)
\end{aligned}
$$

Proof of Theorem 33. We have the following Euler-Lagrange equations [42]

$$\frac{\partial L'_{\rm tot}}{\partial\psi'^{\alpha\beta}} - \frac{\partial}{\partial x'^\sigma}\left(\frac{\partial L'_{\rm tot}}{\partial(\partial'_\sigma\psi'^{\alpha\beta})}\right) = 0. \qquad (203)$$

We have the following results

$$
\begin{aligned}
\frac{\partial(\partial'_\lambda\psi'_{\mu\nu}\partial'^\lambda\psi'^{\mu\nu})}{\partial(\partial'_\sigma\psi'^{\alpha\beta})} &= \frac{\partial(\partial'_\lambda\psi'_{\mu\nu})}{\partial(\partial'_\sigma\psi'^{\alpha\beta})}\partial'^\lambda\psi'^{\mu\nu} \\
&+ \partial'_\lambda\psi'_{\mu\nu}\frac{\partial(\partial'^\lambda\psi'^{\mu\nu})}{\partial(\partial'_\sigma\psi'^{\alpha\beta})}, \qquad (204)
\end{aligned}
$$

$$\psi'_{\mu\nu} = \eta'_{\mu\rho}\eta'_{\nu\tau}\psi'^{\rho\tau}. \qquad (205)$$

Using Eqs. (205), we have

$$
\begin{aligned}
\frac{\partial(\partial'_\lambda\psi'_{\mu\nu})}{\partial(\partial'_\sigma\psi'^{\alpha\beta})} &= \frac{\partial}{\partial(\partial'_\sigma\psi'^{\alpha\beta})}\left(\eta'_{\mu\rho}\eta'_{\nu\tau}\partial'_\lambda\psi'^{\rho\tau}\right) \\
&= \eta'_{\mu\rho}\eta'_{\nu\tau}\frac{\partial(\partial'_\lambda\psi'^{\rho\tau})}{\partial(\partial'_\sigma\psi'^{\alpha\beta})} \\
&= \eta'_{\mu\rho}\eta'_{\nu\tau}\delta^\lambda_\sigma\delta^\rho_\alpha\delta^\tau_\beta \\
&= \eta'_{\mu\alpha}\eta'_{\nu\beta}\delta^\lambda_\sigma. \qquad (206)
\end{aligned}
$$

Using Eqs. (206), the first term on the right hand side of Eqs. (204) can be written as

$$\frac{\partial(\partial'_\lambda\psi'_{\mu\nu})}{\partial(\partial'_\sigma\psi'^{\alpha\beta})}\partial'^\lambda\psi'^{\mu\nu} = \partial'^\sigma\psi'_{\alpha\beta}. \qquad (207)$$

Similarly, the second term on the right hand side of Eqs. (204) can be written as

$$\partial'_\lambda\psi'_{\mu\nu}\frac{\partial(\partial'^\lambda\psi'^{\mu\nu})}{\partial(\partial'_\sigma\psi'^{\alpha\beta})} = \partial'^\sigma\psi'_{\alpha\beta}. \qquad (208)$$

Using Eqs. (207) and Eqs. (208), we have

$$\frac{\partial}{\partial x'^\sigma}\left[\frac{\partial(\partial'_\lambda\psi'_{\mu\nu}\partial'^\lambda\psi'^{\mu\nu})}{\partial(\partial'_\sigma\psi'^{\alpha\beta})}\right] = 2\partial'_\sigma\partial'^\sigma\psi'_{\alpha\beta}. \qquad (209)$$

Similarly, we can verify the following results

$$\frac{\partial}{\partial x'^{\sigma}}\left[\frac{\partial(\partial'_{\lambda}\psi'_{\mu\nu}\partial'^{\mu}\psi'^{\lambda\nu})}{\partial(\partial'_{\sigma}\psi'^{\alpha\beta})}\right] = \partial'^{\sigma}\partial'_{\alpha}\psi'_{\beta\sigma} +\partial'^{\sigma}\partial'_{\beta}\psi'_{\alpha\sigma}, \qquad (210)$$

$$\frac{\partial}{\partial x'^{\sigma}}\left[\frac{\partial(\partial'^{\mu}\psi'_{\mu\nu}\partial'^{\nu}\psi)}{\partial(\partial'_{\sigma}\psi'^{\alpha\beta})}\right] = \partial'_{\alpha}\partial'_{\beta}\psi' +\eta'_{\alpha\beta}\partial'_{\sigma}\partial'_{\lambda}\psi'^{\sigma\lambda}, \qquad (211)$$

$$\frac{\partial}{\partial x'^{\sigma}}\left[\frac{\partial(\partial'_{\lambda}\psi'\partial'^{\lambda}\psi')}{\partial(\partial'_{\sigma}\psi'^{\alpha\beta})}\right] = 2\eta'_{\alpha\beta}\partial'_{\sigma}\partial'^{\sigma}\psi', \qquad (212)$$

$$\frac{\partial L'_{\rm tot}}{\partial\psi'^{\alpha\beta}} = f_0(T'^{m}_{\alpha\beta}+T'^{\Omega(1)}_{\alpha\beta}) \qquad (213)$$

Putting Eq. (187) into Eqs. (203) and using Eqs. (209-213), we obtain Eqs. (202). □

Following Ref. [34], we introduce the following notation in the Fock coordinate system S_F.

$$\Psi'^{\mu\nu} = \partial'_{\lambda}\partial'^{\lambda}\psi'^{\mu\nu} - 2\partial'_{\lambda}\partial'^{\mu}\psi'^{\nu\lambda} - 2\partial'_{\lambda}\partial'^{\nu}\psi'^{\mu\lambda} -6\eta'^{\mu\nu}\partial'_{\sigma}\partial'_{\lambda}\psi'^{\sigma\lambda} - 6\partial'^{\mu}\partial'^{\nu}\psi' - 3\eta'^{\mu\nu}\partial'_{\lambda}\partial'^{\lambda}\psi' \qquad (214)$$

Thus, the field equations (202) can be written as

$$\Psi'^{\mu\nu} = f_0(T'^{\mu\nu}_{m} + T'^{\mu\nu}_{\Omega(1)}). \qquad (215)$$

For convenience, we introduce the following definition of the total energy-momentum tensor $T'^{\mu\nu}$ of the system of the matter, the $\Omega(1)$ substratum and the $\Omega(0)$ substratum in a Fock coordinate system S_F

$$T'^{\mu\nu} = T'^{\mu\nu}_{m} + T'^{\mu\nu}_{\Omega(1)} + T'^{\mu\nu}_{\Omega(0)}, \qquad (216)$$

where $T'^{\mu\nu}_{\Omega(0)}$ is the energy-momentum tensor of the $\Omega(0)$ substratum in the Fock coordinate system S_F.

Adding the term $f_0 T'^{\mu\nu}_{\Omega(0)}$ on both sides of Eqs. (215) and using Eqs. (216), the field equations (215) can be written as

$$\Psi'^{\mu\nu} + f_0 T'^{\mu\nu}_{\Omega(0)} = f_0 T'^{\mu\nu}. \qquad (217)$$

For the total system of matter, the $\Omega(1)$ substratum and the $\Omega(0)$ substratum, the law of conservation of energy and momentum is ([36], p. 169; [38], p. 155)

$$\partial'_{\mu} T'^{\mu\nu} = 0. \qquad (218)$$

Comparing Eqs. (218) and Eqs. (217), we have

$$\partial'_{\mu}(\Psi'^{\mu\nu} + f_0 T'^{\mu\nu}_{\Omega(0)}) = 0. \qquad (219)$$

We introduce the following notation in S_F

$$\Theta'^{\mu\nu} = \partial'_{\lambda}\partial'^{\lambda}\psi'^{\mu\nu} - (\partial'_{\lambda}\partial'^{\mu}\psi'^{\nu\lambda} + \partial'_{\lambda}\partial'^{\nu}\psi'^{\mu\lambda}) +(\partial'^{\mu}\partial'^{\nu}\psi' + \eta'^{\mu\nu}\partial'_{\sigma}\partial'_{\lambda}\psi'^{\sigma\lambda}) - \eta'^{\mu\nu}\partial'_{\lambda}\partial'^{\lambda}\psi'. \qquad (220)$$

We can verify the following result ([4], p. 44; [34])

$$\partial'_{\mu}\Theta'^{\mu\nu} = 0. \qquad (221)$$

From Eqs. (221) and Eqs. (219), we have

$$\partial'_{\mu}\left(\frac{1}{f_0}\Psi'^{\mu\nu} - \frac{b_0}{f_0}\Theta'^{\mu\nu} + T'^{\mu\nu}_{\Omega(0)}\right) = 0. \qquad (222)$$

where b_0 is an arbitrary parameter.

Following Ref. [34], we introduce the following definition of the contravariant energy-momentum tensor $T'^{\mu\nu}_{\Omega}$ of vacuum in S_F

$$T'^{\mu\nu}_{\omega} = \frac{1}{f_0}\Psi'^{\mu\nu} + \frac{1}{f_0}\Theta'^{\mu\nu} + T'^{\mu\nu}_{\Omega(0)}. \qquad (223)$$

Proposition 34 *In the low velocity limit, i.e., $u/c \ll 1$, where $u = |\boldsymbol{u}|$, $\boldsymbol{u}$ is the velocity of the $\Omega(0)$ substratum, the following relationships are valid*

$$\Psi'^{\mu\nu} + \Theta'^{\mu\nu} \approx 0, \qquad (224)$$

where b_0 is a parameter to be determined.

Proof of Proposition 34. In the low velocity limit, i.e., $u/c \ll 1$, where $u = |\mathbf{u}|$, $\mathbf{u}$ is the velocity of the $\Omega(0)$ substratum, the following relationships are valid in an inertial reference frame S_{iner} [46]

$$\Psi^{\alpha\beta} + \Theta^{\alpha\beta} \approx 0. \qquad (225)$$

Similar to the case of inertial reference frames ([43], p. 59-60), we also have the following results in the Fock coordinate system S_F

$$\Psi^{\alpha\beta} = b_{\alpha}{}^{\mu} b_{\beta}{}^{\nu}\Psi'^{\mu\nu}, \qquad (226)$$

$$\Theta^{\alpha\beta} = b_{\alpha}{}^{\mu} b_{\beta}{}^{\nu}\Theta'^{\mu\nu}, \qquad (227)$$

where $b^{\mu}_{\ \nu}$ are coefficients of the Lorentz transformation between the inertial reference frame S_{iner} and the Fock coordinate system S_F.

Putting Eqs. (226) and (227) into Eqs. (225), we have Eqs. (224). □

Using Eqs. (216), Eqs. (223) and Eqs. (224), we have the following estimations of $T'^{\mu\nu} - T'^{\mu\nu}_{\omega}$ in the low velocity limit

$$T'^{\mu\nu} - T'^{\mu\nu}_{\omega} \approx T'^{\mu\nu}_{m} + T'^{\mu\nu}_{\Omega(1)}. \qquad (228)$$

Corollary 35 *The field equations (202) can be written as*

$$\partial'_{\lambda}\partial'^{\lambda}\psi'^{\mu\nu} - \partial'_{\lambda}\partial'^{\mu}\psi'^{\nu\lambda} - \partial'_{\lambda}\partial'^{\nu}\psi'^{\mu\lambda} +\partial'^{\mu}\partial'^{\nu}\psi' + \eta'^{\mu\nu}\partial'_{\sigma}\partial'_{\lambda}\psi'^{\sigma\lambda} -\eta'^{\mu\nu}\partial'_{\lambda}\partial'^{\lambda}\psi' = -f_0(T'^{\mu\nu} - T'^{\mu\nu}_{\omega}). \qquad (229)$$

Proof of Corollary 35. Using Eqs. (223), the field equations (217) can be written as

$$\Theta'^{\mu\nu} = -f_0(T'^{\mu\nu} - T'^{\mu\nu}_{\omega}). \tag{230}$$

Putting Eqs. (220) into Eqs. (230), we obtain Eqs. (229). □

We can verify that the field equations (229) are invariant under the following gauge transformation

$$\psi'^{\mu\nu} \to \psi'^{\mu\nu} + \partial'^{\mu}\Lambda^{\nu} + \partial'^{\nu}\Lambda^{\mu}, \tag{231}$$

where Λ^{μ} is an arbitrary vector field.

We introduce the following definition

$$\phi'^{\mu\nu} = \psi'^{\mu\nu} - \frac{1}{2}\eta'^{\mu\nu}\psi'. \tag{232}$$

Using Eqs. (232), the field equations (229) can be written as

$$\partial'_{\lambda}\partial'^{\lambda}\phi'^{\mu\nu} - \partial'_{\lambda}\partial'^{\mu}\phi'^{\nu\lambda} - \partial'_{\lambda}\partial'^{\nu}\phi^{\mu\lambda} + \eta'^{\mu\nu}\partial'_{\sigma}\partial'_{\lambda}\phi'^{\sigma\lambda} = -f_0(T'^{\mu\nu} - T'^{\mu\nu}_{\omega}). \tag{233}$$

We introduce the following Hilbert gauge condition [34]

$$\partial'_{\mu}\left(\psi'^{\mu\nu} - \frac{1}{2}\eta'^{\mu\nu}\psi'\right) = 0. \tag{234}$$

Using Eqs. (232), the Hilbert gauge condition Eqs. (234) simplifies to

$$\partial'_{\mu}\phi'^{\mu\nu} = 0. \tag{235}$$

If we impose the Hilbert gauge condition Eqs. (235) on the fields, then the field equations Eqs. (233) simplify to

$$\partial'_{\lambda}\partial'^{\lambda}\phi'^{\mu\nu} = -f_0(T'^{\mu\nu} - T'^{\mu\nu}_{\omega}). \tag{236}$$

The field equation (236) can also be written as

$$\eta'^{\alpha\beta}\frac{\partial^2\phi'^{\mu\nu}}{\partial x'^{\alpha}\partial x'^{\beta}} = -f_0(T'^{\mu\nu} - T'^{\mu\nu}_{\omega}). \tag{237}$$

XIV. GENERALIZED EINSTEIN EQUATION IN A SPECIAL CLASS OF NON-INERTIAL REFERENCE FRAMES

Definition 36 *The Einstein tensor $G_{\mu\nu}$ is defined by*

$$G_{\mu\nu} \triangleq R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{238}$$

where $g_{\mu\nu}$ is a metric tensor of a Riemannian spacetime, $R_{\mu\nu}$ is the Ricci tensor, $R \triangleq g^{\mu\nu}R_{\mu\nu}$, $g^{\mu\nu}$ is the corresponding contravariant tensor of $g_{\mu\nu}$ such that $g_{\mu\lambda}g^{\lambda\nu} = \delta^{\nu}_{\mu} = g^{\nu}_{\mu}$ ([44], p. 40).

Similar to Ref. [46], we introduce the following definition of a metric tensor $g_{\mu\nu}$ of a Riemannian spacetime.

Definition 37

$$\tilde{g}^{\mu\nu} \triangleq \sqrt{-g_0}g^{\mu\nu} \triangleq \eta'^{\mu\nu} - 2f_0\phi'^{\mu\nu}, \tag{239}$$

where $g_0 = Det\ g_{\mu\nu}$.

Applying similar methods of V. Fock ([37], p. 422-430), Fock's theorem of the Einstein tensor $G_{\mu\nu}$ in the Galilean coordinates ([37], p. 429) can be generalized to non-inertial coordinate systems (x'^0, x'^1, x'^2, x'^3).

Proposition 38 *The contravariant Einstein tensor $G^{\mu\nu}$ in the non-inertial coordinate systems (x'^0, x'^1, x'^2, x'^3) can be written as*

$$G^{\mu\nu} = \frac{1}{2g_0}\tilde{g}^{\alpha\beta}\frac{\partial^2\tilde{g}^{\mu\nu}}{\partial x'^{\alpha}\partial x'^{\beta}} + \Pi'^{\mu,\alpha\beta}\Pi'^{\nu}_{\alpha\beta} - \frac{1}{2}y'^{\mu}y'^{\nu} + \frac{1}{2}g^{\mu\nu}(L' + B') - B'^{\mu\nu}, \tag{240}$$

where

$$\Pi'^{\mu,\alpha\beta} \triangleq \frac{1}{2g_0}\left(\tilde{g}^{\alpha\lambda}\frac{\partial\tilde{g}^{\mu\beta}}{\partial x'^{\lambda}} + \tilde{g}^{\beta\lambda}\frac{\partial\tilde{g}^{\mu\alpha}}{\partial x'^{\lambda}} - \tilde{g}^{\mu\lambda}\frac{\partial\tilde{g}^{\alpha\beta}}{\partial x'^{\lambda}}\right), \tag{241}$$

$$\Pi'^{\nu}_{\alpha\beta} \triangleq g_{\alpha\lambda}g_{\beta\sigma}\Pi'^{\nu,\lambda\sigma}, \tag{242}$$

$$\Gamma'^{\alpha} \triangleq g^{\sigma\lambda}\Gamma'^{\alpha}_{\sigma\lambda}, \tag{243}$$

$$\Gamma'^{\nu}_{\alpha\beta} \triangleq \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\mu\alpha}}{\partial x'^{\beta}} + \frac{\partial g_{\mu\beta}}{\partial x'^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x'^{\mu}}\right) \tag{244}$$

$$\Gamma'^{\mu\nu} \triangleq \frac{1}{2}\left(g^{\mu\alpha}\frac{\partial\Gamma'^{\nu}}{\partial x'^{\alpha}} + g^{\nu\alpha}\frac{\partial\Gamma'^{\mu}}{\partial x'^{\alpha}} - \frac{\partial g^{\mu\nu}}{\partial x'^{\alpha}}\Gamma'^{\alpha}\right), \tag{245}$$

$$y'_{\beta} \triangleq \frac{\partial(\lg\sqrt{-g_0})}{\partial x'^{\beta}},\ y'^{\alpha} \triangleq g^{\alpha\beta}y'_{\beta}, \tag{246}$$

$$L' \triangleq -\frac{1}{2\sqrt{-g_0}}\Gamma'^{\nu}_{\alpha\beta}\frac{\partial\tilde{g}^{\alpha\beta}}{\partial x'^{\nu}} + \frac{1}{2}y'_{\nu}y'^{\nu}, \tag{247}$$

$$B'^{\mu\nu} \triangleq \Gamma'^{\mu\nu} + \frac{1}{2}(y'^{\mu}\Gamma'^{\nu} + y'^{\nu}\Gamma'^{\mu}),\ B' \triangleq g_{\mu\nu}B'^{\mu\nu}. \tag{248}$$

A proof of Proposition 38 can be found in the Appendix.

Theorem 39 *In the Fock coordinate system S_F, we have the following field equations*

$$G^{\mu\nu} - \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta} - \eta'^{\alpha\beta}\right)\frac{\partial^2(\sqrt{-g_0}g^{\mu\nu})}{\partial x'^{\alpha}\partial x'^{\beta}} - \frac{1}{2g_0}\eta'^{\alpha\beta}\frac{\partial^2\eta'^{\mu\nu}}{\partial x'^{\alpha}\partial x'^{\beta}} - \Pi'^{\mu,\alpha\beta}\Pi'^{\nu}_{\alpha\beta} + \frac{1}{2}y'^{\mu}y'^{\nu} - \frac{1}{2}g^{\mu\nu}(L' + B') + B'^{\mu\nu} = \frac{f_0^2}{g_0}(T'^{\mu\nu} - T'^{\mu\nu}_{\omega}). \tag{249}$$

Proof of Theorem 39. Using Eq.(239), Eq.(240) can be written as

$$\begin{aligned}
G^{\mu\nu} &= \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta}-\eta'^{\alpha\beta}+\eta'^{\alpha\beta}\right)\frac{\partial^2(\eta'^{\mu\nu}-2f_0\phi'^{\mu\nu})}{\partial x'^\alpha\partial x'^\beta}\\
&+\Pi'^{\mu,\alpha\beta}\Pi'^{\nu}_{\alpha\beta}-\frac{1}{2}y'^\mu y'^\nu+\frac{1}{2}g^{\mu\nu}(L'+B')-B'^{\mu\nu}\\
&= \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta}-\eta'^{\alpha\beta}\right)\frac{\partial^2(\eta'^{\mu\nu}-2f_0\phi'^{\mu\nu})}{\partial x'^\alpha\partial x'^\beta}\\
&+\frac{1}{2g_0}\eta'^{\alpha\beta}\frac{\partial^2\eta'^{\mu\nu}}{\partial x'^\alpha\partial x'^\beta}-\frac{f_0}{g_0}\eta'^{\alpha\beta}\frac{\partial^2\phi'^{\mu\nu}}{\partial x'^\alpha\partial x'^\beta}\\
&+\Pi'^{\mu,\alpha\beta}\Pi'^{\nu}_{\alpha\beta}-\frac{1}{2}y'^\mu y'^\nu\\
&+\frac{1}{2}g^{\mu\nu}(L'+B')-B'^{\mu\nu}. \qquad (250)
\end{aligned}$$

Using Eq.(239) and Eq.(237), Eq.(250) can be written as Eq.(249). □

We need to study the relationships between Eqs.(249) and the Einstein field equations. In a harmonic coordinates system, we have ([37], p. 254)

$$\Gamma'^\nu=\Gamma'^{\mu\nu}=B'^{\mu\nu}=B'=0. \qquad (251)$$

Using Eqs.(251) and Eqs.(249), we have the following result.

Corollary 40 *In the Fock coordinate system S_F the field equations (249) can be written as*

$$\begin{aligned}
G^{\mu\nu} &- \frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta}-\eta'^{\alpha\beta}\right)\frac{\partial^2(\sqrt{-g_0}g^{\mu\nu})}{\partial x'^\alpha\partial x'^\beta}\\
&- \frac{1}{2g_0}\eta'^{\alpha\beta}\frac{\partial^2\eta'^{\mu\nu}}{\partial x'^\alpha\partial x'^\beta}-\Pi'^{\mu,\alpha\beta}\Pi'^{\nu}_{\alpha\beta}\\
&+ \frac{1}{2}y'^\mu y'^\nu-\frac{1}{2}g^{\mu\nu}L'=\frac{f_0^2}{g_0}(T'^{\mu\nu}-T'^{\mu\nu}_\omega). \qquad (252)
\end{aligned}$$

Definition 41 *If the following conditions are valid*

$$\eta'^{\mu\nu}\approx\eta^{\mu\nu}, \qquad (253)$$

$$\left|\frac{1}{2}\eta'^{\alpha\beta}\frac{\partial^2\eta'^{\mu\nu}}{\partial x'^\alpha\partial x'^\beta}\right|\ll\left|f_0^2(T'^{\mu\nu}-T'^{\mu\nu}_\omega)\right|, \qquad (254)$$

then we call this reference frame quasi-inertial.

Using Eqs.(254) and Eq.(252), we have the following result.

Corollary 42 *If the reference frame S_F is quasi-inertial, then, the field equations (252) can be written as*

$$\begin{aligned}
&G^{\mu\nu}-\frac{1}{2g_0}\left(\sqrt{-g_0}g^{\alpha\beta}-\eta'^{\alpha\beta}\right)\frac{\partial^2(\sqrt{-g_0}g^{\alpha\beta})}{\partial x'^\alpha\partial x'^\beta}\\
&-\Pi'^{\mu,\alpha\beta}\Pi'^{\nu}_{\alpha\beta}+\frac{1}{2}y'^\mu y'^\nu-\frac{1}{2}g^{\mu\nu}L'\\
&\approx\frac{f_0^2}{g_0}(T'^{\mu\nu}-T'^{\mu\nu}_\omega). \qquad (255)
\end{aligned}$$

Eqs.(255) are only valid approximately in a quasi-inertial Fock coordinate system S_F. Now we consider weak fields.

Definition 43 *If $\phi'^{\mu\nu}$ and their first and higher derivatives satisfy the following conditions*

$$\left|2f_0\phi'^{\mu\nu}\right|\ll 1, \qquad (256)$$

$$\left|\frac{\partial^{j+k}(2f_0\phi'^{\mu\nu})}{\partial(x'^\alpha)^j\partial(x'^\beta)^k}\right|\ll 1, j+k=1,2,3,\cdots \qquad (257)$$

then we call this filed $\phi'^{\mu\nu}$ weak.

Similar to Ref. [46], we have the following result.

Corollary 44 *Suppose that (1) the Fock coordinate system S_F is quasi-inertial; (2) the filed is weak. Then, the field equations (255) reduce to*

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\approx\frac{f_0^2}{g_0}\left(T'^m_{\mu\nu}+T'^{\Omega(1)}_{\mu\nu}\right). \qquad (258)$$

Proof of Corollary 44. According to Definition 43, $f_0\phi'^{\mu\nu}$ and their first and higher derivatives are small quantities of order ε, where $|\varepsilon|\ll 1$ is a small quantity. Since the reference frame is quasi-inertial, Eqs. (253) are valid. Using Eqs. (253), Eqs. (239) can be written as

$$\tilde{g}^{\mu\nu}\triangleq\sqrt{-g_0}g^{\mu\nu}\triangleq\eta^{\mu\nu}-2f_0\phi'^{\mu\nu}. \qquad (259)$$

Since the filed is weak, Eqs. (256) and Eqs. (257) are valid. Thus, using Eqs. (256) and Eqs. (259), we have the following estimations of the order of magnitude of the following quantities

$$\sqrt{-g_0}g^{\mu\nu}-\eta'^{\mu\nu}\sim\varepsilon. \qquad (260)$$

Using Eqs. (257), we have the following estimations

$$\frac{\partial g_{\mu\nu}}{\partial x'^\alpha}\sim\frac{\partial g^{\mu\nu}}{\partial x'^\alpha}\sim\varepsilon. \qquad (261)$$

Applying Eqs. (259) and Eqs. (257), we have the following estimations

$$\frac{\partial^2(\sqrt{-g_0}g^{\alpha\beta})}{\partial x'^\alpha\partial x'^\beta}=\frac{\partial^2(-2f_0\phi'^{\mu\nu})}{\partial x'^\alpha\partial x'^\beta}\sim\varepsilon. \qquad (262)$$

Thus, using Eqs. (260) and Eqs. (262), we have the following estimations

$$\left(\sqrt{-g_0}g^{\alpha\beta}-\eta'^{\alpha\beta}\right)\frac{\partial^2(\sqrt{-g_0}g^{\alpha\beta})}{\partial x'^\alpha\partial x'^\beta}\sim\varepsilon^2. \qquad (263)$$

Applying Eqs. (241), Eqs. (242) and Eqs. (261), we have the following estimations

$$\Pi'^{\mu,\alpha\beta}\sim\Pi'^{\nu}_{\alpha\beta}\sim\varepsilon. \qquad (264)$$

Using Eqs. (246), we have the following relationship ([37], p. 143)

$$y'_\beta = \Gamma'^{\nu}_{\beta\nu}. \tag{265}$$

We also have ([37], p. 143)

$$\Gamma'^{\nu}_{\beta\nu} = \frac{1}{2} g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x'^{\beta}}. \tag{266}$$

Applying Eqs. (265), Eqs. (266) and Eqs. (261), we have the following estimations

$$y'_\beta \sim \varepsilon. \tag{267}$$

Using Eqs. (246) and Eqs. (267), we have the following estimations

$$y'^{\alpha} \sim \varepsilon. \tag{268}$$

Similar to the case of the Galilean coordinates, we have ([37], p. 430)

$$L' = -\frac{1}{2} \Gamma'^{\nu}_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial x'^{\nu}} - \Gamma'^{\alpha} y'_\alpha. \tag{269}$$

Applying Eqs. (251), Eq. (269) can be written as

$$L' = -\frac{1}{2} \Gamma'^{\nu}_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial x'^{\nu}}. \tag{270}$$

Using Eq. (270), Eqs. (244) and Eqs. (261), we have the following estimation

$$L' \sim \varepsilon^2. \tag{271}$$

Applying Eqs. (263), Eqs. (264), Eqs. (268) and Eq. (271), we see that the second to the fifth term on the right side of Eqs. (255) are all small quantities of order ε^2. Ignoring all these small quantities of order ε^2 in Eqs. (255) and using Eqs. (228), we obtain

$$G^{\mu\nu} \approx \frac{f_0^2}{g_0} \left(T'^{m}_{\mu\nu} + T'^{\Omega(1)}_{\mu\nu} \right). \tag{272}$$

Applying the rules of lowering or raising the indexes of tensors, i.e., $G^{\mu\nu} = g^{\mu\sigma} g^{\nu\lambda} G_{\sigma\lambda}$, $T'^{\mu\nu}_{m} = g^{\mu\sigma} g^{\nu\lambda} T'^{m}_{\sigma\lambda}$, $T'^{\mu\nu}_{\Omega(1)} = g^{\mu\sigma} g^{\nu\lambda} T'^{\Omega(1)}_{\sigma\lambda}$, Eqs. (272) can be written as

$$G_{\lambda\sigma} \approx \frac{f_0^2}{g_0} \left(T'^{m}_{\sigma\lambda} + T'^{\Omega(1)}_{\sigma\lambda} \right). \tag{273}$$

Putting Eqs. (238) into Eqs. (273), we obtain Eqs. (258). □

Using Eq. (170), the field equations (258) can be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \approx \frac{1}{g_0} \frac{8\pi\gamma_N}{c^4} \left(T'^{m}_{\mu\nu} + T'^{\Omega(1)}_{\mu\nu} \right). \tag{274}$$

Similar to Ref. [46], we have the following result.

Corollary 45 *Suppose that (1) the Fock coordinate system S_F is quasi-inertial; (2) the filed is weak; (3) $g_0 \approx -1$; ; (4) $T'^{\Omega(1)}_{\mu\nu} \approx 0$. Then, the field equations (258) reduce to*

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \approx -f_0^2 T'^{m}_{\mu\nu}. \tag{275}$$

We introduce the following notation

$$\kappa = f_0^2 = \frac{8\pi\gamma_N}{c^4}. \tag{276}$$

Using Eq. (276), the field equations (258) can be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \approx \frac{\kappa}{g_0} \left(T'^{m}_{\mu\nu} + T'^{\Omega(1)}_{\mu\nu} \right). \tag{277}$$

Using Eq. (276), the field equations (275) can be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \approx -\kappa T'^{m}_{\mu\nu}. \tag{278}$$

We notice that the field equations (278) are the Einstein's equations [1, 2, 36]. Therefore, the field equations (249) are generalizations of the Einstein's equations in some special non-inertial reference frames. Thus, all known experiments of gravitational phenomena which support the theory of general relativity may also be explained by this theory of gravity based on the theory of vacuum mechanics [27, 32, 33, 46].

XV. DISCUSSION

Although the field equations (249) are generalizations of the Einstein's equations, there exists at least the following differences between this theory and Einstein's theory of general relativity.

(1)We prove that the inertial mass of a microscope particle equals it's gravitational mass. This result is an assumption in Einstein's theory of general relativity and is called the principle of equivalence [1, 2, 36].

(2) In the theory of general relativity, the Einstein's equations are assumptions [1, 2, 36]. Although A. Einstein introduced his new concept of gravitational aether ([18], p. 63-113), he did not derive his equations theoretically based on his new concept of the gravitational aether. In our theory, the generalized Einstein's equations (249) are derived by methods of special relativistic continuum mechanics based on some assumptions.

(3) Although the theory of general relativity is a field theory of gravity, the definitions of gravitational fields are not based on continuum mechanics [1, 2, 36, 48–51]. Because of the absence of a continuum, the theory of general relativity may be regarded as a phenomenological theory of gravity. In our theory, gravity is transmitted by the $\Omega(0)$ substratum. The tensorial potential $\psi_{\mu\nu}$ of gravitational fields are defined based on special relativistic continuum mechanics.

(4) In Einstein's theory, the concept of Riemannian spacetime is introduced together with the field equations [1, 2, 36]. The theory of general relativity can not provide a physical definition of the metric tensor of the Riemannian spacetime. In our theory, the background spacetime is the Minkowshi spacetime. However, the initial flat background spacetime is no longer physically observable. According to the equation of motion of a point particle in gravitational field in inertial reference frames [46], to the first order of $f_0\psi_{\mu\nu}$, the physically observable spacetime is a Riemannian spacetime with the metric tensor $g_{\mu\nu}$. The metric tensor $g_{\mu\nu}$ is defined based on the tensorial potential $\psi_{\mu\nu}$ of gravitational fields.

(5) The masses of particles are constants in Einstein's theory of general relativity [1, 2, 36]. In our theory, the masses of particles are functions of time t [27].

(6) The gravitational constant γ_N is a constant in Einstein's theory of general relativity [1, 2, 36]. The theory of general relativity can not provide a derivation of γ_N. In our theory, the parameter γ_N is derived theoretically. From Eq. (151), we see that γ_N depends on time t.

(7) In our theory, the parameter γ_N in Eq. (151) depends on the density ρ_0 of the $\Omega(0)$ substratum. If ρ_0 varies from place to place, i.e., $\rho_0 = \rho_0(t, x, y, z)$, then the space dependence of the gravitational constant γ_N can be seen from Eq. (151).

(8) The Einstein's equations are supposed to be valid in all reference frames [1, 2, 36]. However, in our theory the generalized Einstein's equations (249) are valid only in some special non-inertial reference frames.

(9) The Einstein's equations are rigorous [1, 2, 36]. However, in our theory, Eqs.(278) are valid approximately under some assumptions.

I am curious whether it is possible for us to detect some of these differences by experiments.

XVI. CONCLUSION

We extend our previous theory of gravitation based on a sink flow model of particles by methods of special relativistic fluid mechanics. In inertial reference frames, we construct a tensorial potential of the $\Omega(0)$ substratum. Based on some assumptions, we show that this tensorial potential satisfies the wave equation. Inspired by the equation of motion of a test particle, a definition of a metric tensor of a Riemannian spacetime is introduced. Generalized Einstein's equations in inertial reference frames are derived based on some assumptions. These equations reduce to Einstein's equations in case of weak field in harmonic reference frames. In some special non-inertial reference frames, generalized Einstein's equations are derived based on some assumptions. If the field is weak and the reference frame is quasi-inertial, these generalized Einstein's equations reduce to Einstein's equations. Thus, all known experiments of gravitational phenomena which support the theory of general relativity may also be explained by this theory of gravity. In our theory, gravity is transmitted by the $\Omega(0)$ substratum. The theory of general relativity can not provide a physical definition of the metric tensor of the Riemannian spacetime. In our theory, the background spacetime is the Minkowshi spacetime. However, the flat background spacetime is no longer physically observable. According to the equation of motion of a point particle in gravitational field, to the first order, the physically observable spacetime is a Riemannian spacetime. The metric tensor of this Riemannian spacetime is defined based on the tensorial potential of gravitational fields.

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XVII. APPENDIX

Proof of Proposition 38. The definition of the covariant second rank curvature tensor $R_{\mu\nu}$ is ([37], p. 422)

$$R_{\mu\nu} \triangleq g^{\alpha\beta} R_{\mu\alpha,\beta\nu}, \tag{279}$$

where

$$\begin{aligned} R_{\mu\alpha,\beta\nu} \triangleq\ & \frac{1}{2}\left(\frac{\partial^2 g_{\mu\nu}}{\partial x'^\alpha \partial x'^\beta} + \frac{\partial^2 g_{\alpha\beta}}{\partial x'^\mu \partial x'^\nu} - \frac{\partial^2 g_{\nu\alpha}}{\partial x'^\mu \partial x'^\beta}\right. \\ & \left. - \frac{\partial^2 g_{\mu\beta}}{\partial x'^\nu \partial x'^\alpha}\right) - g_{\rho\sigma}\Gamma'^{\rho}_{\mu\beta}\Gamma'^{\sigma}_{\nu\alpha} \\ & + g_{\rho\sigma}\Gamma'^{\rho}_{\mu\nu}\Gamma'^{\sigma}_{\alpha\beta}, \end{aligned} \tag{280}$$

is the fourth rank curvature tensor.

The contravariant curvature tensor $R^{\mu\nu}$ can be obtained by raising the indices ([37], p. 156)

$$R^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} R_{\rho\sigma}. \tag{281}$$

Following similar methods of V. Fock ([37], p. 425), we have

$$R^{\mu\nu} = \frac{1}{2} g_{\alpha\beta} \frac{\partial^2 g^{\mu\nu}}{\partial x'^\alpha \partial x'^\beta} - \Gamma'^{\mu\nu} + \Gamma'^{\mu,\alpha\beta}\Gamma'^{\nu}_{\alpha\beta}. \tag{282}$$

The definition of the invariant of the curvature tensor is ([37], p. 425)

$$R \triangleq g_{\mu\nu} R^{\mu\nu}. \tag{283}$$

Following similar methods of V. Fock ([37], p. 428), we have

$$R = g^{\alpha\beta} y_{\alpha\beta} - \Gamma'^{\alpha} y_\alpha - \Gamma' - L', \tag{284}$$

where

$$y'_{\alpha\beta} \triangleq \frac{\partial^2 \lg \sqrt{-g_0}}{\partial x'^\alpha \partial x'^\beta}, \tag{285}$$

$$\Gamma' \triangleq g_{\mu\nu}\Gamma'^{\mu\nu}, \tag{286}$$

$$L' \triangleq -\frac{1}{2}\Gamma'^{\nu}_{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x'^\nu} - \Gamma'^\alpha \frac{\partial(\lg\sqrt{-g_0})}{\partial x'^\alpha}, \tag{287}$$

The second derivative of $\tilde{g}^{\mu\nu}$ is ([37], p. 428)

$$\begin{aligned}\frac{\partial^2 \tilde{g}^{\mu\nu}}{\partial x'^\alpha \partial x'^\beta} = \;& \sqrt{-g_0}\left(\frac{\partial^2 g^{\mu\nu}}{\partial x'^\alpha \partial x'^\beta} + y'_\beta \frac{\partial g_{\mu\nu}}{\partial x'^\alpha} + y'_\alpha \frac{\partial g_{\mu\nu}}{\partial x'^\beta}\right.\\ &\left. + y'_{\alpha\beta}g^{\mu\nu} + y'_\alpha y'_\beta g^{\mu\nu}\right). \end{aligned}\tag{288}$$

Multiplying $g^{\alpha\beta}$, Eqs. (288) can be written as ([37], p. 428)

$$\begin{aligned}g^{\alpha\beta}\frac{\partial^2 \tilde{g}^{\mu\nu}}{\partial x'^\alpha \partial x'^\beta} = \;& \sqrt{-g_0}\left(\frac{\partial^2 g^{\mu\nu}}{\partial x'^\alpha \partial x'^\beta} + 2y'^\alpha \frac{\partial g^{\mu\nu}}{\partial x'^\alpha}\right.\\ &\left. + g^{\mu\nu}g^{\alpha\beta}y'_{\alpha\beta} + g^{\mu\nu}y'_\alpha y'^\alpha\right). \end{aligned}\tag{289}$$

Using Eqs. (282) and Eqs. (284), we have ([37], p. 428)

$$\begin{aligned}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;=\;& \frac{1}{2}\left(g^{\alpha\beta}\frac{\partial^2 g_{\mu\nu}}{\partial x'^\alpha \partial x'^\beta} + g^{\mu\nu}g^{\alpha\beta}y'_{\alpha\beta}\right)\\ & + \frac{1}{2}g^{\mu\nu}(\Gamma'^\alpha y'_\alpha + \Gamma' + L')\\ & - \Gamma'^{\mu\nu} + \Gamma'^{\mu,\alpha\beta}\Gamma'^{\nu}_{\alpha\beta} \end{aligned}\tag{290}$$

Comparing Eqs. (290) and Eqs. (289), we have ([37], p. 429)

$$\begin{aligned}R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \;=\;& \frac{1}{2\sqrt{-g_0}}g^{\alpha\beta}\frac{\partial^2 \tilde{g}_{\mu\nu}}{\partial x'^\alpha \partial x'^\beta}\\ & + \frac{1}{2}g^{\mu\nu}(y'_\alpha y'^\alpha + \Gamma'^\alpha y'_\alpha + \Gamma' + L')\\ & - \Gamma'^{\mu\nu} + y'^\alpha \frac{\partial g^{\mu\nu}}{\partial x'^\alpha} + \Gamma'^{\mu,\alpha\beta}\Gamma'^{\nu}_{\alpha\beta} \end{aligned}\tag{291}$$

Using the notations defined in Eqs. (238), Eqs. (241), Eqs. (242) and Eqs. (248), Eqs. (291) can also be written as Eqs. (240) ([37], p. 429-430). □

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[1] S. Weinberg, *Gravitation and cosmology* (John Wiley & sons, New York, 1972).
[2] C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, New York, 1973).
[3] A. Hees and et al., Phys. Rev. Lett. **118**, 211101 (2017).
[4] R. P. Feynman, *Feynman lectures on gravitation, foreword by John Preskill and Kip S. Thorne* (Addison-Wesley Publishing Company, New York, 1995).
[5] J. Maddox, *What remains to be discovered* (The Free Press, New York, 1998).
[6] D. Marsh, Phys. Rev. Lett. **118**, 011302 (2017).
[7] A. Cho, Science **355**, 560 (2017).
[8] A. Einstein, Ann. Math **40**, 922 (1939).
[9] EHTC, The Astrophysical Journal Letters **875**, 1 (2019).
[10] P. Hamilton, M. Jaffe, P. Haslinger, Q. Simmons, H. Müller, and J. Khoury, Science **349**, 849 (2015).
[11] D. Spergel, Science **347**, 1100 (2015).
[12] J. Kim, P. Sikivie, and S. Weinberg, Nature **541**, 464 (2017).
[13] A. Bosma, Nature **543**, 179 (2017).
[14] E. Gibney, Nature **551**, 153 (2017).
[15] E. Whittaker, *A History of the Theories of Aether and Electricity, Revised and enlarged edtion, vol. 1* (Thomas Nelson and Sons Ltd., London, 1951).
[16] T. Hirosige, *History of physics, in Chinese, translated from the Japanese edtion published in 1968* (Qiushi Press, Beijing, 1968).
[17] I. Lakatos, *The methodology of scientific research programmes, Philosophical Papers, Volume I* (Cambridge University Press, New York, 1978).
[18] L. Kostro, *Einstein and the Ether* (Apeiron, Montreal, 2000).
[19] P. A. M. Dirac, *Direction in Physics* (Wiley, New York, 1978).
[20] D. R. Long, Nature **260**, 417 (1976).
[21] B. P. Abbott and et al., Phys. Rev. Lett. **116**, 061102 (2016).
[22] D. Mugnai, A. Ranfagni, and R. Ruggeri, Phys. Rev. Lett. **84**, 4830 (2000).
[23] D. Giovannini, J. Romero, V. Potocek, G. Ferenczi, F. Speirits, S. Barnett, D. Faccio, and M. Padgett, Science **347**, 857 (2015).
[24] J. R. Sambles, Science **347**, 828 (2015).
[25] N. Cornish, D. Blas, and G. Nardini, Phys. Rev. Lett. **119**, 161102 (2017).
[26] A. Einstein, *Letters to Solovine 1906-1955* (Citadel, 2000).
[27] X.-S. Wang, Progress in Physics **4**, 25 (2008).
[28] B. Riemann, *Collected Papers, Translated from the 1892 edition by Roger Baker, Charles Christenson and Henry Orde* (Kendrick Press, Heber, 2004).
[29] H. Poincare, *The science and hypothesis, translated from the French edtion, in Chinese* (Shangwu Press, Beijing, 1997).
[30] A. Einstein and L. Infeld, *The evolution of physics* (Cambridge University Press, London, 1938).
[31] I. G. Currie, *Fundamental Mechanics of Fluids* (Cambridge University Press, 2003), 3rd ed.
[32] X.-S. Wang, Progress in Physics **2**, 111 (2008).
[33] X.-S. Wang, Physics Essays **27**, 398 (2014).
[34] W. Thirring, Annals of Physics **16**, 96 (1961).
[35] S. Weinberg, Physical Review **138**, B988 (1965).
[36] C. Møller, *The theory of relativity, second edition* (Clarendon Press, Oxford, 1972).
[37] V. Fock, *The Theory of Space Time and Gravitation, 2nd Revised Edition, Translated from the Russian by N. Kem-*

mer (Pergamon Press, London, 1964).
[38] W. Rindler, *Introduction to special relativity* (Clarendon Press, Oxford, 1982).
[39] Z.-D. Chen, *Rational mechanics, in Chinese* (Chongqing Press, Chongqing, 2000).
[40] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics, 2nd Edition, translated from the Russian by J. B. Sykes and W. H. Reid* (Pergamon Press, Oxford, 1987).
[41] A. C. Eringen, *Elastodynamics* (Academic Press, New York, 1975).
[42] S. V. Babak and L. P. Grishchuk, Physical Review D **61**, 024038 (1999).
[43] H. C. Ohanian and R. Ruffini, *Gravitation and Spacetime, 3rd edition* (Cambridge University Press, New York, 2013).
[44] L. Liu, *The theory of general relativity, in Chinese, 2nd edition* (Advanced Education Press, Beijing, 2004).
[45] S. N. Gupta, Proc. Phys. Soc. **A65**, 608 (1952).
[46] X.-S. Wang, Reports on Mathematical Physics **84**, 35 (2019).
[47] V. Faraoni, *Special relativity* (Springer, New York, 2013).
[48] C. Truesdell, *The Elements of Continuum Mechanics* (Springer-Verlag, New York, 1966).
[49] Y. C. Fung, *A First Course in Continuum Mechanics* (Prentice-Hall, London, 1977).
[50] A. C. Eringen, *The Elements of Continuum Mechanics* (Robert E. Krieger Pub. Co., Huntington, 1980).
[51] L. D. Landau and Lifshitz, *Fluid Mechanics, translated from the Russian by J.B. Sykes and W.H. Reid.* (Pergamon, New York, 1987).